Rational homotopy theory L2

Commutative differential graded algebras

Def: A <u>commutative differential graded algebra</u> (cdga) over field k is a unital k-alg A s.t.

(I) A is ($\mathbb{Z}$-) graded $A = \bigoplus_{n \in \mathbb{Z}} A^n$. If $a \in A^n$ (a <u>homogeneous</u>) write $|a| = n$.

(II) $|ab| = |a| + |b|$

(III) A <u>graded-commutative</u>: $ab = (-1)^{|a||b|} ba$

(IV) <u>differential</u> $d : A^i \to A^{i+1}$ s.t. $d^2 = 0$

(V) <u>Leibniz rule</u> $d(ab) = (da)b + (-1)^{|a|} a\, db$.

This has <u>unit map</u> $e : k \to A$.

<u>cohomology</u> HA, commutative graded alg

[e.g. any comm. k-alg with $A = A^0$

[e.g. cochain cx of a space $C^*(X)$, mult. by $\cup$ fails (III).

[e.g. de Rham cx of mfd X, $\Omega(X)$, mult by $\wedge$ (wedge product)!

<u>homomorphism of cdgas</u>: linear map preserving grading, product, differential.

So $f : A \to B$ induces $Hf : HA \to HB$

Def: f is a <u>quasi-isomorphism</u> if Hf $\cong$.

<u>category of cdgas</u>: cdgA_k

We will care about $\mathrm{cdgA}_{\mathbb{Q}}^{\geq 0}$, nonnegatively-graded cdgas over $\mathbb{Q}$

<u>Free cdgas</u>

Given a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V^n$ the <u>free graded-comm alg</u> is $S(V) = \underbrace{S(V^{even})}_{\text{symmetric algebra}} \otimes \underbrace{\Lambda(V^{odd})}_{\text{exterior algebra}}$

(same as usual symmetric algebra but accounting for graded-commutativity)

If (V, d) is a cochain cx, we can extend d to $S(V)$ using Leibniz rule to make $S(V)$ into a free cdga

. consider $\cdots \to 0 \to \mathbb{Q}\{t_1, \dots, t_n\} \to \mathbb{Q}\{dt_1, \dots, dt_n\} \to 0 \to \cdots$ cochain ct

$S(V) = \mathbb{Q}[t_1, \dots, t_n] \otimes \Lambda_{\mathbb{Q}}(dt_1, \dots, dt_n)$

call this $\Omega(n)$ "rational version of $\Omega(\mathbb{R}^n)$"

Write $\Omega(n)^k$ for the k-forms. (degree k)

; cdga is semi-free if it's free as a graded commutative algebra

gmented cdgas

f: An augmentation of a cdga A is a hom $\varepsilon: A \to k$ s.t. $\varepsilon \circ \eta = \mathrm{id}_k$

with choice of augmentation ε is an augmented cdga.

tegory of augmented cdgas is $\mathrm{cdgA}_{/k}$, with morphisms (cdga-homs) which preserve augmentation. We care about $\mathrm{cdgA}^{\geq 0}_{/\mathbb{Q}}$

: This is exactly the slice category over $k \in \mathrm{cdgA}_k$

f: Let A augmented cdga. Augmentation ideal is $\bar{A} = \ker \varepsilon$; this is naturally a cdga.

he indecomposables of A are $IA = \bar{A} / \bar{A} \cdot \bar{A}$

llivan's PL deRham functor

stands for "polynomial" or "piecewise linear"

sider $s_i : \Omega(n) \to \Omega(n+1)$, $\partial_i : \Omega(n) \to \Omega(n-1)$

$$s_i t_k = \begin{cases} t_{k+1}, & i<k \\ t_k + t_{k+1}, & i=k \\ t_k, & i>k \end{cases} \qquad \partial_i t_k = \begin{cases} t_{k-1}, & i<k \\ 0, & i=k \\ t_k, & i>k \end{cases} \qquad (t_0 = 1)$$

extended to cdga-homomorphisms.

ese satisfy the usual face (∂_i) & degeneracy (s_i) relations, making $\Omega(*)$ into a simplicial set.

