Modular functors

Idea: modular functors assign invariants to surfaces with ∂ (top, later cx) in a way that's compatible with gluing.

§ Classical version

For gluing to make sense, need ∂ parametrisations.

Def: An extended surface is an or surface with parametrisation of each ∂ component.

We'll consider homeos that preserve param, up to isotopy.

It's sometimes helpful to consider an alternate definition, with marked pts + a tangent vector instead of having ∂ + parametrisation:

Prop: $\left\{\begin{matrix}\text{surfaces} + \partial \text{ param} \\ \text{homeo / isotopy}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{surfaces} + \text{marked pts on each } \partial \text{ comp} \\ \text{homeo / isotopy}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{closed surfaces} + \text{marked \& tangent vectors} \\ \text{homeo / isotopy}\end{matrix}\right\}$

equivalent grpds.

Proof (sketch): First $\simeq$: $\mathrm{Homeo}^+(S^1, *)$ is contractible

Second $\simeq$: Glue in $\odot$ according to ∂ param.

Def: Teichmüller grpd Teich is that grpd defined above.

Def (modular functor, v1): A modular functor consists of:

- An abelian fin. cat. $\mathcal{C}$ & symmetric obj $R \in \mathrm{Ind}\,\mathcal{C}^{\boxtimes 2}$ obj R with $s: R^{op} \xrightarrow{\sim} R$ st. $s s^{op} = id_R$. $((-)^{op}: \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$ swap$)$

  [e.g. $\mathcal{C}$ ss: $R = \bigoplus_i V_i^* \otimes V_i$.

- An assignment $\Sigma \mapsto \left(\tau(\Sigma): \mathcal{C}^{\boxtimes \pi_0(\partial\Sigma)} \to \mathrm{Vect}_k\right)$ $\forall$ ext. surface Σ
  multilinear functor.

- Homeo $(f: \Sigma \xrightarrow{\sim} \Sigma') \longmapsto f_*: \tau(\Sigma) \xrightarrow{\cong} \tau(\Sigma')$ natural iso.

- Isomorphisms $\tau(\emptyset) \xrightarrow{\sim} k$, $\tau(N_1 \sqcup N_2) \xrightarrow{\sim} \tau(N_1) \otimes \tau(N_2)$ ("monoidality")

- Gluing: Given param. of ∂ comps α, β of Σ, can glue them together to get new extended surface $\sqcup_{\alpha\beta} \Sigma$.

  This induces nat. iso $G_{\alpha\beta}: \bigoplus_i \tau(\Sigma)(-, A_i, B_i) \xrightarrow{\cong} \tau(\sqcup_{\alpha\beta}\Sigma)(-)$
  where $R = \bigoplus_i A_i \boxtimes B_i$.

Satisfying some compatibility conditions:

- f_* compatible with $\circ$
- compatibility between all the assignments above

symmetry of $G_{\alpha\beta}$ $\qquad G_{\alpha\beta} = G_{\beta\alpha}$ after identifying $K \to K^{op}$

normalisation $\quad \tau(S^2) = k$

[wr]iting the compatibility conditions is very cumbersome so we want to capture this data in a more efficient way

Towers of groupoids

[De]f: A *tower of groupoids* $\mathcal{T}$ is a symmetric monoidal gpd with ⊗functor $A : \mathcal{T} \to \mathrm{Fin}$ *boundary functor*
(think: $\pi_0 \partial$)

[..]ed *gluing*: $\forall\, \Sigma \in \mathrm{Obj}(\mathcal{T})$ $\forall\, \alpha,\beta \in A(\Sigma)$ have $G_{\alpha\beta}(\Sigma)$

s.t.
- $A(G_{\alpha\beta}(\Sigma)) = A(\Sigma) \setminus \{\alpha\beta\}$
- $G_{\alpha\beta}(- \sqcup \Sigma_1) \cong G_{\alpha\beta}(-) \sqcup \Sigma_1$ canonically
- $G_{\alpha\beta} G_{\gamma\delta}(-) \cong G_{\gamma\delta} G_{\alpha\beta}(-)$ canonically
- functoriality.

e.g. $(\mathrm{Teich}, \sqcup, \emptyset, \pi_0\partial)$, $\quad (\mathrm{Fin}, \sqcup, \emptyset, \mathrm{id})$

e.g. $\mathcal{C}_e$ ab. lin. cat, $R \in \mathrm{Ind}\, \mathcal{C}_e^{\boxtimes 2}$ sym. obj.

$$\mathrm{Fun}(\mathcal{C}_e) := \left\{ \begin{array}{l} (S,F) \quad S \in \mathrm{Fin},\ F : \mathcal{C}_e^{\boxtimes S} \to \mathrm{Vect}_k \\ (\varphi : S_1 \xrightarrow{\cong} S_2,\ f : F_1 \xrightarrow{\cong} \varphi^* F_2) \end{array} \right\}, \quad (\sqcup, \otimes)$$

