Extending the Reshetikhin – Turaev TQFT

Roadmap:

ribbon category $\rightsquigarrow$ ribbon invariant $\rightsquigarrow$ 3-manifold invariant $\rightsquigarrow$ 3D TQFT (surfaces + bordisms) $\rightsquigarrow$ 3D once-extended TQFT (circles, surfaces with ∂, bordisms with corners)

Reshetikhin – Turaev '90

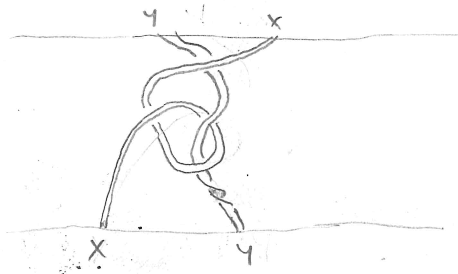
Reshetikhin – Turaev '91

complicated – see later.

Ribbon categories

Tautological definition: "a ribbon category is a category which defines ribbon invariants"

Idea: let's say we have a (framed, oriented) tangle in $\mathbb{R}^2 \times [0,1]$



we want to be able to label ("color") each strand with an object in a category $\mathcal{C}$

& then interpret this diagram as a morphism in $\mathcal{C}$.

what structure does $\mathcal{C}$ need for this to make sense?

- $\mathcal{C}$ is a <u>monoidal category</u> functors $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ (+ conditions) unit $\mathbb{1} \in \mathcal{C}$

so the picture above is some morphism $X \otimes Y \to Y \otimes X$

e.g. $\otimes$ in vector spaces

make sense

- $\mathcal{C}$ is <u>braided</u>

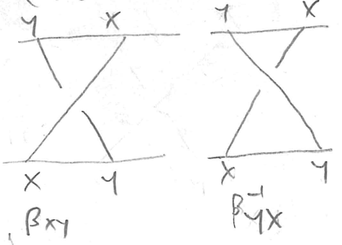
nat iso

$$\beta : \begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[rr, bend left, "\otimes"] \arrow[dr, "\text{swap}"'] & \Downarrow & \mathcal{C} \\ & \mathcal{C} \times \mathcal{C} \arrow[ur, "\otimes"'] & \end{tikzcd}$$

(+ conditions)

i.e. $\forall X, Y \in \mathcal{C}$ we have $\beta_{XY} : X \otimes Y \xrightarrow{\sim} Y \otimes X$

(& so $\beta^{-1}_{YX} : X \otimes Y \xrightarrow{\sim} Y \otimes X$)



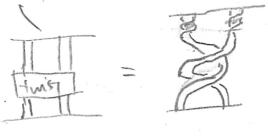
By construction,  (R2)

<u>Lemma</u> (Yang-Baxter eqn)  (R3) <u>Proof</u>: naturality of braiding.

- $\mathcal{C}$ is <u>balanced</u>, i.e. has a <u>twist</u> nat iso

$$\theta : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{C} \end{tikzcd}$$

(+ conditions)

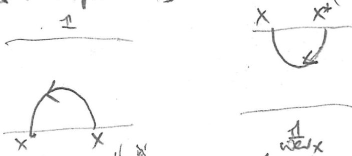


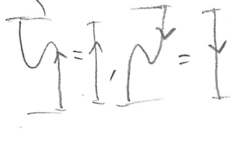
i.e. $\forall X \in \mathcal{C}$ we have $\theta_X : X \xrightarrow{\sim} X$.

- $\mathcal{C}$ is <u>rigid</u>: objects have <u>duals</u>. With the above properties it suffices to have: $\forall X \in \mathcal{C}$ $\exists X^* \in \mathcal{C}$ (left dual) & morphisms (+ conditions)

$ev_X : X^* \otimes X \to \mathbb{1}$

$coev_X : \mathbb{1} \to X \otimes X^*$





β & θ let us define duals in other direction (right duals): 

e.g. duals in vector spaces.

want this to be compatible with the twist: 

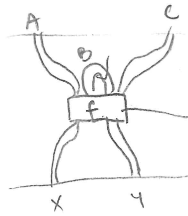
Such a $\mathcal{C}$ is a <u>ribbon category</u>

By construction, each ribbon diagram as drawn above can be interpreted as a morphism in $\mathcal{C}$, and this is "functorial": gluing ribbon diagrams ~ composition.

reover, this is invariant under framed isotopy. We've seen Kex ~ above.

