Noether's theorem & factorisation algebras

Intuitively Noether's thm says "symmetries acting on a classical field theory yield conservation laws".

We want to see what this means in our framework.

Symmetries: These will be <u>local L_∞ algebras</u>.

Def: a <u>L_∞ algebra</u> is a $\mathbb{Z}$-graded v.s. $\mathfrak{g}$ with $\forall n \geq 1$ a multilinear map

$$l_n : \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \to \mathfrak{g} \text{ of degree } 2-n$$

s.t.

- (graded skew symmetry) $l_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = (-1)^{|x_i||x_{i+1}|} l_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$
- (strong htpy Jacobi identity) $\sum_{k=1}^{n} (-1)^k \sum_{\substack{\text{partition of } [n] \\ \text{into } \{i_1 < \cdots < i_k\} \\ \& \{j_{k+1} < \cdots < j_n\}}} \operatorname{sign}(\sigma)\, l_{n-k+1}(l_k(x_{i_1}, \ldots, x_{i_k}), x_{j_{k+1}}, \ldots, x_{j_n})$

$\sigma: r \mapsto \begin{cases} i_r & r \leq k \\ j_r & r \geq k+1 \end{cases}$

think of this as a dgla, where l_1 is the differential, l_2 is $[\,,\,]$, but now Jacobi identity only holds up to homotopy.

Def: A <u>local L_∞ algebra</u> on mfd M consists of

- a graded v.b. $L \to M$
- a differential operator $d : \mathcal{L} \to \mathcal{L}$ of cohom. deg 1, sq. zero. ($\mathcal{L}$ = sheaf of smooth sections)
- $\forall n \geq 2$ polydifferential operators $l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}$ which are alternating, deg $2-n$, & endow $\mathcal{L}$ with the structure of an L_∞ algebra.

<u>Recall</u>: <u>differential operator</u> $D : \mathcal{E}_1 \to \mathcal{E}_2$ map of sheaves of sections of E_1, E_2 which factors as

$$\mathcal{E}_1 \longrightarrow \underbrace{J(E)}_{\text{sheaf of sections of jets}} \longrightarrow \mathcal{E}_2$$

<u>Fact</u>: Let D_M be the sheaf of diff. operators on M with smooth coefficients

differential operator $\mathcal{E} \to \mathcal{F}$ $\longleftrightarrow$ map of D_M-modules $J(E) \to J(F)$

similarly: polydiff. operator $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n \to \mathcal{F}$ $\longleftrightarrow$ map of D_M-modules $J(E_1) \otimes_{C^\infty_M} \cdots \otimes_{C^\infty_M} J(E_n) \to J(F)$

Hence: local L_∞ alg structure on $\mathcal{L}$ is the same thing as L_∞ alg structure on $J(L)$ in the category of D_M-modules. (symm. mon. cat. with mon. structure given by $\otimes_{C^\infty_M}$)

e.g. L_∞ alg $\mathfrak{g}$ acting $\longleftrightarrow$ $\mathfrak{g} \otimes \Omega^*_M$ acting

Classical field theories: <u>Recall</u> <u>free classical field theory</u> on a mfd M consists of

- $\mathbb{Z}$-graded v.b. E with (-1)-shifted symp. structure $\langle\,,\,\rangle$
- diff. operator $Q : \mathcal{E} \to \mathcal{E}$ coh. degree $+1$, sq zero, skew-self-adj wrt $\langle\,,\,\rangle$

s.t. $(\mathcal{E}, Q)$ is elliptic.

This recovers physics, e.g. free scalar field theory recovers E-L equation $(\Delta + m^2)\varphi = 0$

ote: This only recovers linear diff. eq^ns. We require a more general concept:

): A classical field theory on mfd M consists of

local L_∞ alg $\mathcal{L}$ on M

invariant pairing of coh. deg. -3 $\langle , \rangle : L \otimes L \to \mathrm{Dens}_M[-3]$

sym., non-degen., (i.e. induces $E \cong E^![-3]$) $(\alpha_1, \dots, \alpha_{n+1}) \longmapsto \int \langle l_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1} \rangle$ graded antisymmetric for $\alpha_i \in \mathcal{L}_c$ $\forall n \geq 1$

.e. $(\mathcal{L}, d)$ is elliptic.

free field theory is a special case where the l_n are trivial $\forall n \geq 2$

& grading is shifted by $+1$)