: There are 2 gradings on $\Omega(*)$: $\Omega(n)^p$ — p deg in cdga, n dim in sset.

f: PL de Rham functor $A: \mathrm{sSet}^{op} \to \mathrm{cdgA}^{\geq 0}_{\mathbb{Q}}$

$K \longmapsto \mathrm{Hom}_{\mathrm{sSet}}(K, \Omega(*))$

Likewise, define $F: \mathrm{cdgA}^{\geq 0}_{\mathbb{Q}} \to \mathrm{sSet}^{op}$

$B \longmapsto \mathrm{Hom}_{\mathrm{cdgA}^{\geq 0}_{\mathbb{Q}}}(B, \Omega(*))$

g. $A(*) = \Omega(0) = \mathbb{Q}$

g. $A(\Delta^n) \cong \mathbb{Q}[t_0, \dots, t_n] / \sum t_i = 1 \otimes \Lambda_{\mathbb{Q}}(dt_0, \dots, dt_n) / \sum dt_i = 0$ (after base change)

m: $F \dashv A$

Pf: There's a canonical bijection

$$\mathrm{Hom}_{sSet}(K, \mathrm{Hom}_{cdga^{\geq 0}_{\mathbb{Q}}}(B, \Omega(*))) \cong \mathrm{Hom}_{cdga^{\geq 0}_{\mathbb{Q}}}(B, \mathrm{Hom}_{sSet}(K, \Omega(*))) \quad \square$$

: This induces an adjunction $cdgA^{\geq 0}_{\mathbb{Q}} \underset{A}{\overset{F}{\rightleftarrows}} sSet^{op}_{*}$

integration map

The aim is to show $H(AK) \cong H(C^*K)$ (regular cohomology)

Lemma (Poincaré lemma) $\Omega(n) \simeq \mathbb{Q}$ $\forall n$ (quasi-isomorphic)

Pf: $\Omega(1)$ has basis
$$\begin{array}{cc} 0 & 1 \\ 1 \to 0 & \\ t & \longrightarrow dt \end{array}$$
so $H(\Omega(1)) \cong \mathbb{Q}$ (in deg 0) $\cong H(\mathbb{Q})$

In general, $\Omega(n) \cong \Omega(1)^{\otimes n}$; Künneth thm $\Rightarrow H(\Omega(n)) \cong \mathbb{Q}^{\otimes n} \cong \mathbb{Q} \cong H(\mathbb{Q})$ $\square$

Def: formal integration: Let $\omega = p(t_1, \dots, t_p)\,dt_1 \cdots dt_p \in \Omega(p)^p$

Define $\int \omega = \int_{\Delta^p} p(t_1, \dots, t_p)\,dt_1 \cdots dt_p$ where $\Delta^p = \{t_i \mid 0 \leq i \leq p,\ 0 \leq t_i,\ \Sigma t_i = 1\}$

Lemma (Stokes' thm) Let $s \in \Omega(p)^{p-1}$. Then $\underbrace{\int ds}_{\in \Omega(p)^p} = \underbrace{\int \partial s}_{\in \Omega(p-1)^{p-1}}$, where $\partial = \Sigma (-1)^i \partial_i$

Pf: Same as regular Stokes' thm.

Def: The integration map $\rho : AK \longrightarrow C^*K$

$$\begin{array}{ccc} \omega \in AK^p & \longmapsto & \left(\begin{array}{ccc} K_p & \longrightarrow & \mathbb{Q} \\ \Downarrow & & \Downarrow \\ x & \longmapsto & \int \underbrace{\omega(x)}_{\in \Omega(p)^p} \end{array} \right) \\ \omega : K \to \Omega(*)^p & & \end{array}$$

Lemma: ρ is a cochain map & sends $1 \mapsto 1$

: ρ is not a map of algebras (intuitively, this is unlikely as AK is graded-comm but C^*K is not)

Pf: unital ✓

ω is a sSet morphism, so compatible with s, ∂.

For degenerate simplex $sx \in K_p$, $\rho(\omega)(sx) = \int \omega(sx) = \int \underbrace{s\omega(x)}_{\in \Omega(p-1)^p = 0} = 0$.

commutes with differential: $\rho(d\omega)(x) = \int d\omega(x) \overset{\text{Stokes'}}{=} \int \partial \omega(x) = \int \omega(\partial x) = \rho(\omega)(\partial x)$. $\square$

which by definition is $(d\rho)(\omega)(x)$.