1'' is now 

re generally, we can also make sense of diagrams like



$f: X\otimes Y \to A\otimes B\otimes B^* \otimes C$ $\rightsquigarrow$ morphism $X\otimes Y \to A \otimes C$

isotopy invariant by naturality of β, θ.

$\rightsquigarrow$ diagram $D \mapsto$ morphism $F(D)$

shetikhin-Turaev ('90): construction of ribbon categories from quantum groups!

. $\mathrm{Rep}(U_q\mathfrak{sl}_2)$, $q\in\mathbb{C}$, not root of unity.

Recall: $\mathrm{Rep}(\mathfrak{sl}_2) = \{V_n \mid n\in\mathbb{Z}_0^+\}$ $\quad V_n = \mathrm{span}\{x^a y^b \mid a+b=n\}$ $\quad \begin{pmatrix}0&1\\0&0\end{pmatrix} \mapsto x\frac{\partial}{\partial y}$ $\quad \begin{pmatrix}1&0\\0&-1\end{pmatrix} \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

$\begin{pmatrix}0&0\\1&0\end{pmatrix} \mapsto y\frac{\partial}{\partial x}$

$\mathrm{Rep}(U_q\mathfrak{sl}_2)$: similar, but braiding is now non-trivial.

e.g. $q^2\beta^{-1}_{V_1,V_1} - q^2\beta_{V_1,V_1} = (q-q^{-1})\,\mathrm{id}_{V_1\otimes V_1}$

i.e. q^2  $- q^{-2}$  $= (q-q^{-1})$  Jones poly skein relation.

iewing a link as a tangle with no endpts, we get a morphism $\mathbb{1} \to \mathbb{1}$ ($V_0 \to V_0$)

But $\mathrm{Hom}(V_0,V_0)\cong\mathbb{C}$ (Schur's lemma), so this is just a cx no. depending on q $\rightsquigarrow$ Jones poly

manifold invariants

ven a closed conn. 3-mfd M, we can present it in terms of (framed) surgery links in S^3

g.  $\rightsquigarrow S^1\times S^2$.

convert this into a 3-mfd invariant, we need a couple more assumptions on our ribbon category $\mathcal{C}$:

- it's a <u>linear</u> category (hom spaces are vector spaces)
- it's <u>abelian</u> (technical condition, has $\oplus$ & some other stuff)
- it's finite <u>semisimple</u> ($\exists\, \mathbb{1}=X_0,\dots,X_{n-1}$ s.t. each obj is $\cong$ a direct sum of X_i, & $\mathrm{Hom}(X_i,X_j)\in\{\mathbb{C}\ (i=j),\ 0 \text{ o.w.}\}$)
- it's <u>modular</u>:  $=$  $\forall j \Rightarrow i=0$. ("braiding is non-degenerate")

call such a ribbon cat a <u>modular tensor cat.</u>

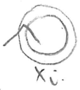
. $\mathrm{Rep}(U_q\mathfrak{sl}_2)$, $q\neq 1$ root of unity..

en, given a surgery link L with m components, we can define

$$\tau(M) := \left(\frac{\Delta}{D}\right)^{\sigma(L)}\left(\frac{1}{D}\right)^{m+1}\sum_{\lambda\in\{X_i\}^m} F(L_\lambda)\cdot\prod_{k=1}^{m}\dim(X_{i_k})$$

$\sigma(L)$: signature of L; $\lambda\in\{X_i\}^m$: colourings of link components with X_i; L_λ: L, but coloured; $\lambda=(X_{i_1},\dots,X_{i_m})$

$\frac{\Delta}{D}$: $\in\mathbb{C}^\times$ depending on $\mathcal{C}$; D is unique; D has 2 choices, $D^2=\sum\dim X_i$

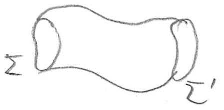
$\dim(X_i) :=$ cx no. assigned to  X_i (cf. dim of vector spaces)

m (Reshetikhin-Turaev '91): This is indep. of orientation of L & invariant under Kirby moves

<u>QFT</u>

b: A (3D, oriented) TQFT is a symmetric monoidal functor $Z:\mathrm{Bord}_{23}^{or} \to \mathrm{Vect}_{\mathbb{C}}$

i.e. it sends closed or. surface Σ $\longmapsto$ vector space $Z(\Sigma)$

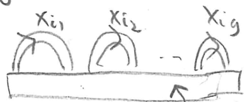
cobordism  $\longmapsto$ linear map $Z(\Sigma) \to Z(\Sigma')$ (respects composition)

disjoint union $\longmapsto$ $\otimes_{\mathbb{C}}$

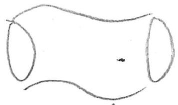
In particular: $\emptyset \longmapsto \mathbb{C}$

so closed 3-mfd $\longmapsto$ linear map $\mathbb{C} \to \mathbb{C}$ i.e. 1×1 matrix i.e. a no.

