Tannakian categories, L4

Aim for today: Recall from L1

<u>Thm</u>: $\mathcal{C}$ <u>tensor category</u> that's <u>finitely $\otimes$-generated</u>. Then:

small, ab., $\mathbb{C}$-lin, $\otimes$ exact & k-lin,
assoc, & comm. constraints,
unit $\mathbb{1}$, rigid, $\mathrm{End}(\mathbb{1}) \cong \mathbb{C}$
(can replace $\mathbb{C}$ with $k=\bar{k}$, char $k=0$)

$\exists$ fin. set which
gen. all objs
via $\oplus$, $\otimes$, $^\vee$, subobj, quotient
(according to Deligne, this is actually not necessary

$$\mathcal{C} \cong \underbrace{\mathrm{Rep}(G,\varepsilon)} \iff \text{(length condition)}$$

will specify later

what is this?

- G is a <u>super group</u> — "super" version of an affine algebraic gp.

Recall that an aff. alg. gp is a representable functor $\mathrm{Alg}_{\mathbb{C}} \to \mathrm{Gp}$
& these correspond to comm. Hopf algs

aff. alg. gps $\quad$ comm. Hopf algs

$$\mathrm{Hom}(A,-) \longleftarrow\!\!| \ A$$
$$G \longmapsto \mathcal{O}(G)$$

Replace "comm. Hopf alg" with supercomm. super Hopf alg.
& $\mathrm{Alg}_{\mathbb{C}}$ with $\mathrm{sAlg}_{\mathbb{C}}$.

Insist that $\mathcal{O}(G)$ f.g. alg (so $\mathrm{Rep}(G,\varepsilon)$ is finitely $\otimes$-gen)

- $\varepsilon \in G(\mathbb{C})$ of order ≤ 2
- $\mathrm{Rep}(G,\varepsilon)$ = f.d. super-representations ρ of G
s.t. $\rho(\varepsilon)$ is the <u>parity aut</u>
$v \mapsto (-1)^{|v|} v$

We will mainly focus on constructing a <u>super fibre functor</u>

$$w : \mathcal{C} \longrightarrow \mathrm{sMod}_R$$

exact $\otimes$-functor $\quad$ supercomm. k-alg

The result will follow from the machinery that was developed in the previous talks.

Schur functors

Recall that irreps of S_n are indexed by Young diagrams a.k.a. partitions
$\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$, $\lambda_1 \geq \dots \geq \lambda_r$, $|\lambda| := \sum \lambda_i = n$.

<u>Schur functor</u>: $\mathcal{C}$ tensor cat, λ partition of n

For $X \in \mathcal{C}$ $\quad S_\lambda(X) := \underbrace{\mathcal{H}om(V_\lambda, X^{\otimes n})^{S_n}}_{\text{canonically iso to "} V_\lambda^\vee \otimes X^{\otimes n} \text{"}}$ — invariants, i.e. image of the idempotent $\frac{1}{n!}\sum_{\sigma \in S_n} \sigma$

so $\mathrm{Hom}(Y, \mathcal{H}om(V_\lambda, X^{\otimes n})) = \mathrm{Hom}(V_\lambda, Y \otimes X^{\otimes n})$

Cts (from this def^n + properties of reps of S_n):

$S_\mu(X) \otimes S_\nu(X) \sim \bigoplus S_\lambda(X)^{[\lambda;\mu,\nu]}$ — multiplicity of $\rho_\mu \otimes \rho_\nu$ in $\mathrm{Res}^{S_{|\lambda|}}_{S_{|\mu|} \times S_{|\nu|}} \rho_\lambda$.

(def^n + properties of char table)

$S_\mu(X) = 0$, $\underbrace{[\mu] \subseteq [\lambda]}_{\text{inclusion of Young diagrams}}$ $\Rightarrow S_\lambda(X) = 0$

(Frobenius reciprocity)

$S_\lambda(X \oplus Y) \sim \bigoplus \left(S_\mu(X) \otimes S_\nu(Y)\right)^{\oplus [\lambda;\mu,\nu]}$

$S_\lambda(X \otimes Y) \sim \bigoplus \left(S_\mu(X) \otimes S_\nu(Y)\right)^{\oplus [V_\mu \otimes V_\nu ; V_\lambda]}$

- In sVect, $S_\lambda(\mathbb{C}^{p|q}) \neq 0 \Leftrightarrow [\lambda] \subseteq$  $\left(\Leftrightarrow \boxed{}^{\,p+1}_{\,q+1} \not\subseteq [\lambda]\right)$