Thm: $K \in sSet$. Then $H(\rho) : H(AK) \xrightarrow{\cong} H(C^*K)$ iso of graded v.s.s

Pf (sketch, K fin dim): Induct on dim. Note

$HA(\Delta[n]) \cong \mathbb{Q} \cong HC^*(\Delta[n])$ & ρ induces this $\cong$

$HA(\coprod_j K_j) \cong \prod_j HA(K_j)$, $HC^*(\coprod_j K_j) \cong \prod_j HC^*(K_j)$ (1st is as A is a right adjoint so sends colimits in sSet to colimits in)

Mayer-Vietoris holds for HA, HC^* (to be proven later)

$$\begin{array}{ccc} K & \xrightarrow{\text{inclusion}} & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$ induces LES.

Then induct: $HA(K_0) \cong HC^*(K_0)$ by 1st 2 points.

Then apply M-V on

$$\begin{tikzcd}
\coprod\limits_{K_n \text{ nondegen}} \partial\Delta[n] \arrow[r] \arrow[d, "\text{inclusion}"] & K_{\leq n-1} \arrow[d] \\
\coprod\limits_{K_n \text{ nondegen}} \Delta[n] \arrow[r] & K_{\leq n} \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

& apply 5-lemma. $\square$

Rk: Because ρ does not preserve products, we haven't solved the comm. cochain problem. The way one solves it is to construct another functor $B: sSet \to dgA^{\geq 0}_{\mathbb{Q}}$ s.t. there are natural quasi-isomorphisms of dgas

$$\begin{tikzcd}
AK \arrow[dr, "\sim"'] & & C^*K \arrow[dl, "\sim"] \\
& BK &
\end{tikzcd}$$

s.t. as morphisms of cochain cxs

$$\begin{tikzcd}
AK \arrow[rr, "\rho"] \arrow[dr, "\sim"'] & & C^*K \arrow[dl, "\sim"] \\
& BK &
\end{tikzcd}$$

see blog post by A. Mathew.

Model categories

To do homotopy theory on cdgas, we need the notion of a homotopy. We can do this by viewing $cdgA^{\geq 0}_{\mathbb{Q}}$ as a model category.

Def: A model category is a category $\mathcal{M}$ with classes $\mathcal{W}$ (weak equivalence), $\mathcal{F}$ (fibrations), $\mathcal{C}$ (cofibrations) s.t.

(MC1) $\mathcal{M}$ has small limits & colimits.

(MC2) 2-out-of-3 property: if 2 of f, g, fg are $\in \mathcal{W}$ so is the 3rd.

(MC3) If f retract of g, i.e.

$$\begin{tikzcd}
A \arrow[r, "\exists"] \arrow[d, "f"] \arrow[rr, bend left, "id_A"] & A' \arrow[r, "\exists"] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, "\exists"'] \arrow[rr, bend right, "id_B"'] \arrow[ur, phantom, "\circlearrowleft"] & B' \arrow[r, "\exists"'] \arrow[ur, phantom, "\circlearrowleft"] & B
\end{tikzcd}$$

& $g \in \mathcal{W} / \mathcal{F} / \mathcal{C}$ then so is f

(MC4) All these squares admit lifts:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\in \mathcal{C}"'] & C \arrow[d, "\in \mathcal{F}\cap\mathcal{W}"] \\
B \arrow[r] \arrow[ur, dashed, "\exists"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, "\in \mathcal{C}\cap\mathcal{W}"'] & C \arrow[d, "\in \mathcal{F}"] \\
B \arrow[r] \arrow[ur, dashed, "\exists"] & D
\end{tikzcd}$$

$\mathcal{F} \cap \mathcal{W}$ = acyclic fibrations

$\mathcal{C} \cap \mathcal{W}$ = acyclic cofibrations

(MC5). $\forall f$ $\exists p \in \mathcal{F} \cap \mathcal{W}$, $i \in \mathcal{C}$, s.t. $f = pi$

- $\forall f$ $\exists q \in \mathcal{F}$, $j \in \mathcal{C}\cap\mathcal{W}$, s.t. $f = qj$

and these factorisations are functorial.

