Differential Cohomology, L3

Chern-Weil Theory

Motivation: Recall Gauss-Bonnet Thm: Σ closed oriented surface

$$\int_\Sigma \kappa \, dA = 2\pi \chi(\Sigma)$$

Note: RHS $= \langle 2\pi \underbrace{e(T\Sigma)}_{\in H^2(\Sigma;\mathbb{R})}, [\Sigma] \rangle$

For the LHS, if R is the Riemann curvature tensor, then its antisymmetric, so locally of form $\begin{pmatrix} 0 & R_{12} \\ -R_{12} & 0 \end{pmatrix} dx_1 \wedge dx_2$.

Then $\kappa = R_{12}$, poly in coeffs of R

So LHS $= \langle$ "$\kappa \, dx_1 \wedge dx_2$", $[\Sigma] \rangle$

In other words, we took a polynomial & got a characteristic class.

The aim is to make this precise.

Connections over vector bundles: Fix vb $\pi: E \to M$.

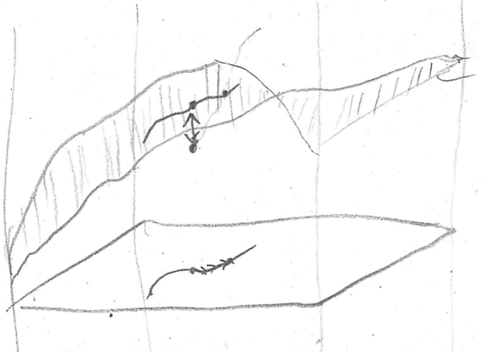
Want a meaningful to differentiate sections $s: M \to E$, i.e. we're looking for a map $\Gamma(E) \to \Gamma(T^*M \otimes E)$

(If you do the obvious thing & work in trivialisations, it's not compatible with transition functions)

Def (Ehresmann) A connection A is a splitting $TE \cong \underbrace{(\ker d\pi)}_{\text{vertical}} \oplus \underbrace{HE}_{\text{horizontal}}$ that's linear: $\forall y \in E \ \forall r \in \mathbb{R}$ s.t. $\overset{\text{multiplyer}}{r_*} HE_y = HE_{ry}$

Given a connection, we define the covariant derivative

$$d^A: \Gamma(E) \to \Gamma(T^*M \otimes E) \quad \text{as follows:}$$



section s

Take section s, point $x \in M$, direction $v \in T_xM$.

Draw path $\varphi: (-\varepsilon, \varepsilon) \to M$, $0 \mapsto x_0$

with $\varphi'(0) = v$

Then: for $t \in (-\varepsilon, \varepsilon)$ take $s(\varphi(t))$ & flow backwards along φ' under the so $HE \cong TM$ to $x^{(t)} \in E_x$

hen the point in E_{x_0} we want is $\lim_{t\to 0} \frac{1}{t}(x_0^{(t)} - s(x_0))$
& can check that this is well-defined, varies smoothly, is linear
n v, so we get a valid elt in $\Gamma(T^*M \otimes E)$ & d^A is smooth, linear.
ever,

: d^A satisfies Leibniz rule: $d^A(fs) = df \otimes s + f d^A s$.
f: Let $f: M \to \mathbb{R}$ smooth. By linearity, applying our definition to fs
ves $\lim_{t\to 0} \frac{1}{t}(f(t) x_0^{(t)} - f(0)s_0(x))$ instead. Apply product rule □

: Linear $\Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying Leibniz rule is the Koszul
finition of a connection; it's equivalent.
ting locally, $d^A s = ds + \overset{\text{(right)}}{A_\alpha s}$ in an open set U_α in which E is trivialised.
onnection is the same data as $A = \{A_\alpha\}$, $A_\alpha \in \Omega^1(U_\alpha; \mathfrak{gl}_n)$
here $A_\alpha = g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha}$ on overlaps
(transition function)

<u>nnections over principle G-bundles</u>. G Lie gp, $\pi: P \to M$ principal
-bundle. Then the definition for v.bs generalises:

f: A <u>connection</u> is a choice of splitting $TP \cong (\ker d\pi) \oplus HP$ s.t.
$\forall g \in G, \forall y \in E$, $HP_{g \cdot y} = g_* HP_y$