($\oplus$ formula + $V_{\nu^T} \sim \mathrm{sgn} \otimes V_\nu$ + properties of $[\lambda;\mu,\nu]$)

- The collection of objs X in $\mathcal{C}$ s.t. $\exists \lambda$, $S_\lambda(X) = 0$ is closed under $\oplus, \otimes$. ✓

Proof: For $\oplus$ ($\otimes$ analogous):

Suppose $[\mu] \subseteq \boxed{}^{\,p+1}_{\,q+1}$ $\quad [\nu] \subseteq \boxed{}^{\,r+1}_{\,s+1}$

Pick any λ containing $\boxed{}^{\,p+r+1}_{\,q+s+1}$

In sVect, $\underbrace{S_\lambda(\mathbb{C}^{p|q} \oplus \mathbb{C}^{r|s})}_{=\,0} \sim \bigoplus \left(S_\mu(\mathbb{C}^{p|q}) \otimes S_\nu(\mathbb{C}^{r|s})\right)^{\oplus [\lambda;\mu,\nu]}$

So $\forall \mu', \nu'$ s.t. $[\lambda;\mu',\nu'] \neq 0$, $\underbrace{\boxed{}^{\,p+1}_{\,q+1} \subseteq [\mu']}_{\Rightarrow [\mu] \subseteq [\mu']}$ or $\underbrace{\boxed{}^{\,r+1}_{\,s+1} \subseteq [\nu']}_{\Rightarrow [\nu] \subseteq [\nu']}$.

Apply $\oplus$ formula in $\mathcal{C}$.

For duals: Note reps of S_n are self-dual

check from def^n that $S_\lambda(X^\vee)$ invertible with $S_\lambda(X)^\vee \sim S_\lambda(X^\vee)$

- The collection of such X is closed under subquotients & extensions.) (exactness of $\otimes$ & $(-)^{S_n}$)
- $S_\lambda(X) = 0 \Rightarrow \mathrm{length}(X) < \infty$

Prop: $\mathcal{C}$ tensor cat, $X \in \mathcal{C}$. Then TFAE:

(A) $\exists \lambda$, $S_\lambda(X) = 0$

(B) $X^{\otimes n}$ fin. length $\forall n$ & $\exists N$, $\forall n \geq 0$ $\mathrm{length}(X^{\otimes n}) \leq N^n$

Proof of (B) $\Rightarrow$ (A): Suppose $S_\lambda(X) \neq 0$ $\forall \lambda$.

Use $X^{\otimes n} \sim \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(X)$

So length$(X^{\otimes n}) \geqslant \sum_{|\lambda|=n} \dim(V_\lambda) \geqslant \left(\sum_{|\lambda|=n} \dim(V_\lambda)^2\right)^{1/2} = (n!)^{1/2}$ # $\square$

For now, we don't know that (A)$\Rightarrow$(B), but we'll take (A) to be the "length condition".

Existence of super fibre functor

We want to show

Prop: If in a tensor cat $\mathcal{C}$ each X satisfies (A), then $\exists$ super-comm alg & exact $\otimes$-functor $\omega: \mathcal{C} \to sMod_R$.

Note: This means each obj has fin. length, so $\forall X, Y$, $\mathrm{Hom}(X,Y) \cong \mathrm{Hom}(\mathbb{1}, X^\vee \otimes Y)$ is fin. dim.

Work in $\mathrm{Ind}\,\mathcal{C}$. Take "algebra" to mean "alg obj in $\mathrm{Ind}\,\mathcal{C}$" (comm + assoc + unital).

Def: A alg. An A-algebra is an alg B with (alg) hom $f: A \to B$

This gives restriction: $\mathrm{Mod}_B \to \mathrm{Mod}_A$, $M \mapsto (M \text{ as } A\text{-module})$ $\dashv$ extension of scalars $\mathrm{Mod}_A \to \mathrm{Mod}_B$, $M \mapsto M_B = B \otimes_A M$

Lemma: M dualisable module over nonzero alg A. Then $\exists$ nonzero A-alg B s.t. $\mathbb{1}_B$ $(= A_B)$ is a direct factor of M_B

$\Longleftrightarrow$ $\mathrm{Sym}^n_A(M) \neq 0$ $\forall n$.