: l_n encode higher-order terms in the E-L eq^ns

g. interacting scalar field theory

$\mathcal{L}(U): C^\infty(U)[-1] \xrightarrow{\Delta} C^\infty(U)[-2]$

pairing: $\langle \varphi_1, \varphi_2 \rangle = \int_U \varphi_1 \varphi_2$

l_n trivial except for $n=3$: $l_3(\underbrace{\varphi_1, \varphi_2, \varphi_3}_{\text{all in deg } 1}) = \underbrace{\varphi_1 \varphi_2 \varphi_3}_{\text{in deg } 2}$

E-L: $d\varphi + \sum \frac{1}{n!} l_n(\varphi, \dots, \varphi) = 0$ i.e. $\Delta \varphi + \frac{1}{3!}\varphi^3 = 0$.

servables: $\mathrm{Obs}^{cl}(U) = CE^*(\mathcal{L}(U))$

Recall: For L_∞ alg $\mathfrak{g}$, $CE^*(\mathfrak{g})$ is $\widehat{\mathrm{Sym}}(\mathfrak{g}[1]^\vee) = \prod_{n=0}^{\infty} ((\mathfrak{g}[1]^\vee)^{\otimes n})_{S_n}$

with differential dual to the l_n

m (Costello-Gwilliam) $\exists$ P_0 factorisation algebra $\widetilde{\mathrm{Obs}}^{cl}$ that's weakly equivalent

as FAs of cdgas) to Obs^{cl}.

the case of free field theories, this is $\widetilde{\mathrm{Obs}}^{cl}(U) = \mathrm{Sym}(\mathcal{E}_c^!(U))$)

tions

the level of Lie algebras, action of $\mathfrak{g}$ on $\mathfrak{h}$ $\rightsquigarrow$ $\mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ $\rightsquigarrow$ semidirect product

ucture on $\mathfrak{g} \oplus \mathfrak{h}$.

milarly, we define

f: Action of L_∞ alg $\mathfrak{g}$ on L_∞ alg $\mathfrak{h}$ is an L_∞ alg structure on $\mathfrak{g} \oplus \mathfrak{h}$, denoted

$\mathfrak{g} \ltimes \mathfrak{h}$, s.t. $0 \to \mathfrak{h} \to \mathfrak{g} \ltimes \mathfrak{h} \to \mathfrak{g} \to 0$ exact. (map of L_∞ algs)

f: Action of local L_∞ alg $\mathcal{L}$ on L_∞ alg $\mathcal{M}$ is a local L_∞ alg structure on $\mathcal{L} \oplus \mathcal{M}$, denoted

$\mathcal{L} \ltimes \mathcal{M}$, s.t. $0 \to \mathcal{M} \to \mathcal{L} \ltimes \mathcal{M} \to \mathcal{L} \to 0$ (maps of sheaves of L_∞ algs)

f: local L_∞ alg acts on classical field theory $\mathcal{M}$ if it preserves the pairing:

$\forall \alpha_1, \dots, \alpha_r \in \mathcal{L}_c$, $\beta_1, \dots, \beta_s \in \mathcal{M}_c$, $\langle l_{r+s}(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s), \beta_{s+1} \rangle$ graded totally antisym in β_i

central extensions associated to actions: Given local L_∞ alg acting on classical field theory $\mathcal{M}$, our version of Noether's thm involves a central extension $\widehat{\mathcal{L}}_c$ of $\mathcal{L}_c$ (presheaf of cptly supported sections)

Def: $CE^*_{red}(\mathcal{J}(L)) = \prod_{n>0} \underbrace{\mathrm{Hom}_{C^\infty_M\text{-Mod}}}_{\text{cts lin. map. of } C^\infty_M\text{-modules}} (\mathcal{J}(L)[1]^{\otimes n}, C^\infty_M)_{S_n}$ (C^∞_M sheaf of smooth functions on M)

Chevalley-Eilenberg cx of $\mathcal{J}(L)$, in the category of D_M-modules. (units different)

$n>0$: quotient out constants, hence "reduced" — "sheaf of Lagrangians"

Def: $CE^*_{red,loc}(\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(\mathcal{J}(L))$ — local Chevalley-Eilenberg complex

Dens_M: sheaf of smooth densities on M

"local action functionals on $\mathcal{L}[1]$" — think of as acting on $\mathcal{L}_c$ via integration; sheaf of dg algebras

Def: $\mathcal{L}$ local L_∞ alg on M. A k-shifted local central extension on M is an L_∞ structure on the presheaf $\widetilde{\mathcal{L}}_c = \mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$ s.t.

- $0 \to \underline{\mathbb{C}}[k] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0$ is a SES of presheaves
- the structure maps $\tilde{\ell}_n : \mathcal{L}_c \to \underline{\mathbb{C}}[k]$ determining the L_∞ structure on $\widetilde{\mathcal{L}}_c$ are local action functionals.