: G-structure gives us something stronger:
re $\ker d\pi \cong \mathfrak{g}$ (trivial $\mathfrak{g}$-bundle over E)
nce a splitting $TP \to \ker d\pi$ is the same data as a section in
$\omega \in \Gamma(T^*P \otimes \mathfrak{g}) = \Omega^1(P; \mathfrak{g})$, <u>connection 1-form</u>
: $\forall g \in G$ $\underset{\text{adjoint rep}}{Ad_g}(\underset{R_g:\text{ right-mult action}}{R_g^*}\omega) = \omega$, $\omega(X_a) = a \; \forall a \in \mathfrak{g}$
vector field arising from a.
: Equivalently, we want $\omega((R_g)_* X) = Ad(g^{-1}) X(\omega) \; \forall$ vector field X
CT decompose X into horizontal & vertical fields.
horizontal: both sides $= 0$.
vertical: At each $y \in P$ $\forall a \in g (\ker d\pi)_y$ $(R_g)_* a = Ad(g^{-1}) a$ □
other statement clear.
: Conversely, any $\omega \in \Omega^1(P; \mathfrak{g})$ for which this holds arises from a
connection.

Curvature of vector bundles: Fix connection A

Let $X, Y \in \Gamma(TM)$

Def: The curvature is $F_A(X,Y)(s) = [d^A X, d^A Y](s) - d^A[X,Y](s)$

Working locally, we get Cartan's structure equation $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$

Using this, we can check that $F_A \in \Omega^2(M; \mathrm{End}(E))$

Moreover, if we extend d^A to exterior covariant derivative $d^A : \Omega^\bullet(M;E) \to \Omega^{\bullet+1}(M;$

using Leibniz rule, then $(d^A)^2 = F_A \wedge -$

Thm (Bianchi identity) $dF_\alpha + [A_\alpha, F_\alpha] = 0$

Curvature of principal bundles: Fix connection 1-form ω. Analogously, define

Def: $\Omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)]$ curvature

This defines $\Omega \in \Omega^2(P; \mathfrak{g})$

e.g. If $G = GL_n$, we may identify $\underline{\mathfrak{g}} \cong \mathrm{End}(E)$ where E is the associated v.b.

Under this identification,

principal connection $\rightsquigarrow$ connection on v.b.

curvature of principal connection $\rightsquigarrow$ curvature

Thm (Bianchi identity) $d\Omega + [\omega, \Omega] = 0$

Invariant polynomials

Recall we applied a polynomial to the curvature tensor. For this to be well-defined, the polynomial has to be invariant under some structure group.

Def: Let G be a Lie gp, $\mathfrak{g}$ its associated Lie alg. A polynomial is an elt $T \in S^\bullet(\mathfrak{g}^\vee)$

e.g. $T \in S^1(\mathfrak{g}^\vee) = \mathfrak{g}^\vee$ is just a linear functional

e.g. $T \in S^2(\mathfrak{g}^\vee)$ is a symmetric bil. form

Def: $T \in S^\bullet(\mathfrak{g}^\vee)$ is invariant if $g \cdot T = T \;\forall g \in G$

The space of invariant polynomials is $S^\bullet(\mathfrak{g}^\vee)^G$

e.g. For $G = GL_n$, tr, det are invariant polynomials

The Chern-Weil homomorphism

For vector bundles: Let $G = GL_n$ or O_n or U_n; $T \in S^k(\mathfrak{g}^\vee)^G$, A connection

Construct $p_T(F_A) = T(\underbrace{F_A, \dots, F_A}_{k}) \in \Omega^{2k}(M;\mathbb{R})$

note this expression is indep of trivialisation as T is invariant

Prop: $p_T(F_A)$ is closed, i.e. $dp_T(F_A) = 0$.

Hence we get homomorphism $S^\bullet(\mathfrak{g}^\vee)^G \longrightarrow H^{2\bullet}_{dR}(M;\mathbb{R})$

Def: This is the <u>Chern-Weil homomorphism</u>

Pf: Locally, $dp_T(F_A) = dT(F_\alpha, \dots, F_\alpha) = T(dF_\alpha, F_\alpha, \dots, F_\alpha) + T(F_\alpha, dF_\alpha, \dots, F_\alpha) + \dots + T(F_\alpha, \dots, dF_\alpha)$

Bianchi
$= T([A_\alpha, F_\alpha], \dots, F_\alpha) + \dots + T(F_\alpha, \dots, [A_\alpha, F_\alpha])$.

