# Notes for Commutative Algebra
# Instructor: Johan de Jong

Henry Liu

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**Abstract**

These are my live-texed notes for the Fall 2016 offering of MATH GR6261 Commutative Algebra. Let me know when you find errors or typos. I'm sure there are plenty.

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# Chapter 1

# Dimension theory

## 1.1 Graded rings and modules

**Definition 1.1.1.** A **graded ring** is a ring $R$ endowed with a direct sum decomposition (as an abelian group) $R = \bigoplus_{d \geq 0} R_d$ such that $R_d \cdot R_e \subset R_{d+e}$.

*Remark.* Note that $R_0 \subset R$ is a subring. Also, the subset $R_+ \coloneqq \bigoplus_{d>0} R_d$ is an ideal. Sometimes we call $R_+$ the **irrelevant ideal**.

**Definition 1.1.2.** An element $f \in R$ is **homogeneous** if $f \in R_d$ for some $d$. This $d$ is called the **degree**.

**Lemma 1.1.3.** *Let $S$ be a graded ring. A set of homogeneous elements $f_i \in S_+$ generates $S$ as an algebra over $S_0$ iff they generate $S_+$ as an ideal.*

*Proof.* If a set $\{f_i\}$ generates $S$ as an algebra, then we can write any $f \in S_+$ as a polynomial in the $f_i$ with the coefficients in $S_0$, whose constant part is zero. But then $f$ is in $S_+$ generated by the $f_i$.

Conversely, we want to write every $f \in S$ as a polynomial in $f_i$ over $S_0$. It suffices to prove this for homogeneous elements, since every $f \in S$ is a sum of homogeneous elements. We induct on the degree $d$.

1. If $d = 0$, then $f \in S_0$ so we are done.

2. If $d > 0$, then $f \in S_+$, so by the hypothesis $f = \sum g_i f_i$ for some $g_i \in S$. Of course, we may replace $g_i$ by its homogeneous piece of degree $d - \deg(f_i)$. Then we can apply the induction hypothesis to the $g_i$ to write them as polynomials in the $f_i$. Hence $f$ is also a polynomial in the $f_i$. □

**Definition 1.1.4.** A ring $R$ is **Noetherian** if every ideal of $R$ is finitely generated. Equivalently, every increasing chain $I_1 \subset I_2 \subset \cdots$ of ideals stabilizes, i.e. there exists an $n$ such that $I_n = I_{n+1} = I_{n+2} = \cdots$.

**Lemma 1.1.5** (Hilbert basis theorem)**.** *If $R$ is a Noetherian ring, then any finitely generated $R$-algebra (i.e. $R[x_1, \ldots, x_n]/J$) is Noetherian.*

**Example 1.1.6.** The polynomial rings $\mathbb{R}[x], \mathbb{C}[x], \mathbb{F}_p[x_1, \ldots, x_{100}]$ are all Noetherian. The ring $\mathbb{Z}$ is also Noetherian, along with any number field.

**Lemma 1.1.7.** *A graded ring $S$ is Noetherian iff $S_0$ is Noetherian and $S_+$ is finitely generated as an ideal.*

*Proof.* If $S$ is Noetherian, then $S_0 = S/S_+$ is Noetherian and $S_+$ is finitely generated by the definition of Noetherian. Conversely, if $S_+ = (f_1, \ldots, f_r)$, then after replacing the $f_i$ by their homogeneous parts, we can assume without loss of generality that the $f_i$ are homogeneous. By lemma 1.1.3, $S$ is finitely generated as an $S_0$-algebra. Since $S_0$ is Noetherian, Hilbert's basis theorem implies $S$ is Noetherian. □

**Definition 1.1.8.** Let $S$ be a graded ring. A **graded $S$-module** is an $S$-module $M$ endowed with a grading $M = \bigoplus_{d\in\mathbb{Z}} M_d$ as an abelian group (note that the grading is over all of $\mathbb{Z}$ now) such that

$$f \in S_d,\ x \in M_e \implies fx \in M_{d+e}.$$

**Example 1.1.9.** Let $S = \mathbb{C}[x]$ and $M = \bigoplus_{d\in\mathbb{Z}} \mathbb{C}z_d$, and make $M$ into an $S$-module by defining

$$x \cdot z_d = z_{d+1}, \quad d \in \mathbb{Z},$$

i.e. multiplication by $x$ is a shift operator. Note that $M$ is not finitely generated as a module.

**Lemma 1.1.10.** *Let $S$ be a graded ring, and $M$ be a graded $S$-module. If $S$ is finitely generated over $S_0$ and $M$ is finitely generated as an $S$-module, then each $M_d$ is a finite $S_0$-module. (Terminology: for $S$-modules,* **finite** *just means finitely generated.)*

*Proof.* Let $x_1, \ldots, x_n \in M$ be generators of $M$ as an $S$-module, i.e. every element in $M$ is a linear combination of the $x_i$ with coefficients in $S$. Again without loss of generality, assume the $x_i$ are homogeneous. Let $f_1, \ldots, f_m \in S_+$ be homogeneous generators of the ideal. Every element $z \in M_d$ can be written as

$$z = \sum a_i x_i, \quad a_i \in S.$$

Replace $a_i$ with its homogeneous part of degree $d - \deg x_i$. By 1.1.3, we can write

$$a_i = \sum_I \alpha_{I,i} f_1^{i_1} \cdots f_m^{i_m}.$$

Hence the generators for $M_d$ as an $S_0$-module are given by

$$f_1^{i_1} \cdots f_m^{i_m} x_i, \quad \text{s.t. } d = \sum i_j \deg(f_j) + \deg(x_i). \qquad \square$$

## 1.2 Numerical polynomials

**Definition 1.2.1.** Let $A$ be an abelian group. An $A$-valued function $f$ defined on sufficiently large integers $n$ is a **numerical polynomial** if there exists an $r \geq 0$ and elements $a_0, \ldots, a_r \in A$ such that

$$f(n) = \sum_{i=0}^{r} \binom{n}{i} a_i, \quad \forall n \gg 0.$$

**Proposition 1.2.2.** *If $P \in \mathbb{Q}[x]$ and $P(n) \in \mathbb{Z}$ for all sufficiently large $n$, then $P = \sum \binom{n}{i} a_i$ for some $a_i \in \mathbb{Z}$.*

**Lemma 1.2.3.** *Suppose $f\colon n \mapsto f(n) \in A$ is a function such that $n \mapsto f(n) - f(n-1)$ is a numerical polynomial. Then $f$ is a numerical polynomial.*

*Proof.* Think of taking repeated differences of sequences: if the sequence of differences is a numerical polynomial, so is the original sequence. $\square$

**Example 1.2.4.** Let $k$ be a field. Consider $k[x_1, \ldots, x_n]$ as a graded ring with $k$ in degree 0 and the $x_i$ in degree 1. Then

$$d \mapsto \dim_k k[x_1, \ldots, x_n]_d$$

is a numerical polynomial, since the dimension is given by $\binom{n+d-1}{d}$.

**Definition 1.2.5.** Let $R$ be a ring. Define the **K-groups** $K_0'(R)$ and $K_0(R)$ as follows. The abelian group $K_0'(R)$ has the following properties:

1. every finite $R$-module $M$ induces an element $[M] \in K_0'(R)$, and $K_0'(R)$ is generated by these $[M]$;
2. every short exact sequence $0 \to M' \to M \to M'' \to 0$ induces a relation $[M] = [M'] + [M'']$, and all relations in $K_0'(R)$ are $\mathbb{Z}$-linear combinations of such relations.

The abelian group $K_0(R)$ is defined similarly, except only for finite projective $R$-modules.

*Remark.* There is an obvious map $K_0(R) \to K_0'(R)$ which is not an isomorphism in general.

**Example 1.2.6.** If $R = k$ is a field, then $\dim\colon K_0'(R) \to \mathbb{Z}$ is an isomorphism; similarly for $K_0(R)$.

**Example 1.2.7.** If $k$ is a field and $R = k[x]$, then every finite projective $R$-module is free because $R$ is a PID. Hence $\mathrm{rank}\colon K_0(R) \to \mathbb{Z}$ is again an isomorphism. As for $K_0'(R)$, the structure theorem for finitely generated modules over a PID says $M = R^r \times R/(d_1) \times \cdots \times R/(d_k)$, but the short exact sequence $0 \to (d_i) \to R \to R/(d_i) \to 0$ shows that $[R/(d_i)] = [R] - [(d_i)] = [R] - [R] = 0$ since $(d_i) \cong R$ (it is free with generator $d_i$). So torsion parts disappear in $K_0'$, and $\mathrm{rank}\colon K_0'(R) \to \mathbb{Z}$ is again an isomorphism.

**Proposition 1.2.8.** *Suppose $S$ is a Noetherian graded ring and $M$ is a finite graded $S$-module. If $S_+$ is generated by elements of degree 1, then*

$$\mathbb{Z} \to K_0'(S_0), \quad n \mapsto [M_n],$$

*(which is well-defined by lemma 1.1.10) is a numerical polynomial.*

*Proof.* We induct on the minimal number of generators of $S_1$. If this number is 0, then $S_+$ is trivial, so scalar multiplication cannot change the degree of elements in $M$, and therefore $M_n = 0$ for $n \gg 0$ (in particular, for $n$ greater than the maximal degree of a generator of $M$ as an $S$-module). Hence clearly $n \mapsto [M_n]$ is a numerical polynomial in this case.

For the induction step, let $x \in S_1$ be part of a minimal generating set so that $S/(x)$ has one less generator. We do a simple case and then generalize.

1. Suppose $x$ is nilpotent on $M$, i.e. $x^r M = 0$ for some $r$. If $r = 1$, i.e. $xM = 0$ then $M$ is an $S/(x)$-module and the induction hypothesis applies. Otherwise we induct on $r$: find a short exact sequence $0 \to M' \to M \to M'' \to 0$ such that $r', r'' < r$, so the result holds for $M'$ and $M''$, and therefore for $[M_d] = [M_d'] + [M_d'']$.
2. If $x$ is not nilpotent on $M$, let $M' \subset M$ be the largest submodule on which $x$ is nilpotent, and consider $0 \to M' \to M \to M/M' \to 0$. It suffices to prove the result for $M/M'$, where multiplication by $x$ is injective. So without loss of generality assume that is the case on $M$. Let $\bar{M} \coloneqq M/xM$. The map $x\colon M \to M$ is not a map of graded $S$-modules since it fails to preserve the grading, but we get a short exact sequence $0 \to M_d \xrightarrow{x} M_{d+1} \to \bar{M}_{d+1} \to 0$. Hence $[M_{d+1}] - [M_d] = [\bar{M}_{d+1}]$. By lemma 1.2.3, we are done. □

**Example 1.2.9.** Let $S = k[X_1, \ldots, X_d]$. Then $S_0 = k$, and we know $K_0'(S_0) = K_0(S_0) \cong \mathbb{Z}$ via dim. Hence any finitely generated graded $k[X_1, \ldots, X_d]$-module $M$ gives a numerical polynomial $n \mapsto \dim_k(M_n)$.

**Lemma 1.2.10.** *Let $k$ be a field, and $I \subset k[X_1, \ldots, X_d]$ is a non-zero graded ideal. Let $M = k[X_1, \ldots, X_d]/I$. Then the numerical polynomial $n \mapsto \dim_k(M_n)$ has degree $< d - 1$.*

*Proof.* By example 1.2.4, the numerical polynomial for $k[X_1, \ldots, X_d]$ is $n \mapsto \binom{n+d-1}{d-1}$. If $f \in I$ is homogeneous of degree $e$ and any degree $n \gg e$, we have $I_n \supset f \cdot k[X_1, \ldots, X_d]_{n-e}$, so that $\dim_k(I_n) > \binom{n-e+d-1}{d-1}$. Subtracting, $\dim_k(M_n) \leq \binom{n+d-1}{d-1} - \binom{n-e+d-1}{d-1}$, which is indeed of degree $< d - 1$. □

## 1.3 Length of modules

Let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$. All modules will be finite $R$-modules.

