Hochschild & cyclic homology.

Throughout: k comm ring ("base ring") A k-alg

Hochschild homology

Let A flat k-alg. The Hochschild complex is

$$HH(A/k) := \quad \underset{0}{A} \xleftarrow{b} \underset{1}{A \otimes_k A} \xleftarrow{b} \underset{2}{A \otimes_k A \otimes_k A} \leftarrow \cdots$$

grading: 0 1 2

where $b: a_0 \otimes \cdots \otimes a_n \mapsto a_0 a_1 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n + \cdots + (-1)^n a_0 \otimes \cdots \otimes a_{n-1} a_n$

$$+ (-1)^{n+1} a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

The Hochschild homology is $HH_*(A/k) := H_*(HH(A/k))$.

e.g. $HH_0(A/k) = A/b(A \otimes_k A) = A/[A,A]$

· If A commutative, $b_1 = 0$, so $HH_1(A/k) = A \otimes_k A / \underbrace{\langle ab \otimes c - a \otimes bc + ca \otimes b \mid a,b,c \in A \rangle}_{\text{cf Leibniz rule } ad(bc) = abdc + cadb}$

$$\Rightarrow \Omega^1(A/k) \xrightarrow{\sim} HH_1(A/k)$$
$$a\,db \longmapsto [a \otimes b].$$

Comparison to bar complex: (recall) The bar complex is

$$B_\cdot(A/k) := \quad \underset{0}{A \otimes_k A} \xleftarrow{b'} \underset{1}{A \otimes_k A \otimes_k A} \xleftarrow{b'} \cdots$$

grading: 0 1

where $b': a_0 \otimes \cdots \otimes a_n \longmapsto a_0 a_1 \otimes \cdots \otimes a_n - \cdots + (-1)^n a_0 \otimes \cdots \otimes a_{n-1} a_n$

Fact: multiplication $\mu: A \otimes_k A \to A$ defines quasi-iso $B_\cdot(A/k) \xrightarrow{\sim} A$ (cx conc. in deg 0)

Moreover $B_\cdot(A/k)$ is a cx of flat $A \otimes A^{op}$-mods
(act on 1st & last terms)

i.e. it's a resolution of A by flat $A \otimes A^{op}$-mods.

Fact: $A \otimes_{A \otimes A^{op}} B_\cdot(A/k) \xrightarrow{\cong} HH(A/k)$ iso of cxes

$$a_0 \otimes (f \otimes a_1 \otimes \cdots \otimes a_n \otimes g) \mapsto g a_0 f \otimes \cdots \otimes a_n.$$

Hence $HH(A/k)$ is a model for the derived tensor product $A \otimes^{\mathbb{L}}_{A \otimes A^{op}} A$ — in general (A not flat) take this to be the def of $HH(A/k)$

& hence $HH_*(A/k) \cong \mathrm{Tor}_*^{A \otimes A^{op}}(A,A)$

simplicial perspective $HH(A/k)$ arises from the simplicial algebra

$$A \leftleftarrows A \otimes_k A \leftleftarrows A \otimes_k A \otimes_k A \;\cdots$$

with face & degeneracy maps

$d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & i<n \\ a_n a_0 \otimes \cdots \otimes a_{n-1} & i=n \end{cases}$

$s_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n$

$HH(A/k)$ is the unnormalised chain cx of this, with $b = \sum_{i=0}^{n} (-1)^i d_i$.

When A commutative, this is a simplicial A-alg (A via first term in $\otimes$),

... formal reasons $\rightsquigarrow$ $HH_*(A/k) := \bigoplus_{n \geq 0} H_*(HH(A/k))$ has the structure of a graded-commutative A-algebra.

But since $\Omega^1_{A/k} \xrightarrow{\sim} HH_1(A/k)$ & $\Omega^1_{A/k}$ generates $\Omega^*_{A/k}$ by multiplication, this induces map of graded A-algs $\Omega^*_{A/k} \to HH_*(A/k)$.