$A : (S,F) \mapsto S$

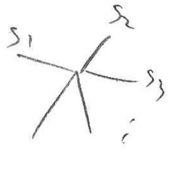
$G_{\alpha\beta}(S,F) = \left(S \setminus \{\alpha\beta\}, \bigoplus_i F(-, A_i, B_i)\right)$

[De]f: A *tower functor* $\mathcal{F}$ between towers of gpds is a functor preserving all the structure $(A, \sqcup, G_{\alpha\beta})$

[De]f: (modular functor v2) A *modular functor* is a tower functor
[τ]$: \mathrm{Teich} \longrightarrow \mathrm{Fun}(\mathcal{C}_e)$ (for some $\mathcal{C}_e$, $R \in \mathrm{Ind}\, \mathcal{C}_e^{\boxtimes 2}$ sym obj)
s.t. $\tau(S^2) \cong \mathrm{id}_{\mathrm{Vect}_k}$

[Th]is is better, but there are still quite some coherences to keep track of.

[Gr]aphs

[Id]ea: View a set as the following graph: 

$S = \{s_1, \dots\}$

corolla →

Note that a tower functor can be thought of as a sym. mon. cat fibred over Fin.

(Explicitly, the fibre over $S \in \mathrm{Fin}$ is $\left\{ \begin{array}{l} (\Sigma, \varphi),\ \Sigma \in \mathcal{T},\ \varphi : A(\Sigma) \xrightarrow{\sim} S \\ f : \Sigma_1 \to \Sigma_2 \text{ s.t. } \begin{tikzcd}[column sep=small] A(\Sigma_1) \arrow[rr, "A(f)"] \arrow[dr, "\varphi_1"'] & \arrow[d, phantom, "\cong"] & A(\Sigma_2) \arrow[dl, "\varphi_2"] \\ & S & \end{tikzcd} \end{array} \right\}$)

Want to add more morphisms to Fin which correspond to gluing.

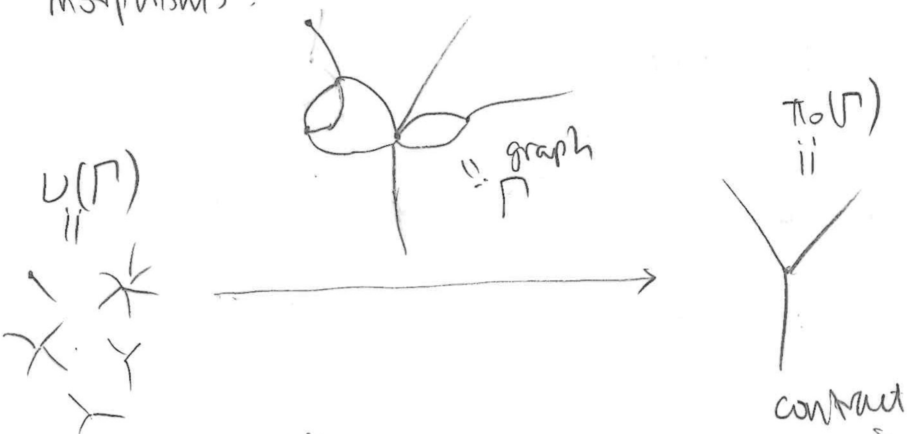
e.g.

think of this as being represented by the graph 

Def: The category Graphs has

- objects: disjoint unions of corollas

- morphisms:



all the local corollas

contract all non-infinite edges

graph: can have loops, multiedge; edges can go to infinity.

composition: "blowing up each vertex" cf. little disks

insert these into the circles.

Idea: Graphs capture all the ways to take $\sqcup$ and glue ∂ comps together.

This is sym mon under $\sqcup$.

Def: A <u>modular operad</u> is a sym. mon. functor $\mathcal{O}$: Graphs $\to$ Cat
viewed as a sym mon (2,1)-ca[t]

(under straightening-unstraightening, think of this as identifying the fibre over each graph)

e.g. <u>modular surface operad</u> Surf : Graphs $\longrightarrow$ Cat

$\mathrm{Surf}(\times\}^{n\,\text{legs}}) = \left\{ \begin{array}{l} \text{conn. surfaces with } n \text{ parametrised } \partial \text{ comps} \\ \text{homeo / isotopy} \end{array} \right\} \left(= \coprod_g \mathrm{Teich}_{g,n} \right)$

morphisms: gluing surfaces

e.g. modular endomorphism operad. Let $\mathcal{S}$ sym mon (2,1)-cat,

$X \in \mathcal{S}$ self-dual. $\Rightarrow \exists$ $K : X \otimes X \to \mathbb{1}$ non-degen sym. pairing
(ev)
(coev) $\Delta : \mathbb{1} \to X \otimes X$