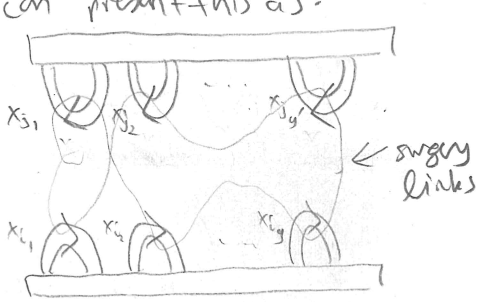
Construction: Think of a genus g surface as the ∂ of the tubular neighborhood

There are n^g ways to colour the g ribbons.
For each colouring 
the set of morphisms that can go in is exactly $\underbrace{\mathrm{Hom}(\mathbb{1}, X_{i_1} \otimes X_{i_1}^* \otimes X_{i_2} \otimes X_{i_2}^* \otimes \cdots \otimes X_{i_g} \otimes X_{i_g}^*)}_{\text{vector space}}$

Define $$Z(\Sigma) := \bigoplus_{(X_{i_1},\dots,X_{i_g}) \in \{X_i\}^g} \mathrm{Hom}(\mathbb{1}, X_{i_1} \otimes X_{i_1}^* \otimes \cdots \otimes X_{i_g} \otimes X_{i_g}^*)$$

Now for cobordisms: given Σ  Σ', glue in handlebodies with embedded ribbon graphs to get $\underline{\text{closed}}$ 3-mfd with embedded ribbon graphs

We can present this as:



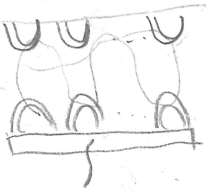
$\subseteq \mathbb{R}^3 \subseteq S^3$.

To define a linear map
$$Z(\Sigma) \longrightarrow Z(\Sigma')$$
$$\| \qquad\qquad \|$$
$$\bigoplus_\mu V_\mu \qquad \bigoplus V_\nu$$

suff. to define linear map $\varphi_{\lambda\mu}: V_\lambda \to V_\mu$ $\forall \lambda, \mu$.

Each choice of μ, ν fixes a colouring of the top & bottom $\cup$, $\cap$

For each colouring λ of the surgery links, we may read off



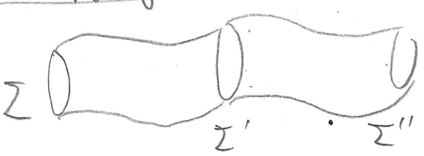
$\rightsquigarrow$ elt of $\mathrm{Hom}(\mathbb{1}, X_{j_1} \otimes X_{j_1}^* \otimes \cdots \otimes X_{j_{g'}} \otimes X_{j_{g'}}^*) = V_\nu$

This gives a linear map $F_\lambda: V_\mu \to V_\nu$

As before, we can now define

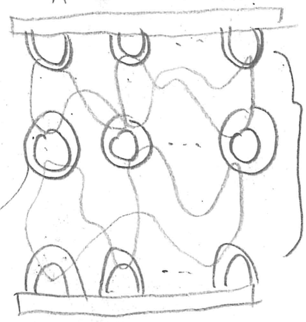
$$\varphi_{\lambda\mu} := \left(\frac{\Delta}{D}\right)^{\sigma(L)} \left(\frac{1}{D}\right)^{m+1} \sum_{\lambda \in \{X_i\}^m} \left(\prod_{k=1}^m \dim(X_{i_k})\right) F_\lambda$$

sketch pf of functoriality:



$\rightsquigarrow$ stack the 2 pictures we had above

(to diagram, surface more complicated in middle.)



$\underline{\text{claim}}$: the 2 surgery links + the new links in the centre give a surgery link for the glued cobordism

Then, we're done since composition of ribbon diagrams $\rightsquigarrow$ comp. of morphisms in $\mathcal{C}$.

