Reeh-Schlieder Theorem & additivity of conformal nets

Recall: A conformal net consists of

- H_0 Hilbert space
- action $U : \mathrm{Diff}^+(S^1) \curvearrowright H_0$ (extending to a proj action of $\mathrm{Diff}^+(S^1)$)
- $\mathcal{A}(I)$ von Neumann alg $\forall$ interval I
- $\Omega \in H_0$ vacuum vector

s.t.:

- $I_1 \subseteq I_2 \Rightarrow \mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$ (isotony)
- $I_1 \cap I_2 = \emptyset \Rightarrow \mathcal{A}(I_1), \mathcal{A}(I_2)$ commute (locality)
- $\mathcal{A}(gI) = U(g)\,\mathcal{A}(I)\,U(g)^* \quad \forall g \in \begin{cases} \text{Möb} \\ \mathrm{Diff}^+(S^1) \end{cases}$ (covariance)
- $R_t := U(z \mapsto z e^{it}) = e^{itL_0}$ for some unbdd +ve operator (+ve energy)
- Ω cyclic for $\bigvee \mathcal{A}(I)$, and $H^{\text{Möb}(S^1)}$ spanned by Ω.

Today:

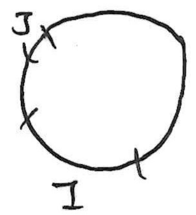
- Reeh-Schlieder thm: Ω cyclic $\forall\, \mathcal{A}(I)$
- additivity: $I = \bigcup_\alpha I_\alpha \Rightarrow \mathcal{A}(I) = \bigvee_\alpha \mathcal{A}(I_\alpha)$.

Reeh-Schlieder thm

Idea: $\mathcal{A}(I)\Omega$ dense $\Leftrightarrow (\mathcal{A}(I)\Omega)^\perp$ trivial.

so want to show $\xi \perp \mathcal{A}(I)\Omega \Rightarrow \xi = 0$.

Lemma: $\xi \perp \mathcal{A}(I)\Omega \Rightarrow \xi \perp \mathcal{A}(J)\Omega$ $\forall$ intervals J which are shorter than I:



Idea of proof: $\exists$ range of values of t which rotate J into I. Then done by some cx analysis.

Proof: Let $a \in \mathcal{A}(J)$. Consider

$$f(t) := \langle \xi, R_t a R_{-t} \Omega \rangle \overset{\text{Möb action fixes } \Omega}{=} \langle \xi, R_t a \Omega \rangle = \langle \xi, e^{itL_0} a \Omega \rangle.$$

This is defined $\forall t \in \mathbb{R}$ & is 0 on some interval. Want: $f(0) = 0$

Positivity of L_0: ~~$\sigma(L_0) \subseteq \mathbb{R}_{\geq 0}$, so whenever $\operatorname{Im} t \geq 0$, $\sigma(itL_0) \subseteq \{\operatorname{Re} z \leq 0\}$~~

~~For these values of t,~~ By spectral thm, $\exists$ unitary iso $H_0 \xrightarrow{\Phi} L^2(X)$ (some meas.)

& ~~[illegible]~~ meas. function $g \geq 0$

s.t. $\begin{array}{ccc} H_0 & \xrightarrow{L_0} & H_0 \\ \Phi \downarrow & \circlearrowright & \downarrow \Phi \\ L^2(X) & \xrightarrow{\cdot g} & L^2(X) \end{array}$ Then $e^{itL_0} \leftrightsquigarrow$ mult by e^{itg} in L^∞ for $\operatorname{Im} t \geq 0$. ~~[illegible]~~

Idea: f cts on $\{\operatorname{Im} t \geq 0\}$ + holo on $\{\operatorname{Im} t > 0\}$ [illegible]

...ent: $t \mapsto e^{itf}$ holo; derivative "should be" ife^{itf}.

...iff. to check $\dfrac{e^{i(t+h)f} - e^{itf}}{h} \xrightarrow[h\to 0]{} ife^{itf}$ in L^∞

$ife^{i(t+h)f} \xrightarrow[h\to 0]{} ife^{itf}$ in L^∞.

This convergence is uniform if $\mathrm{Im}\, t > 0$ (so $\mathrm{Re}(itf) < 0$).
(then exponential term outweighs poly terms)

...nce $t \mapsto e^{itL_0}$ holo on $\{\mathrm{Im}\, t > 0\}$.

...xt, want $t \mapsto e^{itL_0} a\Omega$ cts on $\{\mathrm{Im}\, t \geq 0\}$.