**Definition 1.3.1.** Let $R$ be any ring and $M$ an $R$-module. The **length** of $M$ over $R$ is

$$\operatorname{length}_R(M) \coloneqq \sup\{n : \exists 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M\}.$$

Equivalently, $\operatorname{length}_R(M)$ is the length of any **composition series**: a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_i/M_{i+1}$ is simple for every $i$.

*Remark.* Obviously if $\operatorname{length}_R(M) < \infty$, then any chain can be refined to a maximal chain. It is not obvious that the lengths of every maximal chain are the same (cf. **Jordan–Hölder theorem**).

**Example 1.3.2.** Let $M = R = k[x]$. There exists arbitrarily long sequences $0 \subset (x^{100}) \subset \cdots \subset (x) \subset M$, so $\operatorname{length}_R(M) = \infty$. On the other hand, for $M$ a finite-dimensional $k$-vector space, $\operatorname{length}_k(M) = \dim_k(M)$.

**Lemma 1.3.3.** $\operatorname{length}_R$ *is additive on short exact sequences* $0 \to M' \to M \to M'' \to 0$.

**Lemma 1.3.4.** *Let $R$ be a local ring with maximal ideal $\mathfrak{m}$, and let $M$ be an $R$-module.*

1. *If $M$ is a finite module and $\mathfrak{m}^n M \neq 0$ for every $n$, then $\operatorname{length}_R(M) = \infty$.*
2. *If $\operatorname{length}_R(M) < \infty$, then $\mathfrak{m}^n M = 0$ for some $n$.*

*Proof.* Suppose $\mathfrak{m}^n M \neq 0$ for every $n$. Fix $x \in M$ and pick $f_1, \ldots, f_n \in \mathfrak{m}$ such that $f_1 f_2 \cdots f_k x \neq 0$ for $k \leq n$. Then the terms in the filtration

$$0 \subset Rf_1 f_2 \cdots f_n x \subset \cdots \subset Rf_1 f_2 x \subset Rf_1 x \subset Rx \subset M$$

are distinct: if $Rf_1 f_2 x = Rf_1 x$, then $g f_1 f_2 x = f_1 x$ for some $g \in R$, so that $(1 - g f_2) f_1 x = 0$, i.e. $f_1 x = 0$ since $1 - g f_2$ is a unit, contradicting our choice of $f_1$.

Now note that if $M$ is not a finite $R$-module, $\operatorname{length}_R(M) = \infty$. Hence if $\operatorname{length}_R(M) < \infty$, we must have $M$ finite, and therefore $\mathfrak{m}^n M = 0$ for some $n$. □

**Lemma 1.3.5.** *Let $R$ be a ring, $M$ a finite $R$-module, and $\mathfrak{m} \subset M$ a finitely generated maximal ideal such that $\mathfrak{m}^n M = 0$ for some $n \geq 1$. Then*

$$\operatorname{length}_R(M) = \sum_{i=0}^{n-1} \dim_{R/\mathfrak{m}}(\mathfrak{m}^i M / \mathfrak{m}^{i+1} M).$$

*Proof.* Since $R/\mathfrak{m}$ is a field, it has length 1 as an $R$-module. Also, $\mathfrak{m}^i M / \mathfrak{m}^{i+1} M$ is an $R/\mathfrak{m}$-module since $\mathfrak{m}$ annihilates everything in it. Take the filtration $0 = \mathfrak{m}^n M \subset \cdots \subset \mathfrak{m} M \subset M$. This filtration gives associated short exact sequences $0 \to \mathfrak{m}^{k+1} M \to \mathfrak{m}^k M \to \mathfrak{m}^k M / \mathfrak{m}^{k+1} M \to 0$ on which length is additive. Hence

$$\operatorname{length}_R(M) = \operatorname{length}_R(\mathfrak{m} M) + \dim_{R/\mathfrak{m}}(M/\mathfrak{m}) = \cdots = \operatorname{length}(\mathfrak{m}^n M) + \sum_{i=0}^{n-1} \dim_{R/\mathfrak{m}}(\mathfrak{m}^i M / \mathfrak{m}^{i+1} M). \quad \square$$

## 1.4 Hilbert polynomial

**Proposition 1.4.1.** *Let $I$ be an ideal of definition, i.e. $\mathfrak{m}^r \subset I \subset \mathfrak{m}$. The functions*

$$\varphi_{I,M}(n) \coloneqq \operatorname{length}_R(I^n M / I^{n+1} M)$$
$$\chi_{I,M}(n) \coloneqq \operatorname{length}_R(M / I^{n+1} M)$$

*are numerical polynomials.*

*Remark.* The polynomial $\varphi_{\mathfrak{m},M}$ is called the **Hilbert polynomial** of $M$.

*Proof.* Let $S \coloneqq \bigoplus_{d\geq 0} I^d/I^{d+1}$. This is a Noetherian graded ring generated by $S_1 = I/I^2$ over $S_0 = R/I$. Also, $N \coloneqq \bigoplus_{d\geq 0} I^d M/I^{d+1}M$ is a finitely generated graded $S$-module. By 1.2.8, the map

$$n \mapsto [I^n M/I^{n+1}M] \in K_0'(R/I)$$

is a numerical polynomial. To conclude the proposition is true for $\varphi_{I,M}$, we use the following two facts:

1. any finite $R/I$-module has finite length over $R/I$, so $\text{length}_{R/I}$ is a well-defined function $K_0'(R/I) \to \mathbb{Z}$ by the universal property (or the construction) of the $K$-group;

2. for any finite $R$-module $M$ annihilated by $I$, the length $\text{length}_R(M) = \text{length}_{R/I}(M)$.

This concludes the proof for $\varphi_{I,M}$. We get the result for $\chi_{I,M}$ because $\chi_{I,M}(n) - \chi_{I,M}(n-1) = \varphi_{I,M}(n)$. $\square$

**Example 1.4.2.** Let $R = k[x,y]_{(x,y)}$, and $M = R$. Let $I$ be the maximal ideal in $R$. Then $\kappa = k$, and:

| $n$ | $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ | $\varphi$ | $\chi$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | $x, y$ | 2 | 3 |
| 2 | $x^2, xy, y^2$ | 3 | 6. |

The table is the same for $R = k[[x,y]]$, even though it is not the same ring, but rather the **completion**. If we change $I = \mathfrak{m}^2$, then clearly

$$\chi_{I,R}(n) = \text{length}_R(R/I^{n+1}) = \text{length}_R(R/\mathfrak{m}^{2n+2}) = \chi_{\mathfrak{m},R}(2n+2).$$

It follows that

$$\varphi_{I,R}(n) = \chi_{I,R}(n) - \chi_{I,R}(n+1) = \binom{2n+3}{2} - \binom{2n+1}{2} = 4n+3.$$

**Example 1.4.3.** Let $R = k[x,y]_{(x,y)}$, and $M = R/(x^a + y^b)$ for some $a < b$. Let $I$ be the maximal ideal in $R$. Then $\kappa = k$, and

$$\mathfrak{m}^n M/\mathfrak{m}^{n+1}M = \frac{\mathfrak{m}^n + (x^a + y^b)}{\mathfrak{m}^{n+1} + (x^a + y^b)} = \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1} + \mathfrak{m}^n \cap (x^a + y^b)}$$

will be (for $n \gg 0$) a vector space with basis

$$y^n, xy^{n-1}, \ldots, x^{a-1}y^{n-a+1}$$

since we can get rid of $x^n \equiv x^{n-a}(x^a + y^b) \mod \mathfrak{m}^{n+1}$, and so on. So $\varphi_{\mathfrak{m},M}(n) = a$ for $n \gg 0$. Hence $\chi_{\mathfrak{m},M}(n) = an + c$ for some constant $c$ for $n \gg 0$.

## 1.5 Nakayama, Artin–Rees, and Krull's Intersection

**Lemma 1.5.1** (Nakayama)**.** *Let $R$ be local Noetherian, $\mathfrak{m}$ a maximal ideal, and $M$ a finite $R$-module. Then any of the following equivalent statements are true:*

1. *$\mathfrak{m}M = M$ implies $M = 0$;*

2. *if $x_1, \ldots, x_n \in M$ map to generators of $M/\mathfrak{m}M$ as a $R/\mathfrak{m}$-vector space, then $x_1, \ldots, x_n$ generate $M$ as an $R$-module.*

**Lemma 1.5.2** (Artin–Rees)**.** *Let $R$ be a Noetherian ring and $I \subset R$ a proper ideal. Let $N \subset M$ be finite $R$-modules. Then there exists $c > 0$ such that for all $n \geq c$,*

$$I^n M \cap N = I^{n-c}(I^c M \cap N).$$

*Proof.* Let $S \coloneqq \bigoplus_{d\geq 0} I^d$, called the **Rees algebra**. It is Noetherian. Since $I$ is finitely generated, $\tilde{M} \coloneqq \bigoplus_{d\geq 0} I^d M$ is a finite graded $S$-module, and $\tilde{M}' \coloneqq \bigoplus_{d\geq 0} N \cap I^d M \subset \tilde{M}$ is a graded $S$-submodule. Since $S$ is Noetherian, $\tilde{M}'$ is a finitely generated $S$-module. Let $\xi_j \in N \cap I^{d_j} M$ be the generators (by taking homogeneous parts, if necessary). Then for $n \geq c \coloneqq \max\{d_j\}$,

$$N \cap I^n M = \tilde{M}'_n = \sum S_{n-d_j} \tilde{M}'_{d_j} = \sum I^{n-d_j}(N \cap I^{d_j} M) \subset I^{n-c}(N \cap I^c M). \qquad \square$$

**Theorem 1.5.3** (Krull's intersection theorem)**.** *Let $R$ be a Noetherian local ring, and $I \subset R$ be a proper ideal. Let $M$ be a finite $R$-module. Then $\bigcap_{n\geq 0} I^n M = 0$.*

*Proof.* Set $M' = \bigcap_{n\geq 0} I^n M$ and apply Artin–Rees: there exists $c \geq 0$ such that $M' \cap I^n M = I^{n-c}(M' \cap I^n)$ for $n \geq c$. Take $n = c + 1$. Then this is just $M' = IM'$. Nakayama's lemma gives $M' = 0$. $\square$

## 1.6 Co-length

**Lemma 1.6.1.** *Say $N \subset M$ has* **finite co-length** *if* $\operatorname{length}_R(M/N) < \infty$*.*

1. *If $N \subset M$ is of finite co-length, then there exists $c_1, c_2$ such that for all $n \geq c_2$,*

$$c_1 + \chi_{I,N}(n - c_2) \leq \chi_{I,M}(n) \leq c_1 + \chi_{I,M}(n).$$

*The degree of $\chi_{I,M} - \chi_{I,N}$ is less than the degree of $\chi_{I,M}$ (which is the degree of $\chi_{I,N}$, provided that $M$ does not have finite length.*

2. *If $I, I'$ are two ideals of definition, then there exists $a > 0$ such that $\chi_{I,M}(n) \leq \chi_{I',M}(an)$, so that the degree of $\chi_{I,M}$ is independent of $I$.*

**Definition 1.6.2.** Define $d(M) \in \{-\infty, 0, 1, 2, \ldots\}$ by:

$$d(M) = \begin{cases} -\infty & M = 0 \\ \deg \chi_{I,M} & \text{otherwise.} \end{cases}$$

**Lemma 1.6.3.** *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, then there exists a submodule $N \subset M'$ of finite co-length $\ell$ and $c \geq 0$ such that*

$$\chi_{I,M}(n) = \chi_{I,M''}(n) + \chi_{I,N}(n - c) + \ell$$
$$\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n - c).$$

*Proof.* For every $n \geq 0$, we get a short exact sequence

$$0 \to M'/(M' \cap I^{n+1}M) \to M/I^{n+1}M \to M''/I^{n+1}M'' \to 0.$$

It is clear that $I^{n+1}M' \subset M' \cap I^{n+1}M$, we don't know how much bigger $M' \cap I^{n+1}M$ is. But **Artin–Rees** says that there exists a $c \geq 0$ such that

$$M' \cap I^n M = I^{n-c}(M' \cap I^c M), \quad \forall n \geq c.$$

Set $N = M' \cap I^c M$, which proves the lemma via the sequence

$$\begin{array}{ccccccccc} & & M'/N & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \longrightarrow & M/I^{n+1-c}N & \longrightarrow & M/I^{n+1}M & \longrightarrow & M''/IM'' & \longrightarrow & 0 \\ & & \uparrow & & & & & & \\ & & N/I^{n+1-c}N & & & & & & \\ & & \uparrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

where $M'/N$ has length $\ell$. ☐

**Corollary 1.6.4.** *We have*

$$\max(\deg \chi_{I,M'}, \deg \chi_{I,M''}) = \deg \chi_{I,M}$$
$$\max(d(M'), d(M'')) = d(M)$$
$$\deg(\chi_{I,M} - \chi_{I,M'} - \chi_{I,M''}) < \deg(\chi_{I,M'}).$$

## 1.7 Dimension of local Noetherian rings

**Definition 1.7.1.** Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\dim(R)$ denote the Krull dimension of the ring, as usual. Let $d(R)$ be the degree of $\chi_{\mathfrak{m},R}$. Let $d'(R)$ be the minimal number of generators of an ideal of definition in $R$.