Proof: For each A-alg B let $\mathrm{Fact}(B)$ be the collection of factorisations

$$\begin{tikzcd} \mathbb{1}_B \arrow[r, "\alpha"] \arrow[rr, bend left, "\mathrm{id}_{\mathbb{1}_B}"] & M_B \arrow[r, "\beta"] & \mathbb{1}_B \end{tikzcd}$$

(covariant functor in B)

$$\begin{tikzcd} A \to B \otimes_A M \arrow[d, leftrightarrow, "\text{adjunction}"'] & v: M \to B \text{ in } \mathrm{Mod}_A \arrow[d, leftrightarrow] \\ u: M^\vee \to B \arrow[d, leftrightarrow] & \mathrm{Sym}_A(M) \to B \\ \mathrm{Sym}_A(M^\vee) \to B & \end{tikzcd}$$

(α and β correspond by adjunction to the top row.)

composition:

$$\begin{tikzcd} \mathbb{1}_B \arrow[r, "\mathrm{id}"] & \mathbb{1}_B \\ A = \mathbb{1} \arrow[r, "\text{unit of } B"'] \arrow[u, leftrightarrow, phantom, "\text{adjunction}"] & B \end{tikzcd}$$

chasing def$^{\text{ns}}$, $\beta\alpha = \mathrm{id} \Longleftrightarrow$ $\overset{A}{\mathbb{1}} \xrightarrow{\delta} M \otimes_A M^\vee = M^\vee \otimes_A M \xrightarrow{u} B \otimes_A M \xrightarrow{v} B \otimes_A B \to B$ (this is just $u \otimes v$)

So overall this is equivalent to

$$\mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M^\vee) = \bigoplus_{p,q} \mathrm{Sym}^p_A(M) \otimes \mathrm{Sym}^q_A(M) \xrightarrow{f} B$$

s.t.

$$\begin{tikzcd} & \mathrm{Sym}^0_A(M) \otimes_A \mathrm{Sym}^0_A(M^\vee) \arrow[dr] & & \\ \mathbb{1} \arrow[ur, "\sim"] \arrow[dr, "\delta"'] & & \mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M^\vee) \arrow[r] & B \\ & \mathrm{Sym}^1_A(M) \otimes \mathrm{Sym}^1_A(M^\vee) \arrow[ur] & & \end{tikzcd}$$

are equal.

Let B_0 be the cokernel of

$$\mathrm{Sym}_A(M)\otimes\mathrm{Sym}_A(M^\vee)\xrightarrow{\cdot(\delta-1)}\mathrm{Sym}_A(M)\otimes\mathrm{Sym}_A(M^\vee)$$

By construction $\mathrm{Fact}=\mathrm{Hom}_{\mathrm{Mod}_A}(B_0,-)$

Need to show $\mathrm{Sym}^n_A(M)\neq 0\ \forall n \iff \underbrace{B_0\neq 0}$

$\iff \underbrace{\text{unit } \mathbb{1}\to B}$ is zero

inductive lim of

$\delta^n:\mathbb{1}\to\mathrm{Sym}^n_A(M)\otimes\mathrm{Sym}^n_A(M^\vee)$

this is the morphism for a duality between $\mathrm{Sym}^n_A(M)$ & $\mathrm{Sym}^n_A(M^\vee)$, so is zero $\iff \mathrm{Sym}^n_A M=0$ $\square$

Prop: $\mathcal{C}_e$ tensor cat s.t. each obj has finite length. Suppose $\exists\,\bar{\mathbb{1}}\in\mathcal{C}_e$ s.t. $\bar{\mathbb{1}}\otimes\bar{\mathbb{1}}\cong\mathbb{1}$ & its self braiding is mult by -1. $\forall X\in\mathcal{C}_e$,

(A) $\iff$ (C) $\exists\, p,q$, nonzero alg A s.t. $X_A\cong(\mathbb{1}^p\oplus\bar{\mathbb{1}}^q)_A$

Pf: (C) $\Rightarrow$ (A) Note $\langle\mathbb{1},\bar{\mathbb{1}}\rangle\simeq$ sVect so pick $\lambda=\underbrace{\boxed{}}_{q+1}\ p+1$

Then $S_\lambda(X)_A\cong S_\lambda(\mathbb{1}^p\oplus\bar{\mathbb{1}}^q)_A=0\overset{A\neq 0}{\Longrightarrow}S_\lambda(X)=0$.

(A) $\Rightarrow$ (C): (Idea: Inductively extend by scalars to pull out copies of $\mathbb{1},\bar{\mathbb{1}}$.)

