Conformal nets talk 5

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1 The Reeh–Schlieder Theorem and additivity of conformal nets

The aim of this talk is to prove the following two properties of conformal nets:

Theorem 1.1 (Reeh–Schlieder Theorem). *The vacuum vector Ω is cyclic for $\mathcal{A}(I)$ for each interval I, i.e. $\mathcal{A}(I)\Omega$ is dense.*

Theorem 1.2 (Additivity of conformal nets). *Given a cover $I = \bigcup_\alpha I_\alpha$, the von Neumann algebra $\mathcal{A}(I)$ is generated by the subalgebras $\mathcal{A}(I_\alpha)$, i.e.*

$$\mathcal{A}(I) = \bigvee_\alpha \mathcal{A}(I_\alpha).$$

1.1 Proof of the Reeh–Schlieder Theorem

This is largely based on [Hen14, pp. 16-20].

Fix an interval I. We want to show that the subspace $\mathcal{A}(I)\Omega \subseteq H_0$ is dense. Equivalently, we want to show that its orthogonal complement $(\mathcal{A}(I)\Omega)^\perp$ is trivial. In other words, it suffices to show that

$$\xi \perp \mathcal{A}(I)\Omega \Rightarrow \xi = 0.$$

Starting from $\xi \perp \mathcal{A}(I)\Omega$, we shall deduce further conditions on ξ.

Lemma 1.3. *If $\xi \perp \mathcal{A}(I)\Omega$, then $\xi \perp \mathcal{A}(J)\Omega$ for all intervals J that are shorter than I.*

Remark. The main way in which the condition on J is used is that there exists an interval of values t for which R_t rotates J into I. From this observation, we may conclude via some analysis.

Proof. Let $a \in \mathcal{A}(J)$. We want to show that $\langle \xi, a\Omega\rangle = 0$. Consider the function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(t) := \langle \xi, R_t a R_{-t}\Omega\rangle = \langle \xi, R_t a\Omega\rangle = \left\langle \xi, e^{itL_0} a\Omega\right\rangle.$$

Here, we used the fact that Ω is fixed by the Möbius group action. We want to show that $f(0) = 0$. By the Möbius covariance property, whenever R_t sends J to within I, we have $R_t a R_{-t} \in \mathcal{A}(I)$ and so $f(t) = 0$.

Now, as L_0 is a positive unbounded operator, the assignment $t \mapsto e^{itL_0}a$ is in fact well-defined on $\{\mathrm{Im}(t) \geq 0\}$. Moreover, this map is continuous and holomorphic on the interior.

Thus, the function $f(t) = \left\langle \xi, e^{itL_0} a\Omega\right\rangle$ is defined on the half-plane $\{\mathrm{Im}(t) \geq 0\}$, where it is continuous and holomorphic on the interior. Moreover, there exists some interval $(c, d) \subset \mathbb{R}$ on which f is 0. By the Schwarz Reflection Principle (Lemma 1.4), we may extend $f|_{\{\mathrm{Im}(t)>0\}}$ holomorphically to $\mathbb{C} \setminus ((\infty, c] \cup [d, \infty))$. This is 0 on (c, d), and so must be uniformly zero. Thus, $f|_{\{\mathrm{Im}(t)>0\}}$ is uniformly zero, and hence by continuity, f is uniformly zero.

In particular, $f(0) = \langle \xi, a\Omega \rangle = 0$, which is what we wanted. $\square$

Lemma 1.4 (Schwarz Reflection Principle). *Let f be a holomorphic function defined on the upper half-plane which extends continuously to an interval $(c,d) \subset \mathbb{R}$. Suppose f only takes real values on (c,d). Then, f admits a holomorphic extension to $\mathbb{C} \setminus ((\infty, c] \cup [d, \infty))$.*

In the proof of Lemma 1.3, the only place in which we used the condition on J was the existence of a one-parameter family of Möbius transformations generated by a positve unbounded operator which sends J into I. By choosing a different such family of Möbius transformations, we may generalise this approach.