(MC1) $\Rightarrow$ $\exists$ initial & final objects, $0, 1$

X is cofibrant if $(0 \to X) \in \mathcal{C}$, fibrant if $(X \to 1) \in \mathcal{F}$

[e.g. $cdgA^{\geq 0}_{\mathbb{Q}}$ is a model category with
$\mathcal{W} = \{\text{quasi-isomorphisms}\}$
$\mathcal{F} = \{f \text{ s.t. } f_n \text{ surjective } \forall n\}$
$\mathcal{C} = \{f \text{ having left lifting property wrt. all } g \in \mathcal{F} \cap \mathcal{W}\}$

[e.g. sSet is a model category with
$\mathcal{W} = \{f \text{ s.t. } |f| \text{ is a weak htpy equivalence}\}$
$\mathcal{C} = \{\text{inclusions}\}$
$\mathcal{F} = \{\text{Kan fibrations}\} = \{f \text{ having RLP wrt. all } \Lambda \hookrightarrow \Delta\}$

[e.g. $sSet^{op}$ is a model category with weak equivalences $\mathcal{W}^{op}$, cofibs $\mathcal{F}^{op}$, fibs $\mathcal{C}^{op}$

<u>Homotopies</u>

<u>Def</u>: A <u>cylinder object</u> for $A \in \mathcal{M}$ is an object $A \wedge I$ s.t. the natural map $A \sqcup A \to A$ factors as $A \sqcup A \xrightarrow{i \in \mathcal{C}} A \wedge I \xrightarrow{g \in \mathcal{W}} A$

<u>Note</u>: (MC5) $\Rightarrow$ these exist.

<u>Def</u>: A <u>left homotopy</u> between $f, g : A \to B$ in $\mathcal{M}$ via cylinder object $A \wedge I$ is a map $H : A \wedge I \to B$ s.t.

$$\begin{tikzcd}
A \arrow[dr, "i_1"] \arrow[rrrd, bend left, "f"] & & & \\
& A \sqcup A \arrow[r, "i"] & A \wedge I \arrow[r, "H"] & B \\
A \arrow[ur, "i_2"'] \arrow[rrru, bend right, "g"'] & & &
\end{tikzcd}$$

commutes

Dually, we have a <u>path object</u> B^I for $B \in \mathcal{M}$ s.t. $B \xrightarrow{j \in \mathcal{W}} B^I \xrightarrow{p \in \mathcal{F}} B \times B$; a <u>right homotopy</u> $H : A \to B^I$ s.t.

$$\begin{tikzcd}
& & & & B \\
A \arrow[r, "H"] \arrow[rrrru, bend left, "f"] \arrow[rrrrd, bend right, "g"'] & B^I \arrow[r, "p"] & B \times B \arrow[urr, "p_1"] \arrow[drr, "p_2"'] & & \\
& & & & B
\end{tikzcd}$$

commutes

[e.g. In Top, $A \times I$ is a cylinder object, $Map(I, B)$ is a path object
But $Hom(A \times I, B) \cong Hom(A, Map(I, B))$ so the definitions of left & right htpy coincide.

[e.g. In $cdgA^{\geq 0}_{\mathbb{Q}}$, $\Omega(1) \otimes B$ is a path object under

$$B \cong \mathbb{Q} \otimes B \xrightarrow[\in \mathcal{W};\text{ Künneth}]{e \otimes id} \Omega(1) \otimes B \xrightarrow{(\partial_0 \otimes id, \partial_1 \otimes id)} B \oplus B$$

where $\partial_0 : t \mapsto 0$
$\partial_1 : t \mapsto 1$

composition is Δ; $p \in \mathcal{F}$ by using t to "interpolate values"

Under either left, htpic maps induce the same map

In $cdgA^{\geq 0}$, replace $\Omega(1) \otimes B$ with $\mathbb{Q} \oplus (\Omega(1) \otimes B)$

Lt: If A cofibrant, B fibrant, and $f,g: A \to B$ left-htpic via a cylinder object, then they're left/right htpic via any cylinder/path object. (So $[A,B]$ welldefined)
If A, B fibrant & cofibrant, $(f: A \to B) \in \mathcal{W} \iff f$ htpy equivalence.

notopy categories

lying (M(5)) to $0 \to X$, $X \to 1$ gives us cofibrant/fibrant replacement functors

$$0 \xrightarrow{\in \mathcal{C}} QX \xrightarrow{\in \mathcal{F}\cap\mathcal{W}} X$$