Suppose for some $A\neq 0$ $X_A=\mathbb{1}^r_A\oplus\bar{\mathbb{1}}^s_A\oplus R$ (e.g. $A=\mathbb{1}_{\mathcal{C}_e}$, $r=s=0$ for base case)

- If all $\mathrm{Sym}^n_A(R)\neq 0$ then apply Lemma to get A-alg B s.t. R_B has $\mathbb{1}_B$ as a direct factor, so $X_B=\mathbb{1}^{r+1}_B\oplus\bar{\mathbb{1}}^s\oplus R'$
- If all $\mathrm{Sym}^n_A(\bar{\mathbb{1}}\otimes R)\neq 0$, same reasoning gives $X_B=\mathbb{1}^r_B\oplus\bar{\mathbb{1}}^{s+1}_B\oplus R'$
- Else, $\exists\, m,n$ s.t. $\underbrace{\mathrm{Sym}^{n+1}_A}_{=S_{\text{row } n+1}\text{ trivial rep}} R=\underbrace{\Lambda^{m+1}_A}_{=S_{\text{column } m+1}\text{ sign rep}} R=0$

So $\forall\, k>mn$, all $|\lambda|=k$ have $S_\lambda(R)=0\Rightarrow R^{\otimes k}=0\Rightarrow R=0$

If this never terminates, then $\forall n\ \exists\, p,q$, $p+q=n$, $(\mathbb{1}^p\oplus\bar{\mathbb{1}}^q)_A$ direct factor of X_A for some $A\neq 0$. But $S_\lambda(X)=0\Rightarrow S_\lambda(X)_A=0\Rightarrow S_\lambda(\mathbb{1}^p\oplus\bar{\mathbb{1}}^q)_A=0$

$\overset{A\neq 0}{\Longrightarrow}S_\lambda(\mathbb{1}^p\oplus\bar{\mathbb{1}}^q)=0\Rightarrow|\lambda|\geq(p+1)(q+1)>n$ #. $\square$

Proof of Prop: If there's no $\bar{\mathbb{1}}$, replace $\mathcal{C}$ by "$\mathbb{Z}/2$-gr. obj's of $\mathcal{C}$" to introduce $\bar{\mathbb{1}}$, then restrict to even objs ($\mathcal{C}$). Now:

- $\forall X \in \mathcal{C}\ \exists B \neq 0,\ X_B \cong \mathbb{1}_B^p \oplus \bar{\mathbb{1}}_B^q$
- $\forall$ SES $0 \to X \to Y \to Z \to 0,\ \exists B \neq 0,\ 0 \to X_B \to Y_B \to Z_B \to 0$ split

(similar universal construction)

Take inductive lim of fin. $\otimes$ of these B to get A. ("$A = \otimes B$")

Now $\langle \mathbb{1}, \bar{\mathbb{1}} \rangle \cong \underbrace{\mathrm{sVect}}_{\text{(f.d.)}}$, $\mathrm{Ind}\langle \mathbb{1}, \bar{\mathbb{1}} \rangle \cong \mathrm{sVect}^{\text{any dim}}$

For $M \in \mathrm{Ind}\,A$, let $\rho(M)$ be the largest subobj of M in (identified with super v.s. $\mathrm{Hom}(\mathbb{1}, M) \oplus \mathrm{Hom}(\bar{\mathbb{1}}, M)$)

Then $R = \rho(A)$ is naturally an alg.

Define $\mathcal{C} \to \mathrm{Mod}_R$ This defines functor ω.

$X \mapsto \underbrace{\rho(X_A)}$

by 1st condition on A this is $\cong A \otimes$ (sth in $\langle \mathbb{1}, \bar{\mathbb{1}} \rangle$).

Follows that ω is a $\otimes$-functor.

exactness is by 2nd condition on A.

$\square$

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sVect$_{\mathbb{C}}$ as a categorified alg closure

"$\mathbb{C} = \bar{\mathbb{R}}$" can be described purely in terms of f.d. comm. $\mathbb{R}$-algs:

- $\mathbb{C}$ is a nonzero f.d. comm. $\mathbb{R}$-alg
- each $\mathbb{C} \to A$ of nonzero f.d. comm $\mathbb{R}$-algs is an injection

} i.e. $\mathbb{C}$ is a field

- A nonzero f.d. comm $\mathbb{R}$-alg. has $\exists\ A \to \mathbb{C}$ (alg closed)

& these determine $\mathbb{C}$ up to nonunique $\cong$.

We want to categorify these notions.

Def: A categorified comm. $\mathbb{R}$-alg is a sym mon obj in $2\mathrm{Vect}_{\mathbb{R}}^{\mathrm{pres}}$

2-cat of lin. loc. presentable cats

It's f.d. if it admits a fin. strongly gen. set $\{C_1, \dots, C_n\}$ s.t. $\mathrm{Hom}(C_i, C_j)$ f.d. $\forall i,j$, C_i cpt proj ($\mathrm{Hom}(C_i, -)$ cocts) & all proj objs dualisable

Thm (Johnson-Freyd) sVect$_{\mathbb{C}}$ is the unique (up to nonunique equiv) f.d. categorified comm. $\mathbb{R}$-alg s.t.