Lemma 1.5. *If $\xi \perp \mathcal{A}(I)\Omega$, then $\xi \perp \mathcal{A}(J)\Omega$ for all intervals J that do not contain I.*

Proof. It suffices to find a one-parameter family of Möbius transformations generated by a positve unbounded operator which sent J into I. Pick a point p within the unit disc which is close to a point in $I \setminus J$. Then, working in the Poincaré disc model of the hyperbolic space, the group of rotations around p provides such a one-parameter family. Explicitly, this is the group given by conjugating $\{R_t\}$ by the Möbius transformation which fixes 1 and sends $0 \mapsto p, \infty \mapsto \overline{p}^{-1}$.

Using this family in place of $\{R_t\}$ in the proof of Lemma 1.3, our result follows. $\square$

We now have the tools necessary to prove the Reeh–Schlieder Theorem. Recall that we want to show that if $\xi \perp \mathcal{A}(I)\Omega$, then $\xi = 0$. As Ω is cyclic for the von Neumann algebra $\bigvee_J \mathcal{A}(J)$ generated by the algebras $\mathcal{A}(J)$ assigned to all intervals J of the circle, it suffices to show that for each $a_1 \in I_1, \ldots, a_n \in I_n$ where $I_1, \ldots, I_n$ are intervals, we have

$$\langle \xi, a_1 \ldots a_n \Omega \rangle = 0.$$

Fix such intervals $I_1, \ldots, I_n$ and elements $a_1, \ldots, a_n$. Pick an interval J not containing I that is longer than each of $I_1, \ldots, I_n$. By Lemmas 1.3 and 1.5, we have $\xi \perp \mathcal{A}(J)\Omega$. We will now conclude by using a multivariable version of the proof of Lemma 1.3. Define

$$\begin{aligned} f(t_1, \ldots, t_n) :&= \langle \xi, (R_{t_1} a_1 R_{-t_1})(R_{t_1+t_2} a_2 R_{-t_1-t_2}) \cdots (R_{t_1+\cdots+t_n} a_n R_{-t_1-\cdots-t_n})\Omega \rangle \\ &= \left\langle \xi, e^{it_1 L_0} a_1 e^{it_2 L_0} a_2 \cdots e^{it_n L_0} a_n \Omega \right\rangle . \end{aligned}$$

Then, f is defined on $\{\mathrm{Im}(t_1), \ldots, \mathrm{Im}(t_n) \geq 0\}$. It is continuous and holomorphic on the interior. Moreover, there exist intervals $(c_1, d_1), \ldots, (c_n, d_n) \subset \mathbb{R}$ such that $f(t_1, \ldots, t_n) = 0$ if $t_i \in (c_i, d_i)$ for each $1 \leq i \leq n$.

Fixing $t_2, \ldots, t_n$ and varying t_1, the proof of Lemma 1.3 then shows that $f(t_1, \ldots, t_n) = 0$ if $t_i \in (c_i, d_i)$ for each $2 \leq i \leq n$. Then, fixing t_1 and $t_3, \ldots, t_n$ and varying t_2, we get $f(t_1, \ldots, t_n) = 0$ if $t_i \in (c_i, d_i)$ for each $3 \leq i \leq n$. Repeating inductively, we thus get that f is uniformly zero, and in particular

$$f(0, \ldots, 0) = \langle \xi, a_1 \cdots a_n \Omega \rangle = 0.$$

This completes the proof of the Reeh–Schlieder Theorem. $\square$

We now record a corollary of the Reeh-Schlieder Theorem which will be useful in the proof of additivity.

Corollary 1.6. *The vacuum vector Ω is separating for $\mathcal{A}(I)$ for each interval I, i.e. $a\Omega \neq 0$ for each nonzero $a \in \mathcal{A}(I)$.*

Proof. Suppose $a\Omega = 0$ for some $a \in \mathcal{A}(I)$. For each $b \in \mathcal{A}(I)'$, we have $ab\Omega = ba\Omega = 0$. Thus, $a\mathcal{A}(I)'\Omega = 0$. On the other hand, by Haag duality, $\mathcal{A}(I)' = \mathcal{A}(I')$, so applying the Reeh-Schlieder Theorem to I', we have that $\mathcal{A}(I)'\Omega$ is dense. Hence, we must have $a = 0$. □

1.2 Proof of additivity

This is based on the proof given by Fredenhagen and Jörß [FJ96].