Lemma: $\exists$ bijection $\{k\text{-shifted local central extensions}\}/\cong \;\longleftrightarrow\; H^{k+2}(CE^*_{red,loc}(\mathcal{L})(M))$

(sanity check: elt in deg $k+2$ takes r elts in $\mathcal{J}(L)[1]^{r}$ which sum to $-2-k$ i.e. r inputs in $\mathcal{J}(L)$ with total deg $?-k-$ & spits out a constant; this exactly defines l_r of deg $2-r$; conditions on L_∞ alg $\Leftrightarrow$ cocycle condition)

We want to construct a central (-1)-shifted central extension i.e identify an elt in $H^1(CE^*_{red,loc}(\mathcal{L})(M))$

Sketch: Recall that $\mathcal{M}$ has a (-3)-shifted invariant pairing. This induces a P_0 structure on $CE^*_{red,loc}(\mathcal{M})(M)[-1]$

Idea: For a L_∞ alg h with (-3)-shifted invariant pairing. This induces $h[1]^\vee \cong h[2]$

so we get $h[1]^\vee \otimes h[1]^\vee \cong h[1]^\vee \otimes h[2] \to \mathbb{C}$ of deg 1

$\mathrm{Sym}^1(h[1]^\vee)$ $\quad$ $\mathrm{Sym}^2(h[1]^\vee)$

extend by Leibniz rule to get a P_0 structure on $CE^*(h)$ $\leadsto$ P_0 structure on $CE^*_{red}(h)$

shift to get dgla structure on $CE^*_{red}(h)[-1]$

Analogous result holds for local L_∞ algs (when M not compact, some analysis needed)

Then: construct central ext of dglas

$$0 \to CE^*_{red,loc}(\mathcal{L})(M)[-1] \to \mathrm{Inner\,Act}(\mathcal{L}, \mathcal{M}) \to \mathrm{Act}(\mathcal{L}, \mathcal{M}) \to 0$$

$[\,,\,]$ trivial

$\ker(CE^*_{red,loc}(\mathcal{L} \oplus \mathcal{M})(M)[-1] \to CE^*_{red,loc}(\mathcal{M})(M)[-1])$

dglas

such that an action $\mathcal{L}$ on $\mathcal{M}$ corresponds to a Maurer-Cartan elt of $\mathrm{Act}(\mathcal{L}, \mathcal{M})$ $\quad$ This must have deg 1

...hen we consider the obstruction to lifting S to a Maurer-Cartan elt in InnerAct(L, M). Pick lift $\tilde{S}$; this is $\underbrace{d\tilde{S} + \frac{1}{2}\{\tilde{S}, \tilde{S}\} \in CE^*_{red,loc}(L)(M)[-1]}$

in deg 2 here, so deg 1 in $CE^*_{red,loc}(L)(M)[-1]$.

Note $d\{\tilde{S}, \tilde{S}\} = 0$ as $\tilde{S}$ has deg 1 $\Rightarrow$ defines cocycle $[\alpha] \in H^1(CE^*_{red,loc}(L)(M))$

...ally, we may state Noether's thm!

Thm: Let local L_∞ alg L act on classical field theory M.

Then, $\exists$ map of precosheaves of L_∞-algs

$$\hat{L}_c \rightsquigarrow \widetilde{Obs}^{cl}[-1]$$

sending the central elt $1 \in \mathbb{C}[-1] \subseteq \hat{L}_c$ to $1 \in \widetilde{Obs}^{cl}[-1]$.

Idea of proof: $\forall U \subseteq M$ open $\exists$ inj. (canonical) cochain map

$$\text{InnerAct}(L, M) \longrightarrow C^*_{red}(L_c(U)) \hat{\otimes} \tilde{C}^*(M(U))$$

action $\rightsquigarrow S \in \text{InnerAct}(L, M)$ s.t. $dS + \frac{1}{2}\{S, S\} = \alpha$

$\rightsquigarrow S(U) \in C^*_{red}(L_c(U)) \hat{\otimes} \tilde{C}^*(M(U))$ s.t. $dS(U) + \frac{1}{2}\{S(U), S(U)\} = \alpha(U)$

functionals with smooth first derivative

$\rightsquigarrow \hat{L}_c(U) \rightsquigarrow \widetilde{Obs}^{cl}(U)[-1]$ □

By construction it's a map of precosheaves.

...not obvious how this corresponds to our usual notion of Noether's thm.