Q, R

$$X \xrightarrow{\in \mathcal{C}\cap\mathcal{W}} RX \xrightarrow{\in \mathcal{F}} 1$$

reover, $\forall X$, QRX is fibrant & cofibrant: cofibrant is clear & fibrant is as

$$QRX \xrightarrow{\in \mathcal{F}\cap\mathcal{W}} RX \xrightarrow{\in \mathcal{F}} 1$$

f: The homotopy category $Ho(\mathcal{M})$: Objects = Objects of $\mathcal{M}$
$Hom(X,Y) = [QRX, QRY]$

t: $Ho(\mathcal{M})$ is isomorphic to $\mathcal{M}[\mathcal{W}^{-1}]$
: An adjunction $F \dashv G$ between model categories is a Quillen adjunction if F eserves cofibrations & G preserves fibrations (a.k.a. F, G are left/right Quillen functors)

g. $sSet \underset{Sing}{\overset{|\cdot|}{\rightleftarrows}} CGSp$ → compactly generated spaces

mma: A left Quillen functor sends weak equivalences between cofibrant objects to [illegible]
p: A left Quillen functor $F: \mathcal{M} \to \mathcal{N}$ induces $LF: Ho(\mathcal{M}) \to Ho(\mathcal{N})$, $(LF) = FQ$
A right Quillen functor $G: \mathcal{N} \to \mathcal{M}$ induces $RG: Ho(\mathcal{N}) \to Ho(\mathcal{M})$, $(RG) = GR$
These are derived functors

m: $F \dashv G$ Quillen adjunction. Then $LF \dashv RG$.

m: The adjunction $F \dashv A$ is a Quillen adjunction.

of (sketch): TS A, F take cofibrations to fibrations.

A: TS if $X \hookrightarrow Y$ then any $X \to \Omega(*)^p$ extends to Y.
Want to apply LLP to:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & \Omega(*)^p \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

$\in \mathcal{W}$: $\Omega(*)^p$ is contractible (can write down a homotopy)
$\in \mathcal{F}$: simplicial abelian groups are Kan cts.

Cor: M-V on A: given

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & M \arrow[d] \\ L \arrow[r] & N \end{tikzcd} \xrightarrow{A} \begin{tikzcd} AN \arrow[r] \arrow[d] & AL \arrow[d] \\ AM \arrow[r] & AK \end{tikzcd}$$

in sSet

whence LES.

F: Suffices to show for $(i: B \to C) \in \mathcal{C}$ in $cdgA_{\mathbb{Q}}^{\geq 0}$, $FC \xrightarrow{Fi} FB$ has RLP wrt all $\Lambda \hookrightarrow \Delta$.
As $F \dashv A$, suffices to show $B \to C$ has LLP wrt. all $A\Delta \to A\Lambda$
This is a fibration from the previous half.
This is a weak equivalence as $HA(\Delta) \cong H^*(\Delta) \cong \mathbb{Q} \cong H^*(\Lambda) \cong HA(\Lambda)$ and $1 \mapsto 1$ ∎

inimal models

ll objects in $cdgA_{\mathbb{Q}}^{\geq 0}$ are fibrant. What are the cofibrant objects?

Let B be a cdga. Let $B(n)$ = cdga generated by $B^{\leq n}$ $\quad (B(-1) = k)$

Then define $B(n,0) = B(n-1)$ $\quad$ ($=$ subalg generated by $B^{\leq n}$, dB^{n})

$B(n,p)$ = subalg generated by $B(n,p-1)$, all $x \in B^n$ s.t. $dx \in B(n,p-1)$

Def: M cdga$^{\geq 0}_k$ is <u>minimal</u> if it's <u>semifree</u>, <u>connected</u> ($e \xrightarrow{\cong} M^0$) and

$M(n) = \bigcup M(n,p)$ $\forall n$.

Note: These p have a unique augmentation. Moreover, $dm \in \bar{M}\cdot\bar{M}$ $\forall m$ so $IM \cong H(IM)$ as graded algs.