Explicitly (unwinding product structure on $HH_*(A/k)$) this is

$\varepsilon_n : \Omega^n_{A/k} \to HH_n(A/k)$ — antisymmetrisation maps

$a\, db_1 \wedge \cdots \wedge db_n \mapsto \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} a \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n)}$ — when are these $\cong$? ...

Smooth algebras: Idea: alg analogue of sm. mfd.

Defn: comm k-alg A is smooth if it's flat &

$\forall$ prime ideal $p \lhd A$ $\exists$ $f \notin p$ & factorisation $k \hookrightarrow k[X_1, \ldots, X_n] \xrightarrow[\text{étale}]{\varphi} A_f$

(can cut open along a submfd so there's an étale map to $\mathbb{R}^{n \cdot m}$)

Thm (Hochschild–Kostant–Rosenberg) If A smooth k-alg, then

$\varepsilon_n : \Omega^n_{A/k} \xrightarrow{\cong} HH_n(A/k)$ iso of A-mods $\forall n \geq 0$.

Pf in the case where A is a polynomial alg:

Suffices to show $\Lambda^*_A HH_1(A/k) \xrightarrow{\cong} HH_*(A/k)$.

i.e. $\Lambda^*_A \mathrm{Tor}_1^{A \otimes_k A^{op}}(A, A) \xrightarrow{\cong} \mathrm{Tor}_*^{A \otimes_k A^{op}}(A, A)$

Can verify this directly: Let $A = k[x_1, \ldots, x_n]$, $A \otimes A^{op} = k[y_1, \ldots, y_n, z_1, \ldots, z_n]$

so $A = A \otimes A^{op} / (y_1 - z_1, \ldots, y_n - z_n) = \mathrm{coker}\left(\underbrace{(A \otimes A^{op})^{\oplus n}}_{=M} \xrightarrow{(y_1 - z_1, \ldots, y_n - z_n)} A \otimes A^{op}\right)$

Koszul cx $A \otimes A^{op} \leftarrow M \leftarrow \Lambda^2 M \leftarrow \Lambda^3 M \leftarrow \cdots$

$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^k r_{i_k} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_p}$

is a resolution of A with free $A \otimes A^{op}$ mods.

For the case of ∞-many variables, take colims over inclusions of the subalgs with fin. many vars.

15

Non-flat case: In analogy to the flat case, define

$$HH(A/k) := A \otimes^{\mathbb{L}}_{A\otimes^{\mathbb{L}} A^{op}} A$$

This can be computed as follows: take $\underset{\substack{\pi_0 P_\cdot \xrightarrow{\simeq} A \\ \pi_i P_\cdot = 0 \ \forall i>0}}{\underline{\text{simplicial resolution}}}$ of A by polynomial algs $\underset{\text{simplicial alg}}{P_\cdot} \longrightarrow A$.

$$\text{Then, } HH(A/k) := \underset{\text{diagonal/realisation}}{\Big|} \underbrace{P_\cdot \otimes_{P_\cdot \otimes P_\cdot^{op}} P_\cdot}_{\text{bisimplicial alg}} \Big| = \Big| \underbrace{HH(P_\cdot/k)}_{\text{simplicial alg}} \Big|$$

Each P_n is an A-mod by pulling back along $P_n \to A$

Fact (Dold-Kan): $\underset{\text{normalised chain cx}}{N}: (A\text{-mod})^{\Delta^{op}} \xrightarrow{\sim} Ch_{\geq 0}(A\text{-mod})$ (it's quasi-iso to the unnormalised one)

$\rightsquigarrow$ chain cx of A-mods.