- Each sVect$_{\mathbb{C}} \to \mathcal{C}$ of f.d. cat. comm $\mathbb{R}$-algs is faithful & inj on iso classes of obj's ("categorified field")
- $\forall \mathcal{C}$ f.d. cat. comm $\mathbb{R}$-alg $\exists F : \mathcal{C} \to$ sVect$_{\mathbb{C}}$ "alg closed"

Pf: $sVect_{\mathbb{C}}$ is a cat. field: Let $F: sVect_{\mathbb{C}} \to \mathcal{C}_e$

$\mathbb{1} \mapsto \mathbb{1}_{\mathcal{C}_e}$, $\bar{\mathbb{1}} \mapsto \bar{\mathbb{1}}_{\mathcal{C}_e}$ } diff self braiding, so not $\cong$

these are cpct: $Hom(\mathbb{1}_{\mathcal{C}_e}, -)$, $Hom(\bar{\mathbb{1}}_{\mathcal{C}_e}, -)$ preserve infinite $\oplus$s.

$\Rightarrow$ inj on objects as we can recover the even & odd dim.

Faithfulness of F: If $F(f) = 0$ then as exact seqs in $sVect_{\mathbb{C}}$ split

$F(im(f)) = 0$, so $F(R) = 0$ # with $F(\mathbb{1}) \cong \mathbb{1}_{\mathcal{C}_e}$.

Uniqueness: Clear (univ. property-type argument)

alg closed: $\mathcal{C}_e$ f.d. cat. comm $\mathbb{R}$-alg. $P = \bigoplus C_i$ is cpct & so satisfies (B) as right-exactness of $\otimes$ functor $\Rightarrow$ length$(E \otimes M) \leq$ length$(E \otimes X)$ + length$(E \otimes Y)$ $\forall X, Y, M$ cpct, E ext of X by Y. Hence P satisfies (A)

So our Prop $\leadsto$ $\otimes$-functor $\mathcal{C}_e \to sMod_R$:

Explicitly, we find A by applying the Lemma to P, then let R be the supercomm superalg with even part $End(\mathbb{1}_A) \cong End(\bar{\mathbb{1}}_A)$ & odd part $Hom(\mathbb{1}_A, \bar{\mathbb{1}}_A) \cong Hom(\bar{\mathbb{1}}_A, \mathbb{1}_A)$.

Then by construction R f.d. $\mathbb{R}$-alg, so $\exists R \to \mathbb{C}$. Extension of scalars gives $\mathcal{C}_e \to sMod_R \to sVect_{\mathbb{C}}$ □

Getting to the main thm

Prop: $\exists$ super fibre functor $\omega: \mathcal{C}_e \to sMod_R \Rightarrow \exists$ super fibre functor $\mathcal{C}_e \to sVect$ (i.e. $R = \mathbb{C}$)

(we did this in the special case above under some finiteness assumptions — general case uses a bunch of alg geo & results from EGA)

Then the result follows from the methods in L2.

This also proves (A) $\Rightarrow$ (B): If $\omega(X) \cong \mathbb{C}^{p|q}$ then ω exact & faithful $\Rightarrow$ length$(X^{\otimes d}) \leq$ length$(\omega(X)^{\otimes d}) = (p+q)^d$.

Finite group case Thm: If $\mathcal{C}_e$ is a sym fusion cat then $\exists$ fin gp G, $\varepsilon \in Z(G)$ of order ≤ 2 s.t. $\mathcal{C}_e \cong Rep(G, \varepsilon)$

(ss, fin iso classes of simples)

Proof (sketch) $\mathcal{C}_e$ clearly satisfies (B), so $\exists$ super fib. functor $\mathcal{C}_e \to sVect$.

In applying the machinery from L2, there's a surj map $\omega(\bigoplus C_i) \otimes \omega(\bigoplus C_i)^\vee \to \underbrace{\mathcal{L}(\omega, \omega)}$

Let G° be conn comp of id; invariant under G, $\mathcal{O}(G^\circ) \cong \mathcal{U}(Lie(G^\circ))^\vee$ so this is fin. dim. $\Rightarrow$ res of ss super-rep of G to G° is ss, so trivial. So $\mathcal{C}_e \xrightarrow{\sim} Rep(G/G^\circ, \varepsilon)$.