A <u>minimal model</u> for cdga B is a minimal M and quasi-iso $M \to B$.

e.g. $A(\partial\Delta^{n+1})$: $\quad HA(\partial\Delta^{n+1}) \cong \underset{0}{\mathbb{Q}} \oplus \underset{n}{\mathbb{Q}}$, Let's construct minimal models

If <u>n odd</u>, RHS is already semifree.

If <u>n even</u>, M looks like:

$$\begin{array}{ccccc} 0 & n & 2n-1 \quad 2n & & kn-1 \quad kn \\ 1 \to 0 & 0 \to x \to 0 & 0 \to y \to x^2 \to 0 & & 0 \to x^{k-2}y \to x^k \to 0 \end{array}$$

$dx = 0$ so $d(x^2) = 0$.

odd deg so $y^2 = 0$.

This is semifree.

Check conditions: $\quad dx = 0 \Rightarrow x \in B(n,1)$

Then, $x^2 \in B(n) \subseteq B(2n-1,0) \Rightarrow y \in B(2n-1,1)$.

Rest is clear.

Thm: Every connected cdga B has a minimal model

Proof: Induct on n, then p to construct suitable

$\mathbb{Q} = M(1,0) \subseteq M(1,1) \subseteq \dots \subseteq M(2,0) \subseteq M(2,1) \subseteq \dots$

with (compatible) homs $f_{n,p} : M(n,p) \to B$ s.t. the map on cohomology is $\cong$ in deg $\leq n-1$

and inj in deg $= n$.

Induction step (successors): consider

$$\begin{array}{ccc} M(n,p)^n & \xrightarrow{f_{n,p}} & B^n \ni b_i \text{ reps of basis of coker } H^n(f_{n,p}) \\ & & \ni c_i \text{ s.t. } dc_i = f_{n,p}x_i. \\ d\downarrow & & \downarrow d \\ M(n,p)^{n+1} & \xrightarrow{f_{n,p}} & B^{n+1} \end{array}$$

$x_i \in M(n,p)^{n+1}$ – reps of basis of $\ker H^{n+1}(f_{n,p})$.

Let $M(n,p+1)$ be $M(n,p)$ with added generators m_i, n_i in deg n s.t.

$dm_i = 0$, $\quad dn_i = x_i$; $\quad f_{n,p+1}: m_i \mapsto b_i,\ n_i \mapsto c_i$

(limits) $\quad$ Let $M(n+1,0) = \bigcup_p M(n,p)$.

By construction, this works. $\square$

Lemma: Minimal algebra M is cofibrant in cdgA$^{\geq 0}_{\mathbb{Q}}$

Proof (Sketch): Induct as above.

Fact: cofibrations are stable under pushouts & transfinite composition

Suff. to show $M(n,p-1) \hookrightarrow M(n,p)$ cofibration.

Consider
$$\begin{tikzcd} \bigotimes_{i\in I} \mathbb{Q}\langle \underset{\deg = n+1}{x_i} \rangle \arrow[r] \arrow[d, "x_i \mapsto dy_i"] & M(n,p-1) \arrow[d] \\ \bigotimes_{i\in I} \mathbb{Q}\langle \underset{\deg = n}{y_i}, dy_i\rangle \arrow[r] & M(n,p) \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

$\mathbb{Q}\langle x\rangle \to \mathbb{Q}\langle y, dy\rangle$ is a cofibration:
$$\begin{tikzcd} \mathbb{Q}\langle x\rangle \arrow[r] \arrow[d] & A \arrow[d, "\sim"] \\ \mathbb{Q}\langle y, dy\rangle \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$
$x \mapsto a$, c', $a \mapsto b$, $c' \mapsto c$, $dy \mapsto b$, $y \mapsto c$, $b = dc$

Then $[dc' - a] = 0$
as right map is a quasi-iso. □

Rk: This shows they're cofibrant in $cdga^{\geq 0}_{\mathbb{Q}}$

Homotopy groups

Def: A augmented cdga. The homotopy groups of A are $\pi^k(A) = H^k(\tilde{I}A)$ (recall it's $\bar{A}/\bar{A}\cdot\bar{A}$)

Lemma: $f,g: A \to B$ htpic in $cdgA^{\geq 0}/\mathbb{Q}$ via $B \tilde{\otimes} \Omega(1)$. Then they induce the same map on $\pi(A) \to \pi(B)$