**Theorem 1.7.2.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then*

$$\dim R = 0 \iff d(R) = 0 \iff d'(R) = 0 \iff R \text{ Artinian.}$$

*Proof.* (1) $\iff$ (4) We know $R$ is Artinian iff it is Noetherian and dimension 0.

(1) $\iff$ (3) Note that $d'(R) = 0$ iff $(0)$ is an ideal of definition iff $\mathfrak{m} = \sqrt{(0)}$ iff $\mathfrak{m}$ is minimal iff $\operatorname{Spec} R = \{\mathfrak{m}\}$.

(1) $\iff$ (2) Note that $d(R) = 0$ iff $\chi_{\mathfrak{m},R}$ is eventually constant, iff $\mathfrak{m}^n = \mathfrak{m}^{n+1} = \cdots$ for $n \gg 0$, iff $\mathfrak{m}^n = 0$ by Nakayama, i.e. $(0)$ is an ideal of definition. ☐

**Theorem 1.7.3.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then*

$$\dim R = d(R) = d'(R).$$

*Proof.* First show $d(R) \le d'(R)$. Say $I = (f_1, \dots, f_{d'})$ is an ideal of definition with $d' = d'(R)$. Then we get

$$\bigoplus_{\substack{E=(e_1,\dots,e_{d'}) \\ \sum_{i=1}^{d'} e_i = n}} R/I \to I^n/I^{n+1}, \quad (a_E) \mapsto \sum_E a_E f_1^{e_1} \cdots f_{d'}^{e_{d'}}.$$

Hence $\operatorname{length}_R(I^n/I^{n+1}) \le \operatorname{length}_R(\bigoplus_E R/I) = \binom{d'+n-1}{d'-1} \operatorname{length}_R(R/I)$. It follows that $\deg \varphi_{I,R} \le d' - 1$, and therefore $\deg \chi_{I,R} \le d'$. Hence $d(R) \le d'(R)$.

Now we show $d'(R) \le \dim(R)$. This is clear if $\dim(R) = \infty$, so assume $\dim(R) < \infty$ and induct. We know the base case $\dim(R) = 0$ from last time. If $\dim(R) > 0$, let $p_1, \dots, p_t$ be the minimal primes of $R$. (This is fine because $R$ is Noetherian, hence there are finitely many minimal primes.) Since $\dim(R) > 0$, we know $p_i \subsetneq \mathfrak{m}$ for all $i$. Pick an element $x \in \mathfrak{m}$ not in any of the $p_i$ (using **prime avoidance**; the vanishing set of $x$ transversally cuts each of the irreducible components). Hence $p_i \notin V(x) = \operatorname{Spec}(R/xR)$, i.e. $\dim(R/xR) < \dim(R)$. By the induction hypothesis, there exist $\bar{x}_2, \ldots, \bar{x}_d$ in $R/xR$ which generate the ideal of definition in $R/xR$ with $d \le \dim(R/xR)$. Then $(x, \ldots, x_d)$ is an ideal of definition in $R$, because $V(x, x_2, \ldots, x_d) = V(x_2, \ldots, x_d) = \{\mathfrak{m}_{R/xR}\}$. Hence $d + 1 \le \dim R$ and we are done by induction.

Finally, show $\dim(R) \le d(R)$. We induct on $d(R)$, and we know the base case $d(R) = 0$. Assume $d(R) > 0$; if $\dim(R) = 0$ we are done, so assume $\dim(R) > 0$ as well. Pick

$$p = q_0 \subsetneq q = q_1 \subsetneq \cdots \subsetneq q_e = \mathfrak{m}$$

with $e \ge 1$. (Here $p$ represents a biggest irreducible component; we are working with it.) We must show $e \le d(R)$. Look at

$$0 \to p \to R \to R/p \to 0.$$

For short exact sequences, we know $d(R) \ge d(R/p) > 0$. Pick $x \in q \setminus p$, so that

$$0 \to R/p \xrightarrow{\cdot x} R/p \to R/(xR + p) \to 0.$$

By lemma 1.6.4, $d(R/(xR + p)) < d(R/p)$. By the induction hypothesis, we get $\dim(R/(xR + p)) \le d(R/p) - 1 \le d(R) - 1$. Hence $e - 1 \le d(R) - 1$. □

*Remark.* This theorem is why we built the Hilbert polynomial machinery: somehow we couldn't do the proof of this theorem without it!

**Corollary 1.7.4.** $\dim(R) = d'(R)$ *is less than the minimal number of generators of the maximal ideal, which via Nakayama is equal to* $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.

**Definition 1.7.5.** We say $(R, \mathfrak{m}, \kappa)$ is a **regular local ring** if $R$ is Noetherian and $\dim(R) = d'(R) = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.

**Example 1.7.6.** Let $k$ be a field. Then $R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ is regular of dimension $n$. There are a few ways to see this.

1. $0 \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_n)$ is a chain of primes of length $n$, so the dimension is at least $n$. But $\mathfrak{m} = (x_1, \ldots, x_n)$ is generated by $n$ elements, so the dimension is at most $n$.

2. Use the Hilbert polynomial.

**Corollary 1.7.7** (Krull's Hauptidealsatz)**.** *Let $R$ be a Noetherian ring, and take $x \in R \cap p$ where $p$ is a minimal prime over $(x)$. Then the height of $p$ is at most* $1$.

*Proof.* Consider $R_p$, where the only prime containing $x$ is $pR_p$. Hence if $x$ is not nilpotent, then $V(x) = pR_p$, i.e. $\sqrt{(x)} = pR_p$. Then $(x)$ is an ideal of definition, and also no ideal of definition is trivial because it must contain $x^n$ for some $n$. Hence $\dim R = 1$. It follows that the height of $p$ is 1. Of course, if $x$ is nilpotent, the height of $p$ is 0. □

*Remark.* Geometrically, $p$ is a generic point of an irreducible component of $V(x)$, i.e. if we have an irreducible component cut out by 1 equation, the codimension is at most 1.

**Corollary 1.7.8** (Krull's height theorem)**.** *Let $R$ be Noetherian and $f_1, \ldots, f_r \in R$. Let $p$ be a minimal prime over $(f_1, \ldots, f_r)$. Then the height of $p$ is at most $r$.*

*Proof.* Same proof as for the Hauptidealsatz, which is the height 1 case. □

**Corollary 1.7.9** (Cutting down)**.** *If $(R, \mathfrak{m}, \kappa)$ is Noetherian local and $x \in \mathfrak{m}$, then* $\dim(R/xR) \ge \dim(R) - 1$ *and equality holds if $x$ is not in any minimal prime of $R$.*

*Proof.* Let $n = \dim(R/xR)$. If $x_1, \ldots, x_n \in R$ map to generators of an ideal of definition in $R/xR$, then $x, x_1, \ldots, x_n$ generate an ideal of definition in $R$, i.e. $\dim R \leq \dim(R/xR) + 1$. But if $x$ is not in any minimal prime, then every chain of $R/xR$ is still a chain in $R$, but now we can prepend $(x)$. Hence equality holds. □

**Corollary 1.7.10.** *Let $(R, \mathfrak{m}, \kappa)$ be Noetherian local. If $I = (x_1, \ldots, x_d)$ is an ideal of definition with $d = \dim R$, then $\dim R/(x_1, \ldots, x_i) = d - i$ for all $i$.*

*Proof.* Induct on $d$ and use the previous corollary. □

*Remark.* It follows from all this machinery that $\dim R[x_1, \ldots, x_n] = n$ for $R$ a Noetherian ring. Note that this is *not* a trivial fact at all. For example, $(x^2 - 2, xy - 2, y^2 - 2)$ in $k[x, y]$ is actually generated by $(x^2 - 2, x - y)$.

## 1.8 Annihilators and support

**Definition 1.8.1.** Let $R$ be a ring and $M$ an $R$-module. The **support** of $M$ is

$$\operatorname{supp}(M) \coloneqq \{p \in \operatorname{Spec} R : M_p \neq 0\} \subset \operatorname{Spec} R.$$

(Think of sheaves: these are points where the stalk is non-zero.)

**Lemma 1.8.2.** *$M \neq (0)$ iff $\operatorname{supp}(M) \neq \emptyset$.*

*Proof.* Recall that $M \mapsto \prod_p M_p$ is injective, since if $x/1 = 0 \in M_p$ then $\operatorname{Ann}_R(x) \cap (R \setminus p) \neq \emptyset$, i.e. $\operatorname{Ann}_R(x) \not\subset p$ for every $p$, and therefore must contain a unit. □

**Definition 1.8.3.** Let $R$ be a ring and $M$ an $R$-module. The **annihilator** of $M$ is

$$\operatorname{Ann}_R(M) \coloneqq \operatorname{Ann}(M) \coloneqq \{f \in R : fx = 0 \ \forall x \in M\}.$$

If $x \in M$, let $\operatorname{Ann}_R(x) \coloneqq \{f \in R : fx = 0\} = \operatorname{Ann}_R(Rx)$.

**Lemma 1.8.4.** *If $M$ is a finite $R$-module, then $\operatorname{supp}(M) = V(\operatorname{Ann}_R(M))$ is closed in $\operatorname{Spec} R$.*

*Proof.* Let $I = \operatorname{Ann}_R(M)$, so that $IM = 0$. If $p \in \operatorname{supp}(M)$, then $M_p \neq 0$. so $M_p \neq (IM)_p = I_p M_p$. If $I_p = R_p$, then $M_p \neq RM_p$, which is impossible. Hence $I_p \neq R_p$, i.e. it contains no unit, so $I \subset p$ (since $p$ is the unique maximal ideal in $M_p$, and therefore everything outside it is a unit). Then $p \in V(I)$.

Conversely, if $p \in V(I)$, then $I \subset p$. Then $I_p \neq R_p$. If we could show that $I_p = \operatorname{Ann}(M_p)$ (keep in mind that $M_p$ is an $R_p$-module, so $\operatorname{Ann}(M_p) = \operatorname{Ann}_{R_p}(M_p)$), then we would be done, since then $\operatorname{Ann}(M_p) \neq R_p$, i.e. the annihilator is not the whole ring, and therefore $M \neq 0$. This is where we require $M$ finite.