Then define $\mathrm{End}_K^X : \mathrm{Graphs} \to \mathrm{Cat}$

$$\times\}^{n} \longmapsto \mathrm{Hom}_{\mathcal{S}}(X^{\otimes n}, \mathbb{1})$$

Note: conventions differ here in the literature

On morphisms: pre-comp. with Δ

Prop (Müller–Woike) End_K^X is a modular operad.

e.g. The case we care about: $\mathcal{S} = \mathrm{Rex}^f$ "finite cats"

$\mathrm{Rex}^f := \left\{ \begin{array}{l} \text{ab. lin. cats with f.d. Hom spaces, f. many simples (}\cong\text{), enough projectives,} \\ \quad \text{f. length for each obj} \\ \text{right exact functors} \\ \text{lin nat. transfs.} \end{array} \right\}$ sym mon with Deligne $\boxtimes$.

For $\mathcal{C}$ ss, self-dual, $\Delta : \mathrm{Vect}_k \to \mathcal{C} \boxtimes \mathcal{C}$

must be of form $\mathbb{C} \mapsto \bigoplus X^* \otimes X$.

cf. the R we had in v1

(modular functor, v3) A modular functor is a sym mon. nat. transf

$$\begin{tikzcd} \mathrm{Graphs} \arrow[r, bend left, "\mathrm{surf}"] \arrow[r, bend right, ""'] \arrow[r, phantom, "\Downarrow F"] & \mathrm{Cat} \end{tikzcd}$$

for some $\mathcal{C} \in \mathrm{Rex}^f$

is exactly the $\mathrm{End}_K^{\mathcal{C}}$ same as the prev. def^n, minus the normalisation.

Complex modular functors

We want to repeat the same story for complex curves, but now we have to be careful with $\widetilde{\ }$ gluing.

Recall: $\widetilde{M}_{g,n} \subseteq M_{g,n}$

moduli stack of genus g curves with n marked pts + $(\neq 0)$ tangent vector

$\hookleftarrow$ adding curves whose singularities are ordinary double pts which don't coincide with the marked pts.

s.t. $|\mathrm{Aut}| < \infty$.

Then $D := \overline{\widetilde{M}}_{g,n} \setminus \widetilde{M}_{g,n}$ is a divisor with normal crossings

(i.e. locally looks like $\times$)

We can use these formal properties to define gluing.

We want a gluing functor that's defined on

$$\mathrm{Teich}^{\mathbb{C}} = \coprod_{g,n} \Pi_1(\tilde{\mathcal{M}}_{g,n})$$

More convenient to index marked pts by a set; let

$\mathcal{M}_{*,A}$:= moduli stack of curves with $|A|$ marked pts + tangent vectors indexed by A

$\mathrm{Teich}^{\mathbb{C}}_A := \Pi_1(\tilde{\mathcal{M}}_{*,A})$.

Then: for α, β we can glue the corr pts together to get a curve with a double pt; this gives something in D.

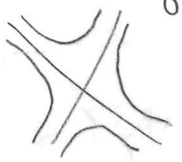
<u>Idea</u>: "project out" from D to get a nonsingular curve.

<u>Lemma</u>: C stable sing. curve with 1 double pt a. Then $N_C D$ (normal bundle at C) is 1-dim & canonically identified with $\underbrace{T'_a C \otimes T''_a C}_{\text{tangent spaces at each component}}$

<u>Proof</u> (sketch): Locally at a, it's $t't'' = 0$. Can deform to $t't'' = q$. then define

$$N_C D \longrightarrow T'_a C \otimes T''_a C$$
$$\partial_q \longmapsto \partial_{t'} \otimes \partial_{t''}$$

defines a $\cong$ & is indep of local coords. $\square$

Picture: 

Now, can define <u>clutching map</u> $S_{\alpha\beta}: \tilde{\mathcal{M}}_{*,A} \longrightarrow N(D)$ — in fact, lies in $N(D) \setminus$ zero section.

$$C \longrightarrow (C/\alpha\sim\beta,\ \overbrace{v_\alpha \otimes v_\beta}^{\neq 0} \in T_\alpha C \times T_\beta C)$$

viewed as vector in normal b

Then pass to Π_1: get $G_{\alpha\beta}: \mathrm{Teich}^{\mathbb{C}}_A \to \Pi_1(N^\times(D)) \xrightarrow[\text{tubular nbhd}]{\text{ind. of}} \Pi_1(\tilde{\mathcal{M}}_{*, A\setminus\{\alpha,\beta\}}) = \mathrm{Teich}_A$

Instead of defining a functor from $\Pi_1(\tilde{\mathcal{M}}_{*,A})$ to $\mathrm{Fun}(\mathcal{C}^{\boxtimes A}, \mathrm{Vect}_k)$, we want a more "stacky" defn working with $\tilde{\mathcal{M}}_{*,A}$ directly.