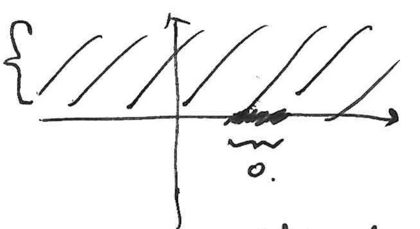
Let $\Phi(a\Omega) = \eta$ then want $t \mapsto e^{itf}\eta \in L^2(X)$ cts.

(Note: $t \mapsto e^{itf} \in L^\infty(X)$ not cts)
(e.g. if $f \notin L^\infty$ then this is not cts at 0)

Note: $t \mapsto e^{itf} \mathbb{1}_{\{f \leq n\}}$ L^∞-cts

$\Rightarrow t \mapsto \underbrace{e^{itf}}_{L^\infty} \underbrace{\mathbb{1}_{\{f\leq n\}} \eta}_{\xrightarrow{n\to\infty} \eta \text{ in } L^2}$ L^2-cts

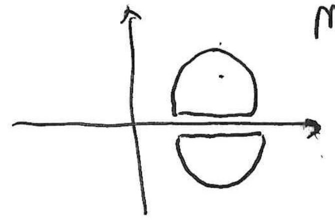
$\Rightarrow t \mapsto e^{itf}\eta$ L^2-cts.

holo  cts

...nce f is of the form:

...pply Schwarz reflection principle: $f \equiv 0$.

idea of proof: extend f to $\mathbb{C}$ via $f(\bar{z}) = \overline{f(z)}$



Morera $\Rightarrow$ holo.

principle of isolated zeros $\Rightarrow$ will zero

...n particular, $t=0$: $f(0) = \langle \xi, a\Omega \rangle = 0$. □

...ummary of the above proof: if J can be ...oved into I via a cts family of Möbius ...ansformations which under U is generated by a +ve unbdd operator, then $\xi \perp A(I)\Omega \Rightarrow \xi \perp A(J)\Omega$.

...ma: $I \subsetneq J \Rightarrow (\xi \perp A(I)\Omega \Rightarrow \xi \perp A(J)\Omega)$

...of: conjugate rotations ~~dilations~~ by this transformation fixing circle & sending 0 to near P.

This is conjugate to rotations.

Repeat proof of prev. lemma. □

<u>Proof of Reeh-Schlieder</u> : Want $\xi \perp \mathcal{A}(I)\Omega \Rightarrow \langle \xi, a_1 \cdots a_n \Omega\rangle = 0$
$\forall \{a_i \in I_i\}$.

Pick $J \not\supset I$, J larger than $I_1, \ldots, I_n$.

Consider $f(t_1, \ldots, t_n) := \langle \xi, R_{t_1} a_1 R_{-t_1} \cdots R_{t_n} a_n R_{-t_n} \Omega\rangle$

$$= \langle \xi, e^{it_1 L_0} a_1 e^{i(t_2 - t_1)L_0} a_2 \cdots e^{i(t_n - t_{n-1})L_0} a_n \Omega$$

$$\overset{z_i = t_i - t_{i-1}}{=} \langle \xi, e^{iz_1 L_0} a_1 \cdots e^{iz_n L_0} a_n \Omega \rangle.$$

$f(z_1, \ldots, z_n) = 0$ for certain intervals in $\mathbb{R}$. $z_i \in K_i$

Inductively apply proof of 1st lemma: $f(z_1, \ldots, z_n) = 0 \;\forall\, \mathrm{Im}\, z_1 \geq 0$
$\forall z_2 \in K_2, \ldots, z_n \in K_n$

$$\Downarrow$$

$f(z_1, \ldots, z_n) = 0 \;\forall\, \mathrm{Im}\, z_1, \mathrm{Im}\, z_2 \geq 0$
$\forall z_3 \in K_3, \ldots, z_n \in K_n$

$$\Downarrow \cdots$$

$f \equiv 0 \Rightarrow f(0, \ldots, 0) = 0$

So $\langle \xi, a\Omega\rangle = 0$ $\square$

<u>Additivity</u>

recall from last week: given $M \subseteq B(\mathcal{H})$, Ω cyclic + separating,

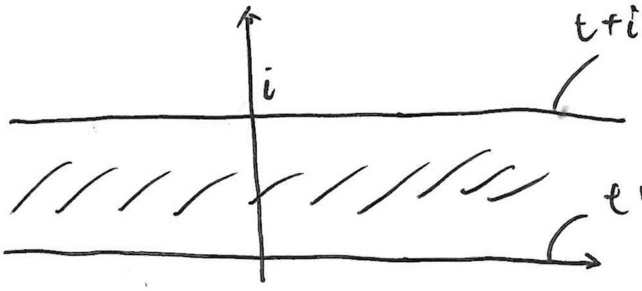
can define $S_M : M\Omega \to M\Omega$ polar decomp. J_M modular conjugation
$m\Omega \mapsto m^*\Omega$. Δ_M modular operator.

which satisfy $\underbrace{\Delta_M^{it} M \Delta_M^{-it} = M}$ $\forall t$ $J_M M J = M'$ Δ^{it}, J fix Ω.

defines a 1-param automorphism gp $\sigma_t : M \to M$
$m \mapsto \Delta_M^{it} m \Delta_M^{-it}$.