Claim: if $M$ is finite, and $S \subset R$ is a multiplicative subset, then $S^{-1} \operatorname{Ann}_R(M) = \operatorname{Ann}_{S^{-1}R}(S^{-1}M)$. Clearly the forward inclusion is obvious. For the converse, say $r/s \in \operatorname{Ann}_{S^{-1}R}(S^{-1}M)$, and let $x_1, \ldots, x_n$ be generators of $M$. Then $(r/s)x_i = 0$ in $S^{-1}M$ means there exist $s_i \in S$ such that $s_i r x_i = 0$ in $M$, and $s_1 r, \ldots, s_n r \in \operatorname{Ann}_R(M)$. Hence $r/s = (s_1 \cdots s_n r)/(s s_1 \cdots s_n) \in S^{-1} \operatorname{Ann}_R(M)$. □

**Example 1.8.5.** We have $\operatorname{supp}(k[x]/(x-5)) = \{(x-5)\}$. Generalizing, $\operatorname{supp}(k[x]/(x-1)\cdots(x-10)) = \{(x-1), \ldots, (x-10)\}$. What about infinitely many points in the support? Try

$$M = \prod_{i=1}^{\infty} k[x]/(x-i),$$

but no polynomial annihilates $M$, because any such polynomial must contain factors $(x-i)$ for every $i$. (This is because localization does not commute with products.) Instead,

$$\operatorname{supp}\left(\bigoplus_{i=1}^{\infty} k[x]/(x-i)\right) = \{(x-1), (x-2), \ldots\}.$$

This is an instance of the general fact that $\operatorname{supp}(\bigoplus_{i=1}^{\infty} M_i) = \bigoplus_{i=1}^{\infty} \operatorname{supp}(M_i)$, since localization commutes with direct sum.

Some useful facts about support:

1. if $M \subset N$ then $\mathrm{supp}(M) \subset \mathrm{supp}(N)$, and if $M \to Q$ is onto, $\mathrm{supp}(Q) \subset \mathrm{supp}(M)$;
2. if $0 \to M_1 \to M_2 \to M_3 \to 0$, then $\mathrm{supp}(M_2) = \mathrm{supp}(M_1) \cup \mathrm{supp}(M_3)$;
3. $\mathrm{supp}(M/IM) = \mathrm{supp}(M) \cap V(I)$ provided $M$ is finite.

**Lemma 1.8.6.** *Let $R$ be a Noetherian ring, and $M$ a finite $R$-module. Then there exists a filtration by submodules*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

*such that $M_i/M_{i-1} \cong R/p_i$ for each $i$, with $p_i$ prime.*

**Example 1.8.7.** Let $R = k[x,y]$ and $M = k[x,y]/(y^2, xy)$. Then $V(y^2, xy) = V(y)$, but $(y^2, xy)$ is not a prime. We get a sequence $0 \to (y)/(y^2, xy) \to M \to k[x,y]/(y) \to 0$. The annihilator of $(y)/(y^2, xy)$ is $(x,y)$, i.e. $(y)/(y^2, xy) \cong R/(x,y)$.

*Proof.* If $M$ is finite, then $0 \subset Rx_1 \subset Rx_1 + Rx_2 \subset \cdots \subset Rx_1 + \cdots + Rx_n = M$. Hence without loss of generality, $M$ is cyclic, i.e. $M = R/I$ for some ideal $I$. Consider the set $S$ of ideals $J$ such that the lemma does not hold for $R/J$, and suppose $S \neq \emptyset$. Then there exists $J \in S$ maximal, and $J$ is not prime. Pick $a, b \in R$ such that $ab \in J$, but $a \notin J$ and $b \notin J$. Then

$$(0) \subset aR/(J \cap aR) \subset R/J.$$

Then the quotients are $R/Ja$ and $R/J'$ where $J' = \mathrm{Ann}_{R/(J\cap aR)}(a)$. Note that $J' \supsetneq J$ and $J'' \coloneqq J + aR \supsetneq J$. Hence we get a filtration of the desired kind on each of the two steps in the above filtration. This gives a filtration of the desired kind on the original module. □

**Corollary 1.8.8.** *In this situation,* $\mathrm{supp}(M) = \bigcup_{i=1}^n V(p_i)$.

*Proof.* We already know the behavior of supp on short exact sequences. □

**Corollary 1.8.9.** *Let $R$ be Noetherian local, $M$ non-zero and finite. Then* $\mathrm{supp}(M) = \{m\}$ *if and only if* $\mathrm{length}_R(M) < \infty$.

*Proof.* If $\mathrm{supp}(R) = \{m\}$, then every step in a filtration has associated graded module $R/m$, which means length is precisely the length of the filtration itself. Conversely, we already have a structure theorem for finite length modules. □

**Corollary 1.8.10.** *Let $R$ be Noetherian, $I \subset R$ an ideal, and $M$ a finite $R$-module. Then $I^n M = 0$ for some $n \geq 1$ if and only if* $\mathrm{supp}(M) \subset V(I)$.

*Proof.* Suppose $\mathrm{supp}(M) \subset V(I)$. Every ideal $p_i$ in the filtration must therefore be contained in $V(I)$, i.e. $I \supset p_i$ for every $i$. Then $I$ kills every graded piece, i.e. $I^n$ kills the entire ring. Conversely, $\mathrm{supp}(M) = V(\mathrm{Ann}(M)) \subset V(I)$. □

**Lemma 1.8.11.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring, and $M$ be a finite $R$-module. Then $d(M) =$* $\dim(\mathrm{supp}(M))$.

*Proof.* Take a filtration as in the lemma. Then $d(M) = \max\{d(R/p_i)\}$ because of the behavior of $d$ over short exact sequences. By the theorem on dimension,

$$\max(\dim(R/p_i)) = \max(\dim V(p_i)) = \dim \bigcup V(p_i) = \dim \mathrm{supp}(M).$$ □

**Lemma 1.8.12.** *Let $R$ be Noetherian and $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite $R$-modules. Then*

$$\dim \mathrm{supp}(M) = \max\{\dim \mathrm{supp}(M'), \dim \mathrm{supp}(M'')\}.$$

## 1.9 Associated primes

**Definition 1.9.1.** Let $R$ be a ring and $M$ a $R$-module. A prime $p$ of $R$ is **associated to** $M$ if there exists $x \in M$ whose annihilator $\mathrm{Ann}_R(x)$ is $p$. The set of associated primes of $M$ is denoted $\mathrm{Ass}_R(M)$.

**Example 1.9.2.** Let $R = k[x, y]$ and $M = k[x, y]/(y^2, xy)$. Then $(y) \in \mathrm{Ass}_R(M)$ because it is the annihilator of $x$. Also, $(x, y) \in \mathrm{Ass}_R(M)$ is killed by $y$. Note that every prime in $\mathrm{Ass}_R(M)$ must contain $\mathrm{Ann}_R(M)$, and hence $\mathrm{Ass}(M) \subset \mathrm{supp}(M)$. (However while $\mathrm{supp}(M) = V(y)$ is infinite, $\mathrm{Ass}(M)$ contains only these two elements!)

**Proposition 1.9.3.** *$\mathrm{Ass}(M) \subset \mathrm{supp}(M)$, and if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence then $\mathrm{Ass}(M') \subset \mathrm{Ass}(M)$ and $\mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$.*

**Lemma 1.9.4.** *Suppose we have $0 \subset M_0 \subset \cdots \subset M_n = M$ with $M_i/M_{i-1} = R/p_i$ for $p_i$ prime. Then*

$$\mathrm{Ass}(M) \subset \{p_1, \ldots, p_n\}.$$

*Proof.* By induction on $n$. Pick $x \in M$ whose annihilator is a prime $p$. Since $p$ is therefore an associated prime, we must show $p$ is one of the $p_i$ arising from the associated graded of the filtration. If $x \in M_{n-1}$, then we are done by induction (since $M_{n-1}$ has a shorter filtration). If not, then $x$ maps to a non-zero element $\bar{x}$ in $M/M_{n-1} = R/p_n$. Then $\mathrm{Ann}_{R/p_n}(\bar{x}) = p_n$, and therefore $p \coloneqq \mathrm{Ann}_R(x) \subset p_n$. If $p = p_n$, we are done. If not, pick $f \in p_n \setminus p$. Then $\mathrm{Ann}_R(fx) = p$ (since if $afx = 0$, then $af \in p$, but $f \notin p$ so $a \in p$). Then $fx \in M_{n-1}$ and therefore we are done by induction. ☐

**Corollary 1.9.5.** *If $R$ is Noetherian and $M$ is finite, then $\mathrm{Ass}(M)$ is finite.*

**Proposition 1.9.6.** *Let $R$ be a Noetherian ring and $M$ be a finite $R$-module. The following sets of primes are the same:*

1. *the primes minimal in $\mathrm{supp}(M)$;*
2. *the primes minimal in $\mathrm{Ass}(M)$;*
3. *for any filtration $0 = M_0 \subset \cdots \subset M_n = M$ with $M_i/M_{i-1} = R/p_i$, the primes minimal in $\{p_1, \ldots, p_n\}$.*

*Proof.* We know that in the situation of (3), $\mathrm{supp}(M) = \bigcup_i V(p_i)$. Hence the sets of (1) and (3) are equal.

(1) $\subset$ (2) Now suppose $p$ is minimal in $\{p_1, \ldots, p_n\}$. Let $i$ be minimal such that $p = p_i$. Pick $x \in M_i \backslash M_{i-1}$. Then $\mathrm{Ann}_R(x) \subset p_i = p$. On the other hand, $p_1 \cdots p_i \subset \mathrm{Ann}(M)$ since $p_k$ kills $M_{k-1}$, and multiplying goes down the filtration killing everything. For $j = 1, \ldots, i-1$, pick $f_j \in p_j$ with $f_j \notin p$ (which is possible by the choice of $i$). Then $\mathrm{Ann}_R(f_1 \cdots f_{i-1} x) = p = p_i$. Hence $p \in \mathrm{Ass}(M) \subset \mathrm{supp}(M)$, so more strongly $p$ is minimal in $\mathrm{Ass}(M)$.

(2) $\subset$ (1) Conversely, if $p$ is minimal in $\mathrm{Ass}(M)$, then since $\mathrm{Ass}(M) \subset \mathrm{supp}(M)$, there exists a minimal $q \in \mathrm{supp}(M)$ with $q \subset p$. We just showed $q \in \mathrm{Ass}(M)$, so $q = p$, i.e. $p$ is minimal in $\mathrm{supp}(M)$. ☐

**Corollary 1.9.7.** *If $R$ is Noetherian and $M$ is finite, then $M = (0)$ iff $\mathrm{Ass}(M) = \emptyset$.*

**Corollary 1.9.8.** *If $R$ is Noetherian and $M$ is finite, then the set of zero divisors on $M$, i.e. $\{f \in R : \exists x \in M \text{ s.t. } fx = 0\}$, is equal to $\bigcup_{p \in \mathrm{Ass}(M)} p$.*

*Proof.* Easy: unwind definition. ☐

**Corollary 1.9.9.** *If $R$ is Noetherian and $I$ is an ideal and $M$ is finite, then TFAE:*

1. *there exists $x \in I$ which is a non-zerodivisor on $M$;*
2. *we have $I \notin p$ for all $p \in \mathrm{Ass}(M)$.*

*Proof.* (1) $\implies$ (2) Use the previous lemma.

(2) $\implies$ (1) By prime avoidance and the previous lemma. ☐

## 1.10 Ext groups

**Lemma 1.10.1.** *If $R$ is a ring and $M$ an $R$-module, then:*

1. *there exists an exact complex*
$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$
*of $R$-modules, with $F_i$ a free $R$-module;*

2. *if $R$ is Noetherian and $M$ is finite, then we can choose the $F_i$ to be finite free.*

*Proof.* For any $R$-module $M$, there is a surjection

$$F_0 \coloneqq \bigoplus_{m \in M} R \to M, \quad I\lambda_m)_{m \in M} \mapsto \sum_m \lambda_m m.$$

Now take $F_1 \to \ker(F_0 \to M)$ to be a surjection by the same method, and so on. □

**Definition 1.10.2.** Such a sequence $\cdots \to F_1 \to F_0 \to M \to 0$ is called a **(free) resolution**.

If $F_\bullet$ is a complex and $\alpha_\bullet \colon F_\bullet \to G_\bullet$ is a morphism of complexes, then there is an induced map $H_i(\alpha_\bullet) \colon H_i(F_\bullet) \to H_i(G_\bullet)$.