Note in general, for $X, X_1, \dots, X_k \in \mathfrak{g}$,

$$T([X,X_1], X_2, \dots, X_k) + T(X_1, [X,X_2], \dots, X_k) + \dots + T(X_1, \dots, [X,X_k]) = 0$$

as $\frac{d}{dt}\Big|_{t=0} T(e^{tX}X_1e^{-tX}, \dots, e^{tX}X_k e^{-tX}) = 0$ by invariance of T.

Replace X_i by $w_i \otimes X_i$ $\forall i$, w_i even form & extend linearly, then

$$T([A,F_1], \dots, F_k) + \dots + T(F_1, \dots, F_{k-1}, [A,F_k]) = 0 \quad \forall A \in \Omega^1(U_\alpha;\mathfrak{g}),\ F_1, \dots, F_k \in \Omega^{\text{even}}(U_\alpha;\mathfrak{g})$$

Hence the expression above is 0 □

Prop: $[p_T(F_A)]$ is indep. of A

Pf (sketch): Let $\tilde{A}_\alpha$ be a connection on the pb
$$\begin{array}{ccc} \tilde{E} & \longrightarrow & E \\ \downarrow & & \downarrow \\ M\times\mathbb{R} & \longrightarrow & M \end{array}$$
given by $\tilde{A}_\alpha = (1-t)A_\alpha + tA'_\alpha \in \Omega^1(U_\alpha \times \mathbb{R}; \mathfrak{g})$. Then

$$p_T(F_A) - p_T(F_{A'}) = i_0^* p_T(F_{\tilde{A}}) - i_1^* p_T(F_{\tilde{A}}) \underbrace{=}_{\text{use fact that } \tilde{A} \text{ is closed}} \dots = 0. \quad □$$

Cor: $[p_T(F_A)] = [p_T(F_{A'})]$ if $E \cong E'$

Prop: This is a ring homomorphism; $[p_{ST}(F_A)] = [p_S(F_A) \wedge p_T(F_A)]$

Def: $[p_T(F_A)]$ is the <u>characteristic class</u> corresponding to T

To see why this is a characteristic class in the usual sense, we have to check it's compatible with pbs

Indeed, for $f: M \to N$, we have

$[p_T(F_{f^*A})] = [f^* p_T(F_A)] = f^*[p_T(F_A)]$.

For principal bundles, we can instead do $T \longmapsto p_T(\Omega)$ & the above [illegible]

Examples

Euler class: $\exists$ Pfaffian polynomial $pf : \mathfrak{o}_{2k} \to \mathbb{R}$ s.t. $pf(X)^2 = \det(X)$

This gives $[P_{pf}(F)] = e(TM)$, assuming Gauss-Bonnet.

Chern class: Let $P_\lambda = \det(\lambda\, id - \frac{1}{2\pi i}(-)) : \mathfrak{u}_k \to \mathbb{R}$.

This can be written as a sum of invariant polynomials

$$P_\lambda = \lambda^k - \lambda^{k-1}\sigma_1 + \lambda^{k-2}\sigma_2 - \cdots \pm \sigma_k$$

σ_i is (constant times) i^{th} symmetric poly of evalues of matrix

Lemma: The σ_i generate the ring of invariant polynomials as a ring

In fact, $\sigma_k = c_k$, kth Chern class.

Proof (sketch): Check that $c = P_{\pm\frac{1}{2\pi i}}$ is multiplicative.

Then, it suffices to show $c_1(\gamma_{\mathbb{C}}) = \sigma_1(\gamma_{\mathbb{C}})$

For $k=2$ (surface) use the fact that all Chern classes of line bundles are scalar multiples of c_1.

$\square$

Check against $\mathbb{CP}^1$ with Gauss-Bonnet.

Pontrjagin class: Similarly, $\det(\lambda\, id - \frac{1}{2\pi}(-)) = \lambda^k - \lambda^{k-1}P_1 + \lambda^{k-2}P_2 - \cdots$