We start off by recording a special case of the additivity property, which we deduce directly from the definition of conformal nets.

Lemma 1.7. *Suppose $I = \bigcup_n I_n$ where $I_1 \subseteq I_2 \subseteq \cdots$ is an increasing sequence of intervals. Then, $\mathcal{A}(I) = \bigvee_n \mathcal{A}(I_n)$.*

Proof. The inclusion $\bigvee_n \mathcal{A}(I_n) \subseteq \mathcal{A}(I)$ is clear by isotony. It suffices to show the reverse inclusion.

Let g be a Möbius transformation such that $\overline{gI} \subset I$, then there exists some n for which $\overline{gI} \subset I_n$. Then, we have $\mathcal{A}(gI) \subseteq \mathcal{A}(I_n) \subseteq \bigvee_n \mathcal{A}(I_n)$. By the Möbius covariance property, we have shown that for each fixed $a \in \mathcal{A}(I)$, $U(g)aU(g)^* \in \bigvee \mathcal{A}(I_n)$.

Take a sequence of such g which converge to the identity, then as U is continuous and $\bigvee \mathcal{A}(I_n)$ is closed, we have $a \in \bigvee_n \mathcal{A}(I_n)$ for each $a \in \mathcal{A}(I)$. □

Before we proceed to the general case, we first have to recall from the previous talk: Given a von Neumann algebra $M \subseteq B(H)$ and a cyclic and separating vector Ω, we can define an antilinear isomorphism $S_M : M\Omega \to M\Omega$ by mapping $m\Omega \mapsto m^*\Omega$. The polar decomposition of this yields *modular conjugation* J_M (an antilinear involution on H) and the *modular operator* Δ_M satisfying

- $\Delta_M^{it} M \Delta_M^{-it} = M$ for all $t \in \mathbb{R}$
- $J_M M J_M = M'$
- Δ_M^{it}, J fix Ω.

In particular, $\sigma_t : m \mapsto \Delta_M^{it} M \Delta_M^{-it}$ defines a 1-parameter automorphism group of M. We take as a black box the following result from Tomita–Takesaki theory:

Theorem 1.8 ([Tak03, Theorem VIII.1.2])**.** *$\{\sigma_t\}$ is the unique 1-parameter family of automorphisms on M satisfying the* Kubo-Martin-Schwinger (KMS) condition*: for every $m_1, m_2 \in M$, there exists a bounded function F continuous on the strip $\{0 \le \mathrm{Im}(z) \le 1\}$ and holomorphic on the interior with boundary values*

$$F(t) = \langle \Omega, \sigma_t(m_1)m_2\Omega \rangle \quad \textit{and} \quad F(t+i) = \langle \Omega, m_2\sigma_t(m_1)\Omega \rangle \quad \forall t \in \mathbb{R}.$$

Recall from the previous talk the Bisognano–Wichmann Theorem: for an interval I, let $D(t)$ be the group of dilations fixing the endpoints of I. (Explicitly, $D(t)$ is the dilation $z \mapsto e^{2\pi t}z$ conjugated by the Möbius transformation sending the endpoints of I to $0, \infty$ and sending the unit circle to the real axis.)

The above results on the modular operator give us another special case of the additivity property:

Lemma 1.9. *Let $J \subseteq I$ be a subinterval. Then $\mathcal{A}(I) = \bigvee_t \mathcal{A}(D(t)J)$.*

The following proof is adapted from [Tak72].