Proof (sketch) $I(\Omega(1)\tilde{\otimes} B) = I(\mathbb{Q} \oplus \Omega(1)\otimes \bar{B}) \cong \Omega(1)\otimes IB$

We get $HIA \to H(\Omega(1)\otimes IB) \to HIB \oplus HIB$
$$\begin{tikzcd} H(\Omega(1)\otimes IB) \arrow[r] & HIB\oplus HIB \\ HIB \arrow[u, "\text{Künneth} \cong"] \arrow[ur, "\Delta"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] \end{tikzcd}$$
□

Lemma: Let $V(n) = \overset{0}{\mathbb{Q}} \oplus \overset{n}{\mathbb{Q}}$ with 0 differential, $B \in cdga^{\geq 0}_{\mathbb{Q}}$ cofibrant.
Then $[B, V(n)] \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n B; \mathbb{Q})$ is a natural bijection.

Note: LHS is well-defined as B cofibrant, $V(n)$ fibrant.

Pf (sketch): $\varphi: [B, V(n)] \to \mathrm{Hom}_{\mathbb{Q}}(\pi^n B, \underbrace{\pi^n V(n)}_{\mathbb{Q}})$ well-defined by previous lemma

φ is surjective by definition.

φ is injective: Suppose $f_* = g_*$ on π^n, then $f_* = g_*$ on $\pi(B) \to \pi(V(n))$
Then $f,g: IB \to IV(n)$ chain homotopic.

$\Rightarrow f,g: \bar{B} \to \bar{B}/\bar{B}\cdot\bar{B} \to \overset{\overline{V(n)}\ =}{\overline{V(n)}/\overline{V(n)}}$ chain htpic, say via $D: \bar{B} \to \overline{V(n)}$, $dD + Dd = f - g$

Extend uniquely to $B \to V(n)$, then $H: B \to \Omega(1)\tilde{\otimes} V(n)$ works □
$x \mapsto dt \otimes Dx + (1-t)\otimes fx + t\otimes gx$

Thm: B cofibrant cdga.
- If B homologically connected, then $\pi_0 FB = *$
- For $n \geq 1$ $\exists$ natural bijection of sets $\pi_n(FB) \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n B; \mathbb{Q})$
- For $n \geq 2$ (or for $n = 1$ if $B(1) = B(1,1)$), this is an isomorphism of groups.

Proof: $\underline{n=0}$: Let B have minimal model M. Then $[M, A*] \cong [M, \mathbb{Q}] \cong *$. (augmentation is unique)
Quillen adjunction $\Rightarrow$ $[*, FM] \cong *$, but this is $\pi_0(FM) \cong \pi_0(FB)$
(F sends weak equivalences between cofibrant cdgas to weak equivalences)

$\underline{n \geq 1}$: $\pi_n FB = [S^n, FB] \cong [B, AS^n] \cong [B, V(n)] \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$.

$V(n) \to AS^n$ quasi-isomorphism

group structure (sketch): coalgebra structure on B induces group structure on $[S^n, FB]$ compatible with regular group structure.
Eckmann Hilton argument $\Rightarrow$ these are the same. $\square$

Cor: $\pi_k(FAS^n) = \begin{cases} \mathbb{Q}, & k=n \text{ or } 2 \mid n,\ k=2n-1 \\ 0 & \text{else.} \end{cases}$

We will (hopefully) see later that $\pi_n(FAS^n) \cong \pi_n(S^n) \otimes \mathbb{Q}$, so this tells us exactly which htpy groups of spheres are not torsion.

Proof: WLOG replace AS^n with min model M.
$\underline{n \text{ odd}}$: $M = AS^n = \overset{0}{\mathbb{Q}} \oplus \underset{x}{\overset{n}{\mathbb{Q}}}$. $\quad IM = \mathbb{Q}\{x\} \Rightarrow H^k(IM) = \begin{cases} \mathbb{Q}, & k=n \\ 0 & \text{else} \end{cases}$
$\underline{n \text{ even}}$: $M = \mathbb{Q}\langle x, y \mid dy = x^2 \rangle$ $\quad IM = \mathbb{Q}\{x, y\} \Rightarrow H^k(IM) = \begin{cases} \mathbb{Q}, & k=n, 2n-1 \\ 0 & \text{else} \end{cases}$ $\square$