**Definition 1.10.3.** We say $\alpha, \beta \colon F_\bullet \to G_\bullet$ are **homotopic** if there is a collection $h_\bullet$ of maps $h_\bullet \colon F_i \to G_{i+1}$, called a **homotopy**, such that
$$\alpha_i - \beta_i = d_G \circ h_i + h_{i-1} \circ d_F.$$

**Lemma 1.10.4.** *If $\alpha_\bullet$ and $\beta_\bullet$ are homotopic, then $H_i(\alpha_\bullet) = H_i(\beta_\bullet)$.*

Goal: show that free resolutions are unique up to homotopy. Then if $\Omega$ from $R$-modules to abelian groups is an additive functor, we can look at $H_i(\Omega(F_\bullet))$.

**Proposition 1.10.5.** *Let $R$ be a ring, $\varphi \colon M \to N$ be a ap of $R$-modules, and $F_\bullet \to M$ be a free resolution. Suppose $G_\bullet \to N$ is a resolution. Then:*

1. *there exists a map of complexes $\alpha \colon F_\bullet \to G_\bullet$ with $H_0(\alpha) = \varphi$;*

2. *if $\alpha, \beta \colon F_\bullet \to G_\bullet$ are two maps with $H_0(\alpha) = H_0(\beta) = \varphi$, then $\alpha, \beta$ are homotopic.*

*Proof.* Using the commutative diagram

$$\begin{array}{ccccccc}
F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \varphi} & & \\
G_1 & \longrightarrow & G_0 & \longrightarrow & N & \longrightarrow & 0,
\end{array}$$

repeatedly use the universal property of free modules. So existence is easy. To show uniqueness up to homotopy, it suffices to show $H_0(\alpha) = 0$ implies $\alpha$ is homotopic to 0. This time we have arrows:

$$\begin{array}{ccccccc}
F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
{\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_0}\downarrow & & \downarrow & & \\
G_1 & \longrightarrow & G_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

and we want maps $h_i \colon F_i \to G_{i+1}$ such that $\alpha_i = d_G \circ h_i + h_{i-1} \circ d_F$. Now do more diagram chasing. □

**Definition 1.10.6.** Let $M, N$ be $R$-modules. View $\mathrm{Hom}_R(M, N)$ as an $R$-module. Define $\mathrm{Ext}^i_R(M, N)$ as follows:

1. pick a free resolution $F_\bullet \to M$;
2. form the cochain complex $\mathrm{Hom}_R^\bullet(F_\bullet, N)$, i.e. the complex whose $n$-th term is $\mathrm{Hom}_R^n(F_\bullet, N) \coloneqq \mathrm{Hom}_R(F_n, N)$;
3. set $\mathrm{Ext}_R^i(M, N) = H^i(\mathrm{Hom}_R^\bullet(F_\bullet, N))$.

**Lemma 1.10.7.** *This is a well-defined bifunctor*

$$\mathsf{Mod}_R^{op} \times \mathsf{Mod}_R \to \mathsf{Mod}_R, \quad (M, N) \mapsto \mathrm{Ext}_R^i(M, N).$$

*Proof.* For functoriality in $M$, suppose we have $\varphi\colon M \to M'$ and free resolutions $F_\bullet \to M$ and $F'_\bullet \to M'$. Then pick $\alpha\colon F_\bullet \to F'_\bullet$ with $H_0(\alpha) = \varphi$. This gives a map of complexes

$$\alpha^t\colon \mathrm{Hom}_R(F'_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N).$$

Hence we get induced maps $H^i(\alpha^t)\colon \mathrm{Ext}_R^i(M', N) \to \mathrm{Ext}_R^i(M, N)$. This map $H^I(\alpha^t)$ is independent of the choice of $\alpha$, since if $\beta$ is another choice, then we know $\alpha$ is homotopic to $\beta$ via a family $h_i\colon F_i \to F'_{i+1}$, so that $H^i(\alpha^t)$ is homotopic to $H^i(\beta^t)$ via the family $H^i(h_i^t)$. □

**Lemma 1.10.8.** $\mathrm{Ext}_R^n(M, N) = 0$ *for* $n < 0$*, and* $\mathrm{Ext}_R^0(M, N) = \mathrm{Hom}_R(M, N)$*.*

*Proof.* Note that $\mathrm{Hom}_R(-, N)$ is left exact, so the sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$$

is exact because $F_1 \to F_0 \to M \to 0$ is exact. □

**Lemma 1.10.9.** *A short exact sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$ *of $R$-modules gives a long exact sequence*

$$0 \to \mathrm{Ext}_R^0(M_3, N) \to \mathrm{Ext}_R^0(M_2, N) \to \mathrm{Ext}_R^0(M_1, N) \to \mathrm{Ext}_R^1(M_3, N) \to \mathrm{Ext}_R^1(M_2, N) \to \cdots.$$

*Proof.* We can find free resolutions

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
F_{1,\bullet} & \longrightarrow & M_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
F_{2,\bullet} & \longrightarrow & M_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
F_{3,\bullet} & \longrightarrow & M_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

such that $0 \to F_{1,\bullet} \to F_{2,\bullet} \to F_{3,\bullet} \to 0$ is a short exact sequence of complexes (i.e. it is exact in every degree). Since $F_{3,\bullet}$ is free, the sequence splits. Then $0 \to \mathrm{Hom}(F_{1,\bullet}, N) \to \mathrm{Hom}(F_{2,\bullet}, N) \to \mathrm{Hom}(F_{3,\bullet}, N) \to 0$ is also (split) exact. Fact from homological algebra: if $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short exact sequence of complexes, then we get a long exact sequence of cohomology

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \xrightarrow{\delta} H^{i+1}(A^\bullet) \to \cdots$$

where the $\delta$ are called **boundary maps** via the snake lemma. □

**Lemma 1.10.10.** *If $0 \to N_1 \to N_2 \to N_3 \to 0$ is a short exact sequence of $R$-modules, ther eis a long exact sequence*

$$0 \to \mathrm{Ext}^0_R(M, N_1) \to \mathrm{Ext}^0_R(M, N_2) \to \mathrm{Ext}^0_R(M, N_3) \to \mathrm{Ext}^1_R(M, N_1) \to \mathrm{Ext}^1_R(M, N_2) \to \cdots$$

*Proof.* Note that $0 \to \mathrm{Hom}(F_\bullet, N_1) \to \mathrm{Hom}(F_\bullet, N_2) \to \mathrm{Hom}(F_\bullet, N_3) \to 0$ is short exact. □

**Lemma 1.10.11.** *If $F$ is a free $R$-module, then $\mathrm{Ext}^i_R(F, N) = 0$ for $i > 0$.*

*Proof.* Choose the resolution $0 \to F \to F \to 0$. □

**Lemma 1.10.12** (Dimension shifting)**.** *If $0 \to K \to F \to M \to 0$ is a short exact sequence with $F$ free (or projective), then*

1. *there is an exact sequence $0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F, N) \to \mathrm{Hom}_R(K, N) \to \mathrm{Ext}^1_R(M, N) \to 0$;*
2. *$\mathrm{Ext}^{i+1}_R(M, N) \cong \mathrm{Ext}^i_R(K, N)$ for $i > 0$.*

*Proof.* Apply the long exact sequence and use the previous lemma. □

**Example 1.10.13.** Let $R$ be a ring, and $f \in R$ a non-zerodivisor. We compute $\mathrm{Ext}^i_R(R/fR, N)$ using the short exact sequence (or resolution) $0 \to R \xrightarrow{f} R \to R/fR \to 0$, which gives via the lemma

$$0 \to \mathrm{Ext}^0_R(R/fR, N) \to N \xrightarrow{f} N \to \mathrm{Ext}^1_R(R/fR, N) \to 0$$

and $\mathrm{Ext}^i_R(R/fR, N) = 0$ for $i > 1$.

**Lemma 1.10.14.** *If $R$ is a ring, $x \in R$, and $M, N$ are $R$-modules, then:*

1. *multiplication by $x$ on $M$ induces multiplication on $\mathrm{Ext}^i_R(M, N)$ via functoriality in the first variable;*
2. *multiplication by $x$ on $N$ induces multiplication on $\mathrm{Ext}^i_R(M, N)$ via functoriality in the second variable.*

## 1.11 Regular sequences

**Definition 1.11.1.** Let $R$ be a ring and $M$ an $R$-module. A sequence $f_1, \ldots, f_r$ of $R$ is called an **$M$-regular sequence** if:

1. $f_i$ is a non-zerodivisor on the module $M/(f_1, \ldots, f_{i-1})$;
2. $M/(f_1, \ldots, f_r)$ is non-zero.

**Example 1.11.2.** Note weirdness: the property depends on the order of the elements. Let $R = k[x, y, z]$. Then

$$(f_1, f_2, f_3) = (x, y(1-x), z(1-x))$$

is regular. (Terminology: If $R$ is the module, we drop the $R$, i.e. **regular** means "$R$-regular".) But

$$(f_1, f_2, f_3) = (y(1-x), z(1-x), x)$$

is not regular! Even worse, we can make a (non-Noetherian) local example:

**Lemma 1.11.3.** *Let $R$ be Noetherian local and $M$ finite. If $x_1, \ldots, x_r$ is $M$-regular, then any permutation of them is still $M$-regular.*

*Proof.* It suffices to prove this for $r = 2$ since any permutation is a composition of transpositions. By hypothesis, $x_1$ is a non-zerodivisor. Set $K = \ker(M \xrightarrow{x_2} M)$. Then the snake lemma applied to the diagram (given by the lower two rows)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{x_1} & K & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{x} & M & \longrightarrow & M/xM & \longrightarrow & 0 \\ & & {\scriptstyle x_2}\downarrow & & {\scriptstyle x_2}\downarrow & & {\scriptstyle x_2}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & M/x_2M & \longrightarrow & 0 \end{array}$$

shows that $K \xrightarrow{x_1} K$ is an isomorphism. Then $x_1K = K$, so that $\mathfrak{m}_R K = K$, i.e. $K = 0$ by Nakayama. Therefore $x_1 \colon M \to M$ is injective. Similarly, looking at cokernels, $M/x_2M \xrightarrow{x_1} M/x_2M$ is injective, i.e. $x_2, x_1$ is $M$-regular. ☐

**Definition 1.11.4.** Let $R$ be a ring and $I$ an ideal and $M$ finite. The ***I*-depth** of $M$, denoted $\operatorname{depth}_I(M)$, is:

1. if $IM \neq M$, then it is the maximum length of $M$-regular sequences of elements of $I$;
2. if $IM = M$, then define it to be $\infty$.

If $(R, \mathfrak{m})$ is local, we write $\operatorname{depth}(M) = \operatorname{depth}_{\mathfrak{m}}(M)$.

Key fact: in the situation of the definition, if $f \in I$ is a non-zerodivisor on $M$, then $\operatorname{depth}_I(M/fM) = \operatorname{depth}_I(M) - 1$. We will prove this in the local case using Ext.

**Lemma 1.11.5.** *Let $R$ be a ring, $I \subset R$ an ideal, and $M$ a finite $R$-module.q Then $\operatorname{depth}_I(M)$ is equal to the supremum of lengths of sequences $f_1, \dots, f_r \in I$ such that $f_i$ is a non-zerodivisor on $M/(f_1, \cdots, f_{i-1})M$ for $i = 1, \dots, r$.*

*Remark.* If $M = 0$, then $1, 1, \dots$ is an infinite sequence of non-zerodivisors, so indeed $\operatorname{depth}_I(M) = \infty$.