Prop (Hochschild–Kostant–Rosenberg filtration): A comm. k-alg. Then (chain cx) $HH(A/k)$ admits natural filtration

$$0 \subseteq F^0 \subseteq F^1 \subseteq \cdots \subseteq HH(A/k)$$

s.t. $\operatorname{colim} F^n \xrightarrow{\sim} HH(A/k)$ & $\underset{\text{graded piece}}{F^i/F^{i-1}} \cong \mathbb{L}^i_{A/k}[i]$

(sketch) Proof: $HH(P_n/k)$ simplicial alg $\underset{\text{Postnikov filtration}}{\rightsquigarrow} \tau_{\leq 0} HH(P_n/k) \to \tau_{\leq 1} HH(P_n/k) \to \cdots$

take diagonal of each $\underset{\text{(bisimplicial alg)}}{\tau_{\leq i} HH(P_\cdot/k)}$, then apply N: $F^0 \subseteq F^1 \subseteq \cdots \subseteq HH(A/k)$

In fact, each $F^i/F^{i-1} = N\Big(\Big| \underbrace{\tau_{\leq i} HH(P_\cdot/k) / \tau_{\leq i-1} HH(P_\cdot/k)}_{\substack{\pi_i \cong \pi_i(HH(P_\cdot/k)) = HH_i(P_\cdot/k) \overset{HKR}{\cong} \Omega^i_{P_\cdot/k} \\ \pi_j = 0 \ \forall j \neq i}} \Big|\Big)$

$$= N\Big(\big| \overbrace{\Omega^i_{P_\cdot/k} \otimes^{\mathbb{L}}_{P_\cdot} P_\cdot}^{\Omega^i_{P_\cdot/k} \otimes P_\cdot} \big|\Big)[i] = N\big(\big|\Omega^i_{P_\cdot/k} \otimes A\big|\big)[i] =: \mathbb{L}^i_{A/k}[i]$$

recall this is a simplicial resolution of A

can replace $\otimes^{\mathbb{L}}$ with $\otimes$ as each $\Omega^i_{P_n/k}$ is a free P_n-module

: Let A comm, flat k-alg, $A \to A'$ étale. Then

(Th) $HH(A/k) \otimes_A A' \xrightarrow{\sim} HH_n(A'/k)$ iso $\forall n \geq 0$.

Pf: $A \to A'$ flat so want $HH(A/k) \otimes^{\mathbb{L}}_A A' \to HH(A'/k)$

HKR filtration
$\Longleftrightarrow$ $\mathbb{L}^i_{A/k} \otimes_A A' \to \mathbb{L}^i_{A'/k}$

which is true by properties of the cotangent cx. $\square$

This lets us prove the HKR thm for (general) smooth k-algs: work Zariski locally on $\operatorname{Spec} A$ & use $k[x_1, \ldots, x_n] \to A_f$ étale

cor used again here.

Cor used to go $HH_n(A/k) \otimes_A A_f \cong HH_n(A_f/k)$

Cyclic homology

Def: A cyclic object in a cat $\mathcal{C}$ is a simplicial obj $X_\bullet$ s.t. $\forall n$ $X_n \circlearrowleft \mathbb{Z}/_{n+1} = \langle t_n | t_n^{n+1} \rangle$

satisfying
$$t_n d_i = \begin{cases} d_{i-1} t_{n-1} & 1 \leq i \leq n \\ d_n & i = 0 \end{cases} \qquad t_n s_i = \begin{cases} d_{i-1} s_{n-1} & 1 \leq i \leq n \\ s_n t^2_{n+1} & i = 0 \end{cases}$$

Eg. $HH(A/k)$ (in classical setting) with

$t_n: a_0 \otimes \cdots \otimes a_n = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$

A mixed complex $(A_\bullet, b, B)$ is:

$$A_\bullet = \cdots \underset{B}{\overset{b}{\rightleftarrows}} A_{n-1} \underset{B}{\overset{b}{\rightleftarrows}} A_n \underset{B}{\overset{b}{\rightleftarrows}} A_{n+1} \underset{B}{\overset{b}{\rightleftarrows}} \cdots$$

A_n k-modules $\forall n$

s.t. $b^2 = 0$, $B^2 = 0$, $Bb = -bB$.