Specifically, we want to assign each obj of $\mathcal{C}^{\boxtimes A}$ to some "nice" v.b. over $\tilde{\mathcal{M}}_{*}$

Def: $\mathcal{RS}(\overline{\mathcal{M}}, \mathcal{M}) := \left\{ \begin{array}{l} \text{f.d. v.b.s on } \overline{\mathcal{M}} \text{ with flat connection on } \mathcal{M} \text{ with regular singularities} \\ \quad \text{at } D = \overline{\mathcal{M}} \setminus \mathcal{M} \\ \text{meromorphic morphisms} \end{array} \right\}$

regular singularities: (E, ∇) in local coordinates around $D = z_1 \cdots z_k$:

conn.
$$\left.\begin{array}{ll} \nabla_i = \dfrac{\partial}{\partial z_i} + \dfrac{A_i(z)}{z_i} & i \leq k \\ \nabla_i = \dfrac{\partial}{\partial z_i} + A_i(z) & i > k. \end{array}\right\} \text{"logarithmic singularity"}$$

o: (modular functor v4) A cx modular functor is:

- A $\otimes$-functor $\mathcal{C}^{\boxtimes A} \longrightarrow RS(\widetilde{\overline{M}}_{*,A}, \widetilde{M}_{*,A})$ $\forall A \in \mathrm{Fin}$.

$\qquad W \longmapsto \langle W \rangle$.

- gluing: iso $G_{\alpha,\beta}: \langle W \boxtimes R \rangle \xrightarrow{\sim} S^*_{\alpha\beta} \underbrace{S_{\gamma_D}\langle W \rangle}_{\text{restrict to } RS(N^\circ D,\, N^* D)}$ $\qquad \forall \alpha, \beta$.

- propagation of vacua: $S_{\alpha\beta}$ only defined in the stable case so need to handle forgetting disks separately
  (forgetting $\longleftrightarrow$ removing marked pt.)

  $\exists\, \mathbb{1} \in \mathcal{C}$, $\forall \alpha$, $G_\alpha: \langle W \boxtimes \mathbb{1} \rangle \xrightarrow{\sim} S^*_\alpha \langle W \rangle$

  (compatibility conditions)

Thm: Def$^{\text{ns}}$ of cx modular functor & top. modular functor are equiv.

Pf (sketch): tower functor $\mathrm{Teich} \longrightarrow \mathrm{Fun}(\mathcal{C})$

$\Longleftrightarrow$ tower functor $\mathrm{Teich}^{\mathrm{stab}} \longrightarrow \mathrm{Fun}(\mathcal{C})$ + condition on removing marked pts

Thm: $\mathrm{Teich}^{\mathrm{stab}} \simeq \mathrm{Teich}^{c}$ (remove $(g,n) = (0,0), (1,0), (0,1)$ as towers of grids)

$\Longleftrightarrow$ tower functor $\mathrm{Teich}^{c} \longrightarrow \mathrm{Fun}(\mathcal{C})$ + ——— " ———

$\Longleftrightarrow$ assignment of each $W \in \mathcal{C}^{\boxtimes A}$ to a local system on $M_{*,A}$ + ——— " ———

Thm (Riemann-Hilbert corr.) $RS(\overline{M}, M) \xrightarrow{\sim} \{\text{local sys. on } M\}$

$\Longleftrightarrow$ $\mathcal{C}^{\boxtimes A} \to RS(\overline{M}_{*,A}, M_{*,A})$ s.t. $(-)$ + propagation of vacua. $\square$

Anomalies

...rent def$^{\text{n}}$(s) too restrictive — $\exists$ "framing anomaly" in examples that we can construct.

Solution: weaken def$^{\text{n}}$(s) slightly by replacing Surf with an extension.

This is a modular spread Q with nat. transf $Q \to \mathrm{Surf}$ s.t.

- $\forall \Gamma \in \mathrm{Graphs}$ $\forall \Sigma \in \mathrm{Surf}(\Gamma)$ hofib of $Q(\Gamma) \to \mathrm{Surf}(\Gamma)$ connected.
- On genus 0 part, $\exists$ section satisfying some properties ("insertion of vacua")

...e case we care about is some $Q_c = \mathrm{Surf}^{c}$, $c \in \mathbb{C}$, central charge of VOA.

...stead of defining this directly, we instead describe the corresponding change in the def$^{\text{n}}$ of cx modular functor:

$RS(\overline{M}, M) \longleftrightarrow$ modules over $\mathcal{D} := \{\partial \in \mathcal{D}_{\overline{M}} \mid \partial I \subseteq I\}$. (by def of log singularity)

(I: sheaf of functions on $\overline{M}$ vanishing at D.)

... this ... $\mathcal{D}$-module twisted over the Atiyah alg (defined last week)