<u>Fact</u> : σ_t is the unique 1-param family of automorphisms s.t. $\forall m_1, m_2 \in M$,

$t + i \mapsto \langle \Omega, m_2 \Delta^{it} m_1 \Delta^{-it} \Omega\rangle$

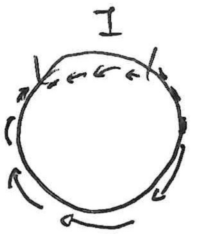


$t \mapsto \langle \Omega, \Delta^{it} m_1 \Delta^{-it} m_2 \Omega\rangle$

extends cts to the strip s.t. its holo on the interior.

(KMS)
Kubo-Martin-Schwinger condition

the context of conformal nets, the Δ^{it} arises naturally from the Möb action:



dilations are conjugate to $\{z \mapsto e^{2\pi t} z \mid t \in \mathbb{R}\}$.

Bisognano Wichmann : $\Delta^{it}_{\mathcal{A}(I)} = U(D(t))$.

w suppose ~~[illegible]~~ we have $I = \bigcup_n I_n$ where $I_1 \subseteq I_2 \subseteq \cdots$

aim : $\mathcal{A}(I) = \bigvee_n \mathcal{A}(I_n)$.

of : let $g \in \text{Möb}$ s.t. $\overline{gI} \subseteq I$, then $\exists n$, $\overline{gI} \subseteq I_n$.

So $\mathcal{A}(gI) \subseteq \mathcal{A}(I_n) \subseteq \bigvee \mathcal{A}(I_n)$. i.e. $\forall a \in \mathcal{A}(I)$ $U(g)aU(g)^* \in \bigvee \mathcal{A}(I_n)$

Take $g \to id$, then U cts, $\bigvee \mathcal{A}(I_n)$ closed $\Rightarrow$ $a \in \bigvee_n \mathcal{A}(I_n)$ $\square$

general if $I = \bigcup I_\alpha$, construct sequence of intervals $J_1 \subseteq J_2 \subseteq \cdots$

s.t. $I = \bigcup_n J_n$.

n $\overline{J_n}$ is covered by fin. many I_α, say $I_{\alpha_1}, \dots, I_{\alpha_m}$.

Sent $\mathcal{A}(J_n) \subseteq \bigvee_{j=1}^{m} \mathcal{A}(I_{\alpha_j} \cap J)$ then we'd have $\mathcal{A}(J_n) \subseteq \bigvee \mathcal{A}(I_\alpha)$ th

$\Rightarrow \mathcal{A}(I) = \bigvee \mathcal{A}(J_n) \subseteq \bigvee \mathcal{A}(I_\alpha)$. so WLOG indexing set $\{\alpha\}$ is finite.

k (small) interval $J \subseteq I$ s.t. $\forall t \exists j$ $D(t)J \subseteq I_{\alpha_j}$.

hen consider $M := \bigvee_{t \in \mathbb{R}} \mathcal{A}(D(t)J) \subseteq \bigvee \mathcal{A}(I_{\alpha_j})$.

this is fixed by conj. by $\Delta^{it}_{\mathcal{A}(I)}$ (by def^n).

But $M \to M$, $m \mapsto \Delta^{it}_{\mathcal{A}(I)} \, m \, \Delta^{-it}_{\mathcal{A}(I)}$ satisfies the KMS condition

$\Rightarrow \Delta^{it}_{\mathcal{A}(I)} = \Delta^{it}_M$. As $S_{\mathcal{A}(I)} = S_M$ this means $J_{\mathcal{A}(I)} = J_M$.

ally: $M \subseteq \mathcal{A}(I)$ $\Rightarrow \mathcal{A}(I)' \subseteq M'$ $\Rightarrow \underbrace{\mathcal{A}(I)'}_{J_{\mathcal{A}(I)}\mathcal{A}(I)J_{\mathcal{A}(I)}}\Omega \subseteq \underbrace{M'}_{J_M M J_M}\Omega$ $\Rightarrow \mathcal{A}(I)\Omega \subseteq M\Omega$ $\xRightarrow{\Omega \text{ sep.}}$ $\mathcal{A}(I) \subseteq M$ $\square$