Eg. $(HH(A/k), b, B)$ $\qquad B = (1-(-1)^n t_n) s N$ $\qquad$ Connes operator

$N: A^{\otimes_k n} \longrightarrow A^{\otimes_k n}$ $\qquad N = \sum_{i=0}^{n-1} ((-1)^{n-1} t_{n-1})^i$ $\qquad$ "norm"

$s: A^{\otimes_k n} \longrightarrow A^{\otimes_k (n+1)}$ $\qquad s = 1 \otimes -$ $\qquad$ "extra degeneracy"

Rmk: B comes from the dR differential:

$$\begin{array}{ccc} HH_n(A/k) & \xrightarrow{B} & HH_{n+1}(A/k) \\ \uparrow \varepsilon_n & & \uparrow \varepsilon_{n+1} \\ \Omega^n_{A/k} & \xrightarrow{d} & \Omega^{n+1}_{A/k} \end{array}$$

Given a mixed cx, we can construct the BP, BC, BC^- bicomplexes.

Then $HP(A_\bullet, b, B) := \operatorname{Tot}(BP)$ $\qquad$ periodic cyclic homology

$HC(A_\bullet, b, B) := \operatorname{Tot}(BC)$ $\qquad$ cyclic homology

$HC^-(A_\bullet, b, B) := \operatorname{Tot}(BC^-)$ $\qquad$ negative cyclic homology

$$BP = \begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots & A_4 \arrow[l] \arrow[d, "b"'] & A_3 \arrow[l, "B"'] \arrow[d, "b"'] & A_2 \arrow[l, "B"'] \arrow[d, "b"'] & \cdots \arrow[l] \\
\cdots & A_3 \arrow[l] \arrow[d, "b"'] & A_2 \arrow[l, "B"'] \arrow[d, "b"'] & A_1 \arrow[l, "B"'] \arrow[d, "b"'] & \cdots \arrow[l] \\
\cdots & A_2 \arrow[l] \arrow[d, "b"'] & A_1 \arrow[l, "B"'] \arrow[d, "b"'] & A_0 \arrow[l, "B"'] & \\
\cdots & A_1 \arrow[l] \arrow[d] & A_0 \arrow[l, "B"'] & (0,0) & \\
 & A_0 & & &
\end{tikzcd}$$

BC

BC^-

There's an analogue of the HKR thm & filtration:

<u>Thm (Connes, Loday-Quillen, Feigin-Tsygan)</u> Let k comm ring, $\mathbb{Q} \subseteq k$, A sm k-alg. Then $\exists$ natural equivalences

$$HP(A/k) \simeq \prod_{i \in \mathbb{Z}} \Omega_{A/k}[2i] \qquad HC(A/k) \simeq \prod_{i \geq 0} \Omega^{\leq i}_{A/k}[2i].$$

$$HC^-(A/k) \simeq \prod_{i \in \mathbb{Z}} \Omega^{\geq i}_{A/k}[2i].$$

<u>Proof</u>: Let $\pi : A^{\otimes_k (n+1)} \longrightarrow \Omega^n_{A/k}$ This satisfies:

$a \otimes b_1 \otimes \cdots \otimes b_n \mapsto \frac{1}{n!} a\, db_1 \wedge \cdots \wedge db_n$

- $$\begin{tikzcd} A_k^{\otimes n} \arrow[d, "\pi"'] \arrow[r, phantom, "\Omega"] & A_k^{\otimes(n+1)} \arrow[l, "b"'] \arrow[d, "\pi"] \\ \Omega^{n-1}_{A/k} & \Omega^n_{A/k} \arrow[l, "0"] \end{tikzcd}$$
- $$\begin{tikzcd} A_k^{\otimes n} \arrow[r, "B"] \arrow[d, "\pi"'] & A_k^{\otimes(n+1)} \arrow[d, "\pi"] \\ \Omega^{n-1}_{A/k} \arrow[r, "d"] & \Omega^n_{A/k} \end{tikzcd}$$
- $\Omega^n_{A/k} \xrightarrow{\varepsilon_n} HH_n(A/k) \xrightarrow{H_n(\pi)} \Omega^n_{A/k}$ is $id_{\Omega^n_{A/k}}$.