**Lemma 1.11.6.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $M \neq 0$ a finite $R$-module. Then $\dim(\operatorname{supp}(M)) \geq \operatorname{depth}(M)$.*

*Proof.* Proof by induction on $\dim(\operatorname{supp}(M))$. The base case is $\dim(\operatorname{supp}(M)) = 0$, so that $\operatorname{supp}(M) = \{\mathfrak{m}\}$. Then $\operatorname{Ass}(M) = \{\mathfrak{m}\}$, so there cannot be a non-zerodivisor in $\mathfrak{m}$; there must be an element $x \in M$ killed by everything in $\mathfrak{m}$. Hence $\operatorname{depth}(M) = 0$ because there are no non-zerodivisors.

(Induction step) Assume $\dim(\operatorname{supp}(M)) > 0$, and let $f_1, \dots, f_d$ be a sequence of elements in $M$ such that $f_i$ is a non-zerodivisor on $M/(f_1, \dots, f_{i-1})$. We must show $\dim(\operatorname{supp}(M)) \geq d$. By a previous lemma, $\dim(\operatorname{supp}(M/f_1M)) = \dim(\operatorname{supp}(M)) - 1$, since $f_1$ a non-zerodivisor implies $f_1 \notin p$ for every $p \in \operatorname{Ass}(M)$, which implies $f_1$ is not a minimal prime in $\operatorname{supp}(M)$ by 1.9.6, which means $\dim(\operatorname{supp}(M/f_1M)) = \dim(\operatorname{supp}(M) \cap V(f_1)) < \dim(\operatorname{supp}(M))$. By induction, $d - 1 \leq \dim(\operatorname{supp}(M/f_1M))$ since $f_2, \dots, f_d$ is still a regular sequence in $M/f_1M$, so $d \leq \dim(\operatorname{supp}(M))$. ☐

**Lemma 1.11.7.** *Let $R$ be Noetherian, $I \subset R$ an ideal, and $M$ a finite $R$-module such that $IM \neq M$. Then $\operatorname{depth}_I(M) < \infty$, allowing us to do induction on depth.*

*Proof.* Idea: the previous lemma says that in the Noetherian local case, $\operatorname{depth}(M) \leq \dim(\operatorname{supp}(M))$. So it suffices in the non-local case to localize at certain primes. ☐

**Lemma 1.11.8.** *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring, and $M$ a finite $R$-module. Then*

$$\operatorname{depth}(M) = \min\{i \in \mathbb{Z} : \operatorname{Ext}^i_R(\kappa, M) \neq 0\}.$$

*Proof.* Call the minimal integer $i(M)$. Suppose $i(M) = 0$. Then $\mathrm{Hom}_R(\kappa, M) \neq 0$, i.e. the image of $1 \in \kappa$ is killed by the maximal ideal since it is a copy of $R/\mathfrak{m}$ sitting inside $M$. Hence $\mathfrak{m} \in \mathrm{Ass}(M)$ and $\mathrm{depth}(M) = 0$. Conversely, if $\mathrm{depth}(M) = 0$. Then every $f \in \mathfrak{m}$ is a zero-divisor, so by prime avoidance, $\mathfrak{m} \in \mathrm{Ass}(M)$. Hence $i(M) = 0$. Essentially:

$$i(M) = 0 \iff \mathrm{Hom}_R(\kappa, M) = 0 \iff \mathfrak{m} \in \mathrm{Ass}(M) \iff \mathrm{depth}(M) = 0.$$

Assume now that $i(M), \mathrm{depth}(M) > 0$. There exists a non-zerodivisor $f \in \mathfrak{m}$ such that $\mathrm{depth}(M/fM) = \mathrm{depth}(M) - 1$, by picking a maximal $M$-regular sequence $f_1, \ldots, f_{\mathrm{depth}(M)}$ and set $f = f_1$. Then the short exact sequence

$$0 \to M \xrightarrow{\cdot f} M \to M/fM \to 0$$

gives a long exact sequence of cohomology

$$0 \to \mathrm{Ext}^0_R(\kappa, M) \xrightarrow{\cdot f} \mathrm{Ext}^0_R(\kappa, M) \to \mathrm{Ext}^0_R(\kappa, M/fM) \to \mathrm{Ext}^1_R(\kappa, M) \xrightarrow{\cdot f} \mathrm{Ext}^1_R(\kappa, M) \to \cdots.$$

Using a previous lemma, these multiplication maps by $f$, which originally came from $M$, also comes from $\kappa$. But multiplication on $\kappa$ is zero (by choice of $f$), so all the maps $\mathrm{Ext}^i_R(\kappa, M) \xrightarrow{\cdot f} \mathrm{Ext}^i_R(\kappa, M)$ are zero maps. Hence we get short exact sequences

$$0 \to \mathrm{Ext}^i_R(\kappa, M) \to \mathrm{Ext}^i_R(\kappa, M/fM) \to \mathrm{Ext}^{i+1}_R(\kappa, M) \to 0.$$

If $\mathrm{Ext}^{i+1}_R(\kappa, M)$ is the smallest non-zero Ext, then $\mathrm{Ext}^i_R(\kappa, M) = 0$ and therefore $\mathrm{Ext}^i_R(\kappa, M/fM) \neq 0$. Hence $i(M) - 1 = i(M/fM)$. By induction, $i(M/fM) = \mathrm{depth}(M/fM) = \mathrm{depth}(M) - 1$. □

**Lemma 1.11.9.** *Let $(R, \mathfrak{m}, \kappa)$ be a local Noetherian ring, and $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of finite R-modules. Then:*

1. $\mathrm{depth}(N) \geq \min\{\mathrm{depth}(N'), \mathrm{depth}(N'')\}$*;*
2. $\mathrm{depth}(N'') \geq \min\{\mathrm{depth}(N), \mathrm{depth}(N') - 1\}$*;*
3. $\mathrm{depth}(N') \geq \min\{\mathrm{depth}(N), \mathrm{depth}(N'') + 1\}$*;*

*Proof.* Write down the long exact sequence of Ext coming from the short exact sequence. Then, for example, $\mathrm{Ext}^i_R(\kappa, N)$ is sandwiched between $\mathrm{Ext}^i_R(\kappa, N')$ and $\mathrm{Ext}^i_R(\kappa, N'')$, so one of them has to be non-zero in order for $\mathrm{Ext}^i_R(\kappa, N)$ to be non-zero. This proves (1). □

**Lemma 1.11.10.** *Let $R$ be local Noetherian, $M$ a non-zero finite $R$-module.*

1. *If $x \in \mathfrak{m}$ is a non-zerodivisor on $M$, then* $\mathrm{depth}(M/xM) = \mathrm{depth}(M) - 1$*.*
2. *Any $M$-regular sequence $x_1, \ldots, x_r \in \mathfrak{m}$ can be extended to a maximal one.*

*Proof.* Apply the previous lemma to the short exact sequence $0 \to M \xrightarrow{\cdot x} M \to M/xM \to 0$. Then note that $\mathrm{depth}(M/xM) < \mathrm{depth}(M)$ because we can always lift regular sequences in $M/xM$ and prepend $x$. The second claim follows from induction using the first. □

*Remark.* We actually already proved that there exists some $f \in \mathfrak{m}$ satisfying this lemma; now we've showed every $x \in \mathfrak{m}$ satisfies the lemma.

**Lemma 1.11.11.** *Let $(R, \mathfrak{m})$ be local Noetherian, and $M$ a finite $R$-module. Pick $x \in \mathfrak{m}$ and $p \in \mathrm{Ass}(M)$ and $q$ minimal over $(x) + p$. Then $q \in \mathrm{Ass}(M/x^n M)$ for some $n$.*

*Proof.* Pick $N \subset M$ such that $N = R/p$. By Artin–Rees, $N \cap x^n M \subset xN$ for some $n \geq 1$. Let $\bar{N} \subset M/x^n M$ be the image of $N$. It is enough to show that $q \in \mathrm{Ass}(\bar{N})$. By construction, there is a surjection $\bar{N} \to M/xN = R/((x) + p)$. Then $q$ is in the support of $R/((x) + p)$, so it is in the support of $\bar{N}$. On the other hand, $x^n$ and $p$ kill $\bar{N}$, so $\mathrm{supp}(\bar{N}) \subset V((x^n) + p) = V((x) + p)$. So $q$ is minimal in $\mathrm{supp}(\bar{N})$, i.e. it is minimal in $\mathrm{Ass}(\bar{N})$. (This has to do with embedded primes.) □

*Remark.* We say a prime is **embedded** if it is an associated prime but is not minimal in the support of $M$. (This will be important later on in the context of Cohen–Macaulay modules.)

**Lemma 1.11.12.** *Let $(R, \mathfrak{m})$ be local Noetherian, and $M$ a finite $R$-module. Then $p \in \mathrm{Ass}(M)$ gives* $\mathrm{depth}(M) \le \dim(R/p)$.

*Proof.* Use induction on $\mathrm{depth}\, M$. We skip the base case. Assume $\mathrm{depth}\, M \ge 1$. Pick $x \in M$ a non-zerodivisor (i.e. $x$ is not in any associated prime of $M$), and $q$ minimal over $(x) + p$ with $\dim R/q = \dim R/p - 1$. By the previous lemma, $q \in \mathrm{Ass}(M/x^n M)$ for some $n \ge 1$. By the induction hypothesis, $\mathrm{depth}(M) - 1 = \dim M/x^n M \le \dim(R/q) = \dim(R/p) - 1$ where in the first equality we used that $x^n$ is a non-zerodivisor. ☐

## 1.12 Cohen–Macaulay modules

**Definition 1.12.1.** Let $R$ be a Noetherian local ring, and $M$ a finite $R$-module. We say $M$ is **Cohen–Macaulay (CM)** if $\mathrm{depth}\, M = \dim \mathrm{supp}\, M$.

*Remark.* Suppose we defined property $P$ for $(R, M)$ where $R$ is local Noetherian and $M$ an $R$-module. We would like to say $(R, M)$ has property $P$ for $R$ non-local Noetherian iff $(R_p, M_p)$ has $P$ for all $p \in \mathrm{Spec}\, R$. But then we must sanity check that $(R_p, M_p)$ has $P$ in the local Noetherian case too!

**Lemma 1.12.2.** *Let $(R, \mathfrak{m})$ be local Noetherian, $M$ finite over $R$, and $x \in \mathfrak{m}$ a non-zerodivisor on $M$. Then $M$ is CM iff $M/xM$ is CM.*

*Proof.* We proved lemmas that show both sides of $\dim \mathrm{supp}\, M = \mathrm{depth}\, M$ drop by exactly 1 when we mod out by $x$. ☐

**Lemma 1.12.3.** *Let $R \to S$ be a surjective (local) homomorphism of local Noetherian rings, and $M$ be a finite $S$-module. Then $M$ is CM over $S$ iff $M$ is CM over $R$.*

**Lemma 1.12.4** (Unmixedness)**.** *Let $(R, \mathfrak{m})$ be local Noetherian, and $M$ finite CM. If $p \in \mathrm{Ass}(M)$, then* $\dim(R/p) = \dim \mathrm{supp}\, M$*, and $p$ is a minimal prime in* $\mathrm{supp}\, M$.

*Proof.* We have $\mathrm{depth}\, M \le \dim R/p$ from a previous lemma. But $\dim R/p \le \dim \mathrm{supp}\, M$, and $\dim \mathrm{supp}\, M = \mathrm{depth}\, M$ by the CM property. We can't have a smaller prime because then $\dim R/p$ would be bigger. ☐

**Lemma 1.12.5.** *Let $(R, \mathfrak{m})$ be local Noetherian. Assume there exists a finite CM module $M$ over $R$ with* $\mathrm{supp}\, M = \mathrm{Spec}\, R$. *Then any maximal chain of primes $p_0 \subset \cdots \subset p_n$ has length $n = \dim R$.*

*Proof.* Induct on $\dim R$. If $\dim R = 0$, it is clear. Assume $\dim R > 0$, so that $n > 0$. Using prime avoidance, choose $x \in p_1$ such that $x$ is not in any of the minimal primes of $R$. In particular, $x \notin p_0$. Then $\dim R/xR = \dim R - 1$. By previous lemmas, $M/xM$ is CM over $R/xR$. (Outline: $x$ is a non-zerodivisor because $x$ is not in any associated prime, because those are precisely the minimal primes of $R$ by the unmixedness lemma and the fact that $\mathrm{supp}\, M = \mathrm{Spec}\, R$.) Then

$$\mathrm{supp}(M/xM) = \mathrm{supp}(M) \cap V(x) = \mathrm{Spec}(R) \cap V(x) = \mathrm{Spec}(R/xR).$$

Because our chain $p_0 \subset \cdots \subset p_n$ is maximal, we know $p_1$ is minimal over $(x) + p_0$ since $x \notin p_0$. Then $p_1 \in \mathrm{Ass}(M/x^n M)$ for some $n \ge 1$. We can replace $x$ with $x^n$ and get $p_1 \in \mathrm{Ass}(M/xM)$. Since $\mathrm{supp}\, M/xM = \mathrm{Spec}\, R/xR$, we get $p_1$ is minimal in $R/xR$ (by unmixedness). It follows that $p_1/(x) \in \mathrm{Ass}_{R/xR}(M/xM)$. By a previous lemma, $p_1/(x)$ is a minimal prime of $R/xR$.