$\underline{\text{Caveat}}$: the terms at the front make it not exactly functorial

But we can replace $\mathrm{Bord}_{2,3}^{or}$ by a similar category $\mathrm{Bord}_{2,3}^{sig}$ (same category but with

ℤ times more morphisms) we can define a genuine sym. mon functor $\mathrm{Bord}^{\sigma}_{3} \to \mathrm{Vect}_{\mathbb{C}}$

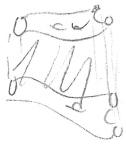
ending to 1-mflds

: A once-extended (3D oriented) TQFT is a sym. mon. functor $Z: \mathrm{Bord}^{or}_{123} \to 2\mathrm{Vect}_{\mathbb{C}}$

bicategory, i.e. we have objs, morphisms, & morphisms between morphisms ("2-morphisms")

objs: closed or. 1-mflds

morphisms:  surfaces with ∂

2-morphisms  3-mflds with corners

at is $2\mathrm{Vect}_{\mathbb{C}}$?

's think about how we got $\mathrm{Vect}_{\mathbb{C}}$. We wanted some category with an obj $* \circlearrowleft \mathbb{C}$, $\mathrm{Hom}_{\mathcal{C}}(*,*) = \mathbb{C}$

so if $\emptyset \mapsto *$ then closed 3-mfld $\mapsto$ cx no.

want this to have $\oplus$, so we have objs $*\oplus*$, $*\oplus*\oplus*$, ...

cause of how $\oplus$ works, $\mathrm{Hom}_{\mathcal{C}}(X\oplus Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X,Z) \oplus \mathrm{Hom}_{\mathcal{C}}(Y,Z)$

$\mathrm{Hom}_{\mathcal{C}}(X, Y\oplus Z) \cong \mathrm{Hom}_{\mathcal{C}}(X,Y) \oplus \mathrm{Hom}_{\mathcal{C}}(X,Z)$

$\Rightarrow \mathrm{Hom}_{\mathcal{C}}(*^{\oplus n}, *^{\oplus m}) \cong \mathbb{C}^{mn}$ ⟵ m×n matrices

've exactly constructed the category of (fin. dim.) vector spaces
↑ remove this to allow for infinite $\oplus$.

's repeat. Now we have a bicat with obj $* \circlearrowleft \mathrm{Vect}_{\mathbb{C}}$

en this should have objects $*^{\oplus 2}, *^{\oplus 3}, \dots$ and

$\mathrm{Hom}_{\mathcal{C}}(*^{\oplus n}, *^{\oplus m})$ = "m×n matrices of $\mathrm{Vect}_{\mathbb{C}}$"

i.e. morphisms are m×n matrices of vector spaces

2-morphisms —— " —— linear maps.

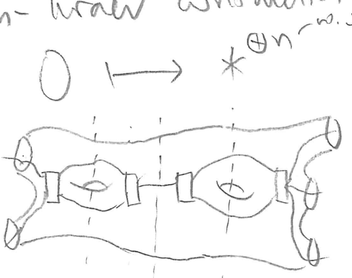
this turns out to be the bicat of ab. lin. cats & lin. functors.

n (Schommer-Pries '11, Tsumura '14, Bartlett-Douglas-Schommer-Pries-Vicary '15, De Renzi '21, L. '26??):
+ Sytillidis ??

Schommer-Pries '11: stated with no proof
Tsumura '14: only connected surfaces

The Reshetikhin-Turaev construction extends to a sym. mon. functor $Z: \mathrm{Bord}^{sig}_{123} \to 2\mathrm{Vect}_{\mathbb{C}}$.

ea: circle $O \mapsto *^{\oplus n}$ ⟵ no. of simples of $\mathcal{C}$ so k circles $\mapsto *^{\oplus n^k}$

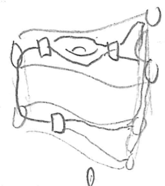
surfaces: 

want a $\underbrace{n^k \times n^l}$ matrix of vector spaces

choice of index ⇝ colouring of "end" ribbons

the vector space in this entry is

$\bigoplus_{\text{colourings of internal edges}} \bigotimes_{\text{internal }\square\text{s}} \mathrm{Hom}(\mathbb{1}, \dots)$

3-mflds:  "join up" to form larger ribbon graphs; similar construction to regular version.

n" (BDSV '15 + ...) $\{\text{sym mon functors } \mathrm{Bord}^{\{or, sig\}}_{123} \to 2\mathrm{Vect}_{\mathbb{C}}\} \xleftrightarrow{1:1} \begin{cases} \{\text{MTCs s.t. } \Delta^2 = D^2\} \\ \{\text{MTCs + choice of } D\} \end{cases}$

n (L. '26??) $\{\text{sym mon functors } \mathrm{Bord}^{sig/2}_{123} \to 2\mathrm{Vect}_{\mathbb{C}}\} \xleftrightarrow{1:1} \{\text{MTCs}\}$