Proof. As $D(t)J \subseteq I$ for all t, we have $\mathcal{A}(D(t)J) \subseteq \mathcal{A}(I)$ by isotony, and so $M := \bigvee_t \mathcal{A}(D(t)J) \subseteq \mathcal{A}(I)$. It suffices to show that $\mathcal{A}(I) \subseteq M$.

By the Bisognano–Wichmann Theorem, we have $\Delta_{\mathcal{A}(I)}^{it} = U(D(t))$. By Möbius convariance, conjugation by this sends each $\mathcal{A}(D(t')J)$ to $\mathcal{A}(D(t+t')J)$, so overall, conjugation by $\Delta_{\mathcal{A}(I)}^{it}$ sends M to itself. This satisfies the KMS condition for M since $M \subseteq \mathcal{A}(I)$, and so by Theorem 1.8, we have $\Delta_M^{it} m \Delta_M^{-it} = \Delta_{\mathcal{A}(I)}^{it} m \Delta_{\mathcal{A}(I)}^{-it}$ for each $m \in M$. Applying both sides to Ω and noting that $\Delta_M^{-it}, \Delta_{\mathcal{A}(I)}^{-it}$ both fix Ω, we thus have $\Delta_M^{it} m\Omega = \Delta_{\mathcal{A}(I)}^{it} m\Omega$ for each $m \in M$. Since Ω is separating for $\mathcal{A}(I)$, we thus have that Δ_M is a restriction of $\Delta_{\mathcal{A}(I)}$. On the other hand, as $S_M = S_{\mathcal{A}(I)}|_M$ by definition, we then have $J_M = J_{\mathcal{A}(I)}$ and we may write both of these as J.

Now, as $M \subseteq \mathcal{A}(I)$, we have

$$J\mathcal{A}(I)J = \mathcal{A}(I)' \subseteq M' = JMJ.$$

Considering the orbits of Ω under these and noting that J is an involution fixing Ω, we have $\mathcal{A}(I)\Omega \subseteq M\Omega$. Then, as Ω is separating for $\mathcal{A}(I)$, it is separating for M as well, and hence we have $\mathcal{A}(I) \subseteq M$ as desired. □

Finally, we may combine the two special cases above to deduce the addivity property in general. Let $I = \bigcup_\alpha I_\alpha$. As before, the inclusion $\bigvee_\alpha \mathcal{A}(I_\alpha) \subseteq \mathcal{A}(I)$ is clear by isotony; it suffices to show the reverse inclusion.

Construct an increasing sequence of subintervals $J_1 \subsetneq J_2 \subsetneq \cdots$ of I such that $I = \bigcup_n J_n$. Then, by Lemma 1.7, we have $\mathcal{A}(I) = \bigvee_n \mathcal{A}(J_n)$. It then suffices to show that $\mathcal{A}(J_n) \subseteq \bigvee_\alpha \mathcal{A}(I_\alpha)$ for each n.

For each given n, $\overline{J_n} \subset I$ is covered by finitely many I_α, say $I_{\alpha_1}, \ldots, I_{\alpha_m}$. In particular, we have $J_n = \bigcup_i (I_{\alpha_i} \cap J_n)$, so it suffices to show the additivity property for finite covers, and then we will have

$$\mathcal{A}(J_n) \subseteq \bigvee_i \mathcal{A}(I_{\alpha_i} \cap J_n) \subseteq \bigvee_i \mathcal{A}(I_{\alpha_i}) \subseteq \bigvee_\alpha \mathcal{A}(I_\alpha).$$

So, now suppose $I = \bigcup_\alpha I_\alpha$ for some finite union of intervals I_α. Then, there exists a subinterval $J \subseteq I$ such that for each t, there exists some α for which $D(t)J \subseteq I_\alpha$.

But now we have $\mathcal{A}(D(t)J) \subseteq \bigvee_\alpha \mathcal{A}(I_\alpha)$ for each t, so by Lemma 1.9,

$$\mathcal{A}(I) = \bigvee_t \mathcal{A}(D(t)J) \subseteq \bigvee_\alpha \mathcal{A}(I_\alpha),$$

which is what we wanted. □

References

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