Now consider the chain $p_1/(x) \subset \cdots \subset p_n/(x)$. It is a maximal chain in $R/xR$ since $p_1/(x)$ is minimal. By the induction hypothesis, we get $n - 1 = \dim R/xR = \dim R - 1$. ☐

**Example 1.12.6.** Let $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ and $M = R$. We know $\dim R = n$, and $x_1, \ldots, x_n$ is an $R$-regular sequence. So $R$ is CM over $R$. (We actually want to show this for the whole ring $k[x_1, \ldots, x_n]$, i.e. when we localize at any maximal ideal.)

**Corollary 1.12.7.** *Let $(R, \mathfrak{m})$ be local Noetherian. Assume there exists a CM module $M$ with $\operatorname{Spec} R = \operatorname{supp} M$. Then for all $p \in \operatorname{Spec} R$, we have $\dim(R) = \dim(R_p) + \dim R/p$.*

**Example 1.12.8.** Suppose we have a ring $R$ and an $R$-module $M$. Then we can make a new ring $R \oplus M$ given by $(r, m) \cdot (r', m') = (rr', rm' + r'm)$. Then $M$ is an ideal of square 0 in $R \oplus M$. We can take $M$ to be CM, but as an $(R \oplus M)$-module.

Explicitly, $k[x, y]_{(x,y)} \oplus k$ has depth 0 and $k[x, y]_{(x,y)}$ is a CM module over it, with support the whole ring. What went wrong is we took a well-behaved space $k[x, y]_{(x,y)}$ and literally just added an embedded prime, destroying CM-ness.

We have two goals:

1. if $(R, \mathfrak{m})$ is local Noetherian and $M$ is finite CM, then we want to show that $M_p$ is CM over $R_p$ for every $p \in \operatorname{Spec} R$ (this will allow us to define CM-ness for an arbitrary finite $M$ over a Noetherian $R$);
2. if $R$ is Noetherian and $M$ is finite CM, then $M \otimes_R R[x_1, \ldots, x_n]$ is CM over $R[x_1, \ldots, x_n]$.

*Remark.* If $P$ is a property of (Noetherian) rings, then we say "$R$ is **universally** $P$" if any finite-type $R$-algebra has $P$. For example, no ring is universally CM; that makes no sense. However it will turn out that CM rings are universally catenary.

**Lemma 1.12.9.** *Let $(R, \mathfrak{m})$ be local Noetherian and $M$ finite over $R$ and CM. For any prime $p \subset R$, we have $M_p$ is CM over $R_p$.*

*Proof.* By a simple induction argument. Reduce to the case where $p \subsetneq \mathfrak{m}$ and there is no prime strictly in between. If $M_p = 0$, then there is nothing to prove, so assume $M_p \neq 0$. Then $\dim \operatorname{supp} M_p \leq \dim \operatorname{supp} M - 1$. So it is enough to show $\operatorname{depth} M_p \geq \operatorname{depth} M - 1$. By induction on $\operatorname{depth} M$, we will show there exists an $M$-regular sequence $f_1, \ldots, f_{\operatorname{depth}(M)-1} \in p$. Since localization is exact, this $M$-regular sequence maps to an $M_p$-regular sequence in $pR_p$, and so we are done.

The base case is trivial. So assume $\operatorname{depth}(M) \geq 2$. Let $I = \operatorname{Ann}(M)$, so $V(I) = \operatorname{supp} M$. Since $M$ is CM, every irreducible component in $V(I)$ has the same depth. So every maximal chain in $V(I)$ has length $\geq 2$. Therefore $p$ is not contained in any associated prime of $M$, because $\operatorname{Ass}(M)$ consists of minimal primes in $R/I$. By prime avoidance, we can find $f_1 \in p$ and $f_1$ not in any associated prime. Hence $f_1$ is a nonzerodivisor on $M$ and is the first element of our $M$-regular sequence. Then $\operatorname{depth}(M/f_1M) = \operatorname{depth}(M) - 1$ and we are done by induction. ☐

**Definition 1.12.10.** Let $R$ be Noetherian and $M$ finite over $R$. We say $M$ is **Cohen–Macaulay** iff $M_p$ is CM over $R_p$ for all $p \in \operatorname{Spec} R$.

**Lemma 1.12.11.** *Let $A$ be a local ring, and $\mathfrak{m} \subset A[x]$ a maximal ideal such that $A \cap \mathfrak{m}$ is the maximal ideal of $A$. Then there exists $f \in \mathfrak{m}$ monic.*

*Proof.* Let $\kappa := A/\mathfrak{m}_A$ be the residue field of $A$. Then $\mathfrak{m}_A A[x] \subset \mathfrak{m}$, and $\mathfrak{m}_A A[x] = \ker(A[x] \to \kappa[x])$. So $\mathfrak{m}$ is the inverse image of some maximal ideal $\bar{\mathfrak{m}} \subset \kappa[x]$. By the structure of ideals in $\kappa[x]$, we can pick a monic $x^e + \bar{a}_1 x^{e-1} + \cdots + \bar{a}_e \in \bar{\mathfrak{m}}$. Then choose any lift $f \in \mathfrak{m}$ back to $\mathfrak{m}$. ☐

**Lemma 1.12.12.** *Let $R$ be Noetherian and $M$ finite CM over $R$. Then $M \otimes_R R[x_1, \ldots, x_n]$ is CM over $R[x_1, \ldots, x_n]$.*

*Remark.* Since Johan doesn't want to spend time explaining tensor product, we will prove the following corollary instead. It is the one that is commonly used.

**Lemma 1.12.13.** *Let $R$ be Noetherian and CM. Then $R[x_1, \ldots, x_n]$ is CM.*

*Proof.* It suffices to prove the $n = 1$ case. Let $\mathfrak{m} \subset R[x]$ be a maximal ideal. It is enough to show $R[x]_\mathfrak{m}$ is CM. Let $p = R \cap \mathfrak{m}$ and $f_1, \ldots, f_d \in pR_p$ be a regular sequence of length $d = \dim R_p$ (since $R$ is CM). Then $f_1, \ldots, f_d$ is still a regular sequence in $R_p[x]$, with final quotient $R_p[x]/(f_1, \ldots, f_d) = (R_p/(f_1, \ldots, f_d))[x]$. We know $\mathrm{supp}(R_p/(f_1, \ldots, f_d)) = \{pR_p\}$ (since the support is zero-dimensional, and $R_p$ is local). Hence $\mathrm{supp}(R_p[x]/(f_1, \ldots, f_d))$ consists of all primes in $R_p[x]$ lying over $p$.

By the previous lemma, pick $f = x^e + a_1 x^{e-1} + \cdots + a_e \in \mathfrak{m}$ with $a_i \in R_p$. Then look at

$$R_p/(f_1, \ldots, f_d)[x] \xrightarrow{\cdot f} (R_p/(f_1, \ldots, f_d))[x], \quad \bigoplus_{n \geq 0} (-)x^n \to \bigoplus_{n \geq 0} (-)x^n,$$

which is injective. Since localization is exact, $f_1, \ldots, f_d, f$ forms a regular sequence in $R[x]_\mathfrak{m}$ and moreover,

$$\mathrm{supp}(R[x]_\mathfrak{m}/(f_1, \ldots, f_d, f)) = \{\mathfrak{m}R[x]_\mathfrak{m}\}. \qquad \square$$

*Remark.* Read: section on Cohen–Macaulay rings excluding the last 3 lemmas.

## 1.13 Catenary rings

**Definition 1.13.1.** Let $R$ be a Noetherian ring. Then $R$ is **catenary** iff for any $p \subset q$, every maximal chain of primes $p = p_0 \subset p_1 \subset \cdots \subset p_e = q$ has the same length. Equivalently, for every $p \subset q \subset r$, we have $\mathrm{height}(r/p) = \mathrm{height}(q/p) + \mathrm{height}(r/q)$. (Recall that $\mathrm{height}(p) \coloneqq \dim(R_p)$.) It is **universally catenary** if every finite type $R$-algebra is catenary.

**Lemma 1.13.2.** *Properties of being catenary:*

1. *$R$ is catenary iff $R_\mathfrak{m}$ is catenary for all maximal primes $\mathfrak{m}$;*
2. *$R$ is catenary iff $R/p$ is catenary for all minimal primes $p$;*
3. *$R$ catenary implies $S^{-1}R$ and $R/I$ catenary.*

*Remark.* Catenary does not imply "equidimensional." For example, we can have one irreducible component of dimension 2 and another of dimension 1. The lack of "loops" in the poset $\mathrm{Spec}\, R$ ensures we are still catenary.

**Definition 1.13.3.** Let $X$ be a topological space, and $Y \subset X$ be an irreducible subset. The **codimension** $\mathrm{codim}(Y, X)$ of $Y$ in $X$ is the supremum of lengths of chains $Y = Y_0 \subset \cdots \subset Y_n = X$ of irreducible closed subsets.

**Example 1.13.4.** If $X = \mathrm{Spec}\, R$, then irreducible loci in $X$ correspond to prime ideals. Then

$$\mathrm{codim}(V(p), \mathrm{Spec}\, R) = \sup\{n : \exists p = p_0 \supset p_1 \supset \cdots \supset p_n\} = \dim(R_p) = \mathrm{height}(p).$$

**Definition 1.13.5.** A topological space $X$ is **catenary** iff for all $Y, Z \subset X$ irreducible closed, we have

1. $\mathrm{codim}(Y, Z) < \infty$ (so that there are maximal chains);
2. every maximal chain $Y = Y_0 \subset \cdots \subset Y_n = Z$ of irreducible closed subsets has the same length.

**Lemma 1.13.6.** *$R$ is catenary iff* $\mathrm{Spec}\, R$ *is catenary.*

| $R$ ring | $X$ space |
|---|---|
| $R$ catenary $\implies$ $R/I$ catenary | $X$ catenary $\implies$ any closed subset is catenary |
| $R$ catenary $\implies$ $S^{-1}R$ catenary | $X$ catenary $\implies$ any open subset is catenary. |

**Lemma 1.13.7.** *Let $R$ be a ring. The following are equivalent:*

1. *$R$ is (universally) catenary;*

2. *$R/p$ is (universally) catenary for every minimal prime $p$.*

3. *$R_{\mathfrak{m}}$ is (universally) catenary for every maximal prime $\mathfrak{m}$.*

**Theorem 1.13.8.** *If $R$ is a Noetherian CM ring, then $R$ is universally catenary.*

**Example 1.13.9.** Important examples include $R = k$ a field, and $R = \mathbb{Z}$ the integers. In fact it turns out $R$ Noetherian and $\dim R \leq 1$ is CM, and therefore universally catenary. If $R$ is CM, then (Johan thinks) $R[[x_1, \ldots, x_n]]$ is CM. (It is certainly true for $R = k$ a field.)