Hence $\pi : (HH(A/k), b, B) \overset{q.i.}{\xrightarrow{\sim}} (\Omega_{A/k}, 0, d)$ of mixed cxes

$\leadsto$ iso on HC, HC^-, HP, which can be computed directly. $\square$

<u>Prop (HKR filtration)</u>: $HC(A/k)$ admits natural filtration

$HH(A/k) = F^0 \supseteq F^1 \supseteq \cdots$ s.t. $F^i/F^{i+1} \simeq \bigoplus_n \mathbb{L}^i_{A/k}[i+2n]$ $\forall i$, $(HH(A/k))_n \xrightarrow{\sim} \lim_i HH(A/k)_n / F^i$ $\forall i$.

<u>Proof</u> (sketch) in the case where A is a polynomial alg:

consider filtration on bicomplex

$$F^i := \begin{tikzcd} A^{\otimes(i+3)} & A_k^{\otimes(i+2)} \arrow[l, "B"'] \arrow[d, "b"] & \ker b \arrow[l, "B"'] \arrow[d, "b"] \\ A^{\otimes(i+2)} \arrow[u, leftarrow, "b"] & \ker b \arrow[l, "B"'] \arrow[r] & 0 \end{tikzcd}$$

corresponding filtration:

en $F^i/F^{i+1} = \mathrm{Tot}\left(\begin{smallmatrix} A^{\otimes i+2}/\ker b \leftarrow \ker b \leftarrow 0 \\ \downarrow \quad\quad \downarrow \\ \ker b \leftarrow 0 \\ \downarrow \\ 0 \end{smallmatrix}\right) = \bigoplus_{n\geq 0} \Omega^i_{A/k}[i+2n]$

eral A: similar $P_\cdot$ resolution this is 0 as argument? $\forall a \in \ker b$ $bBa = -Bba = 0 \Rightarrow Ba \in \ker b$. $\square$

r $k = p$ case: There is a version of this for char p.

m k perfect, char $k = p$, A smooth k-alg. Then HP, HC, HC^- admit natural complete filtrations $H(\,) \supseteq \cdots \supseteq F^1 \supseteq F^0 \supseteq F^{-1} \supseteq \cdots$ whose i^{th} gr pieces are (this is somewhat analogous to the HKR one)

$\Omega^\cdot_{R/k}[2i]$ $\quad\quad \Omega^{\leq i}_{R/k}[2i]$ $\quad\quad \Omega^{\geq i}_{R/k}[2i]$.

: This is a weaker statement than the previous Thm.

e's a sketch of the ideas that go into proving this.

f: $\mathcal{F}: \mathrm{Alg}_k \to \mathcal{D}$ (∞-cat with sifted colims) satisfies flat descent if $\forall$

faithfully flat $A \to B$, e.g. $D(\mathbb{Z})$, Sp

$$\mathcal{F}(A) \xrightarrow{\sim} \lim \mathcal{F}(\underbrace{\check{C}ech(B/A)})$$

Čech nerve $B \rightrightarrows B\otimes_A B \rightrightarrows B\otimes_A B \otimes_A B \cdots$

t: If R is a smooth k-alg, k perfect of char p, then $R \to R_{perf} := \mathrm{colim}(R \xrightarrow{\varphi} R \xrightarrow{\varphi} \cdots)$ (Frobenius; colimit perfection) is faithfully flat.

mma: $\mathbb{L}^i_{-/k}$ (Bhatt), $\underbrace{HH(-/k), HC(-/k)}_{\text{HKR filtration}}$, $\underbrace{HC^-(-/k)}_{\text{technical computation}}$, $HP(-/k)$ satisfy flat descent.