Conservation laws

Def: A conserved current is a map of precosheaves $J: \bar{\Omega}^*_c[1] \to \widetilde{Obs}^{cl}$

distributional forms

If $N \subseteq M$ cpct codim 1 oriented submfd then the delta distribution on N is an elt $[N] \in \bar{\Omega}^1(U)$ defined on any neighborhood U of N. (compatible with inclusions $\rightsquigarrow$ gives elt of $\widetilde{Obs}^{cl}(M)$)

Suppose we have cobordism

codim 0 submfd.

Then $dJ[M] = J[N] - J[N'] \in \widetilde{Obs}^{cl}(M) \Rightarrow$ cohom. class $[J[N]]$ is cobordism invariant.

special case: spacetime

$\uparrow t \rightsquigarrow$ usual concept of conserved current

In the usual version of Noether's thm, we have an $\mathbb{R}$-family of symmetries, so the local L_∞ alg is Ω^*_M.

In this case, can check that central ext is trivial, so we get

$$\Omega^*_{M,c}[1] \longrightarrow \widetilde{\mathrm{Obs}}^{cl}$$

which for formal reasons gives $\overline{\Omega}^*_{M,c}[1] \longrightarrow \widetilde{\mathrm{Obs}}^{cl}$

(conserved current).

Extension to factorisation algs

Define $\mathbb{U}^{P_0}_\alpha(\mathcal{L})(U) = \underbrace{\mathrm{Sym}^*(\hat{\mathcal{L}}_c(U)[1])}_{\text{enveloping } P_0 \text{ alg}} \otimes_{\mathbb{C}[c]} \underbrace{\mathbb{C}_{c=1}}_{\text{specialise central elt to 1}}$

Then replace $\hat{\mathcal{L}}_c$ by this in Noether's thm to get a homomorphism of P_0 FAs $\mathbb{U}^{P_0}_\alpha(\mathcal{L}) \longrightarrow \widetilde{\mathrm{Obs}}^{cl}$.

Quantum version

classical field theory $\rightsquigarrow$ quantum field theory

definition is rather involved, but it gives a BD FA of quantum observables as in previous lectures

recall: BD-alg is a deformation of P_0 alg

Get map $\underbrace{\mathbb{U}^{BD}_\alpha(\mathcal{L}_c)}_{\text{twisted enveloping BD factorisation alg}} \longrightarrow \widetilde{\mathrm{Obs}}^q$ of BD FAs.

angular momentum for free scalar theory

first order formulation: $S(p,q) = \int (p,\dot{q})\,dt + \frac{1}{2}\int (p,p)\,dt$

take $M = \mathbb{R}$ (time) $\qquad V \cong \mathbb{R}^N$ with inner product $(\,,\,)$

classical field theory is given by

$$\begin{array}{ccccccc} & Q^0 = q, & 1 & & 2 & , Q^1 & \\ 0 & \longrightarrow & C^\infty(\mathbb{R})\otimes V & \xrightarrow{\ l_1 \otimes d\ } & \Omega^1(\mathbb{R})\otimes V & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ & & C^\infty(\mathbb{R})\otimes V & & \Omega^1(\mathbb{R})\otimes V & & \\ & P^0 = p & & & & P^1 & \end{array}$$

differential: $\begin{matrix} Q \\ P \end{matrix} \longmapsto \begin{matrix} dQ - P\,dt \\ dP \end{matrix}$

all other l_n are 0

invariant pairing induced by: $\langle Q, P \rangle = \int Q \wedge P$

we recover action $S(Q,P) = \int P \wedge dQ + \frac{1}{2}\int (P \wedge P) \wedge dt$

which gives eq^n above when restricted to fields in deg 0?

symmetries: $\mathcal{L} = \Omega^* \otimes \mathfrak{so}(V)$

L_∞ action is given by the Maurer-Cartan elt

$$S^{eq}(P,Q,\alpha) = \int (P, dQ) + \int (P, \alpha\cdot Q) + \frac{1}{2}\int (P,P)\,dt$$

$$\in \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) = C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / CE^*_{red,loc}(\mathcal{M})$$

intuitively: $0 \to \mathcal{M} \to \mathcal{L} \ltimes \mathcal{M} \to \mathcal{L} \to 0$

$\forall \alpha \in \mathcal{L}$, look at pre-image. This functional on Q, P defines L_∞ structure on the pre-image.

$\Downarrow$

conserved current: $J : \mathcal{L}_c(\mathbb{R}) \longrightarrow \underbrace{\widetilde{\mathrm{Obs}}^q}_{\text{recall this is } \simeq CE^*(\mathcal{M}[-1])}[-1]$

$$\alpha \longmapsto \int \langle P, \alpha \cdot Q \rangle$$

put in $\alpha = a \cdot \delta_t$: $\langle p(t), a\cdot q(t) \rangle$ constant which is conservation of angular momentum.