'satisfying flat descent is a derived limit so closed under derived limits'

naturally gives filtration via truncation.

mining these lets us express $HH(R/k)$ etc. as a limit of a cosimplicial diagram

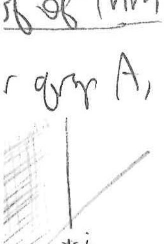
: An $\mathbb{F}_p$-alg A is quasiregular semiperfect (qrsp) iff the Frobenius is surj & $\mathbb{L}_{A/\mathbb{F}_p}$ is a flat A-mod supp. in homological deg 1.

: k perfect field, char p, R sm. k-alg. $\underbrace{R_{perf} \otimes_R \cdots \otimes_R R_{perf}}$ is qrsp $\forall n$.

ma: A qrsp $\mathbb{F}_p$-alg. Then $HH_*(A/k)$, $HC_*(A/k)$, $HC^-_*(A/k)$, $HP_*(A/k)$ are all supp. in even degrees. (spectral seq. from HKR filtration)

(sketch) f of Thm for HP: By periodicity, suff. to show $\mathrm{Tot}\left(HP_0(\underbrace{\check{C}ech(R_{perf}/R)}_{\text{each entry is qrsp}}/k)\right) \simeq \Omega^\cdot_{R/k}$

r qrsp A, $HP(A/k)$ has natural filtration coming from



graded pieces are exactly $HH(A/k)[2i]$

On $HP_0(A/k)$, this induces filtration with gr. pieces $\simeq HH_{2i}(A/k)$ $\overset{\text{spectral seq from HKR fil.}}{\parallel}$ $\mathbb{L}^i_{A/k}[i]$

So LHS has a filtration with gr. pieces

$$\mathrm{Tot}\left(\mathbb{L}^i_{R_{perf}/k} \rightrightarrows \mathbb{L}^i_{R_{perf}\otimes_R R_{perf}/k} \cdots\right)[i] \underset{\mathbb{L}_{-/k}\text{ flat}}{\simeq} \mathbb{L}^i_{R/k}[i] \simeq \Omega^i_{R/k}[i]$$

HC, HC^- follow from direct computation from studying $0 \to BC^- \to BP \to BC \to 0$

The quasisyntomic site

Def: $A \to B$ of $\mathbb{F}_p$ algs is quasisyntomic if it's flat & $\forall$ B-mod M, $\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} M$ is supp. in cohom. degrees $[-1,0]$.

It's a quasisyntomic cover if it's also faithfully flat.

$QSyn \subseteq Alg_{\mathbb{F}_p}$ full subcat of quasisyntomic algs

Lemma: $A \in QSyn \iff \exists$ qrsp $\mathbb{F}_p$ alg S, quasisyntomic cover $A \to S$

If so, all $S \otimes_A \cdots \otimes_A S$ are qrsp.

this is the main property we needed of $R \to R_{perf}$

Fact: $QSyn^{op}$ is a site with covers being quasisyntomic covers

Suppose $\mathcal{F}$ is a presheaf of ab. grps on $QSyn^{op}$ s.t.

(*) $\forall$ qsyn cover $A \to S$ of qrsp algs $0 \to \mathcal{F}(A) \to \mathcal{F}(S) \to \mathcal{F}(S \otimes_A S) \to$ is exact.

Then some formal arguments show that $\mathcal{F}|_{QRSP}$ is an ab. sheaf on $QRSP^{op}$ & it has no cohom. on any $A \in QRSP^{op}$.

full subcat of qrsp algs

Lemma: $\pi_n(\mathbb{L}^i_{-/\mathbb{F}_p})$, $HH_n(-/\mathbb{F}_p)$, $HC_n(-/\mathbb{F}_p)$, $HC_n^-(-/\mathbb{F}_p)$, $HP_n(-/\mathbb{F}_p)$ satisfy (*) $\forall$ i, n.