*Proof.* Note that $R$ is universally catenary iff $R[x_1, \ldots, x_n]$ is catenary for all $n$. We saw that $R$ CM implies $R[x_1, \ldots, x_n]$ CM. Hence it is enough to show $R$ CM implies $R$ catenary. In particular, by the preceding lemma, it is enough to show $R$ local CM implies $R$ catenary. We saw that in a Noetherian CM local ring $R$, every maximal chain of primes has the same length. This implies $R$ is catenary. □

**Example 1.13.10** (Nagata)**.** There is a Noetherian local domain of dimension 3 which is not catenary. Idea: find a Noetherian ring $R$ with exactly two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ with $\dim(R_{\mathfrak{m}_1}) = 2$ and $\dim(R_{\mathfrak{m}_2}) = 3$ such that there is an isomorphism of residue fields $\alpha\colon R/\mathfrak{m}_1 \to R/\mathfrak{m}_2$. Then set

$$R' \coloneqq \{f \in R : \alpha(f \bmod \mathfrak{m}_1) = f \bmod \mathfrak{m}_2\}.$$

We can show that:

1. $R'$ is a local domain with maximal ideal $\mathfrak{m} = R' \cap \mathfrak{m}_1 = R' \cap \mathfrak{m}_2$;

2. $R'$ is Noetherian (tricky; use Eakin's theorem);

3. $\operatorname{Spec} R'$ is $\operatorname{Spec} R$ with $\mathfrak{m}_1$ and $\mathfrak{m}_2$ identified.

Choose chains $(0) \subset p \subset \mathfrak{m}_1$ and $(0) \subset p' \subset p'' \subset \mathfrak{m}_2$ in $R$. Then in $R'$ we have two chains of different length from $(0)$ to $R' \cap \mathfrak{m}_1 = R' \cap \mathfrak{m}_2$, and therefore $R'$ is not catenary.

**Definition 1.13.11.** Let $X$ be a topological space. We say $X$ is **sober** iff every irreducible closed subset has a unique generic point. (For example, every spectrum is sober.) Assume $X$ is sober, and $x, y \in X$ with $x \rightsquigarrow y$, i.e. $x$ specializes to $y$. We say this is an **immediate specialization** if $x \neq y$ and there is no $z \in X \setminus \{x, y\}$ such that $x \rightsquigarrow z \rightsquigarrow y$. A **dimension function** is a map $\delta\colon X \to \mathbb{Z}$ such that:

1. if $x \rightsquigarrow y$ and $x \neq y$, then $\delta(x) > \delta(y)$;

2. if $x \rightsquigarrow y$ is immediate, then $\delta(x) = \delta(y) + 1$.

*Remark.* Dimension functions do not exist for non-catenary rings.

**Example 1.13.12.** If $R$ is Noetherian local catenary, then $p \mapsto \dim(R/p)$ is a dimension function.

**Lemma 1.13.13.** *Let $X$ be sober and $\delta$ a dimension function. Then $X$ is catenary and for all $x \rightsquigarrow y$ in $X$,*

$$\operatorname{codim}(\overline{\{y\}}, \overline{\{x\}}) = \delta(x) - \delta(y).$$

**Proposition 1.13.14.** *Facts about dimension functions:*

1. *any two dimension functions $\delta, \delta'$ on a connected Noetherian sober topological space differ by a constant;*

2. *if $X$ is a Noetherian sober catenary topological space, then dimension functions exist locally (i.e. around every point there is an open subset with a dimension function).*

The goal of the next few lectures is to use transcendence degree trdeg to construct dimension functions. Another goal is to show that the map $\operatorname{Spec}(k[x_1, \ldots, x_n]) \to bZ$ given by $p \mapsto \operatorname{trdeg}_k \kappa(p)$ is a dimension function (where $\kappa(p)$ is the residue field $\operatorname{Frac}(R/p)$).

## 1.14 The dimension formula

**Definition 1.14.1.** Let $K/k$ be a field extension. Say $x_i \in K$ for $i \in I$ is **algebraically independent** over $k$ if there is no non-zero polynomial in $k[x_i : i \in I]$ such that $p(x_i) = 0$. A **transcendence basis** of $K/k$ is a maximal set $\{x_i\}_{i\in I}$ of algebraically independent elements in $K$. Equivalently, $\{x_i\}_{i\in I}$ is algebraically independent, and $K/k(x_i : i \in I)$ is algebraic. The **transcendence degree** $\operatorname{trdeg}_k(K)$ is the cardinality of a transcendence basis.

**Lemma 1.14.2.** *If $L/K$ and $K/k$ are field extensions, then*

$$\operatorname{trdeg}_k(L) = \operatorname{trdeg}_K(L) + \operatorname{trdeg}_k(K).$$

**Lemma 1.14.3.** *If $K/k$ is finitely generated as a field, then $\operatorname{trdeg}_k(K) < \infty$.*

**Theorem 1.14.4.** *Let $R \to S$ be a ring, and $q \in \operatorname{Spec} S$ be a prime lying over $p \in \operatorname{Spec} R$. Assume $R$ is Noetherian, $R$ and $S$ are domains, $S$ is of finite type, and $R \subset S$, i.e. the ring map is injective. Then*

$$\operatorname{height}(q) \leq \operatorname{height}(p) + \operatorname{trdeg}_R(S) - \operatorname{trdeg}_{\kappa(p)} \kappa(q)$$

*with equality if $R$ is universally catenary.*

*Remark.* Here, $\operatorname{trdeg}_R(S)$ is the transcendence degree of the fraction field $\operatorname{Frac}(S)$ over the fraction field $\operatorname{Frac}(R)$.

**Lemma 1.14.5.** *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local Noetherian rings. Then $\dim(S) \leq \dim(R) + \dim(S/\mathfrak{m}S)$.*

*Proof.* Pick $f_1, \ldots, f_{\dim R}$ in $\mathfrak{m}_R$ a minimal set of generators for an ideal of definition in $R$. Pick elements $\bar{g}_1, \ldots, \bar{g}_{\dim(S/\mathfrak{m}S)} \in \mathfrak{m}_S/\mathfrak{m}S$ generating an ideal of definition in $S/\mathfrak{m}S$. Find $g_i \in \mathfrak{m}_S$ lifting $\bar{g}_i$. Then $f_1, \ldots, f_{\dim(R)}, g_1, \ldots, g_{\dim S/\mathfrak{m}S}$ generate an ideal of definition in $S$, since

$$\{\mathfrak{m}\} = V(f_1, \ldots, f_{\dim R}) \implies V(f_1 S + \cdots + f_{\dim R} S) = V(\mathfrak{m}S) = \operatorname{Spec}(S/\mathfrak{m}S)$$

and $V(g_1, \ldots, g_{\dim(S/\mathfrak{m}S)}) \cap V(\mathfrak{m}S) = V(\bar{g}_1, \ldots, \bar{g}_{\dim S/\mathfrak{m}S})$. □

*Remark.* Equality holds, as in the following lemma, whenever $R \to S$ is a flat local homomorphism of local Noetherian rings.

**Lemma 1.14.6.** *Let $(R, \mathfrak{m})$ be local Noetherian, $q \in \operatorname{Spec} R[x]$ a prime ideal lying over $\mathfrak{m}$. With $S = R[x]_q$, we have*

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S).$$

*Proof.* We already have the inequality. Since $\mathfrak{m}R[x] \subset q$, either $q = \mathfrak{m}R[x]$ or $q = (\mathfrak{m}, f)$ where $f \in R[x]$ maps to an irreducible polynomial $\bar{f}$ in $\kappa(\mathfrak{m})[x]$ (i.e. either it is a generic point or it is a closed point of the fiber in $\operatorname{Spec} R[x]$ over $\mathfrak{m}$).

1. If $q = \mathfrak{m}R[x]$, then $S/\mathfrak{m}S = \kappa(\mathfrak{m})(x)$ is a field. Pick $p_0 \subset \cdots \subset p_{\dim R} = \mathfrak{m}$ a maximal chain in $R$, and lift. Then
$$p_0 R[x] \subset p_1 R[x] \subset \cdots \subset p_{\dim R} R[x] = \mathfrak{m}R[x]$$
is a chain of primes in $S$, and therefore $\dim(S) \geq \dim(R)$.

2. If $q = (\mathfrak{m}, f)$, then $S/\mathfrak{m}S$ is the local ring of $\kappa(\mathfrak{m})[x]$ at $(\bar{f})$, and therefore $\dim(S/\mathfrak{m}S) = 1$. By the same argument as above, $\dim S \geq \dim R + 1$. □

*Proof of dimension formula.* Use induction on the number of generators of $S$ as an $R$-algebra. Using the additivity of transcendence degree, we can reduce to the case where $S$ is generated by one element over $R$. With more than one generator, split $R \subset S$ into $R \subset S' \subset S$ with $q$ lying above $q'$ lying above $p$ such that $R \subset S'$ and $S' \subset S$ both have fewer generators. Then

$$\operatorname{height}(q) \leq \operatorname{height}(q') + \operatorname{trdeg}_R S' - \operatorname{trdeg}_{\kappa(R)} \kappa(S'), \quad \operatorname{height}(q') \leq \operatorname{height}(p) + \operatorname{trdeg}_{S'} S - \operatorname{trdeg}_{\kappa(S')} \kappa(S).$$

So the only case we need to prove is $S = R[x]/I$ for some ideal $I$.

1. If $I = 0$, then $S = R[x]$, and $\operatorname{height}(q) = \dim(R[x]_q)$. Since $x$ is transcendental over $R$, we have $\operatorname{trdeg}_R(S) = 1$. By localizing the preceding lemmas at $p$ in $R$, i.e. to $R_p \to R_p[x]$, we get $\dim(R[x]_q) = \dim(R_p) + \dim(R[x]_q/pR[x]_q)$. But the second dimension is either 0 or 1, depending on whether $q$ is a closed point or a generic point in the fiber. We know $\operatorname{trdeg}_R R[x] = 1$, and $\operatorname{trdeg}_{\kappa(p)} \kappa(q)$ is either 1 or 0, from the proof of the preceding lemmas. Hence we are done.

2. If $I \neq 0$, then $x$ is no longer transcendental over $R$, so $\operatorname{trdeg}_R(S) = 0$. Also, $S_q = R[x]_{\tilde{q}}/I_{\tilde{q}}$ where $\tilde{q}$ is the inverse image of $q$ in $R[x]$. Since $I \neq 0$, we know $I_{\tilde{q}} \neq 0$, and therefore $\dim S_q < \dim R[x]_{\tilde{q}}$. By case (i), we have the dimension formula for $R[x]_{\tilde{q}}$:

   $$\dim R[x]_{\tilde{q}} = \dim(R_p) + 1 - \operatorname{trdeg}_{\kappa(p)}(\kappa(q)).$$

   Hence we have that

   $$\dim S_q \leq \dim(R_p) + 0 - \operatorname{trdeg}_{\kappa(p)}(\kappa(q)).$$

It remains to show equality holds when $R$ is universally catenary. We only need this in the second case, because the first case is already an equality. Since $S$ is a domain, $I$ is a prime ideal. We know $\operatorname{height}(I) = 1$ because it lies over $(0)$ and corresponds to a prime in $\operatorname{Frac}(R)[x]$. Since $R$ is universally catenary, all chains in $S_q = R[x]_{\tilde{q}}/I_{\tilde{q}}$ have the same length, and these chains correspond to chains in $\dim(R[x]_{\tilde{q}})$ with length 1 or greater. Hence $\dim S_q = \dim(R[x]_{\tilde{q}}) - 1$, and we are done. $\square$

*Remark.* Actually, the dimension formula holds with equality if and only if $R$ is universally catenary.

**Theorem 1.14.7.** *Let $R$ be a universally catenary ring and $S$ be a finite-type $R$-algebra. Assume that $\delta\colon \operatorname{Spec}(R) \to \mathbb{Z}$ is a dimension function. Then*

$$\delta_S\colon \operatorname{Spec}(S) \to \mathbb{Z}, \quad q \mapsto \delta(p) + \operatorname{trdeg}_{\kappa(p)} \kappa(q),$$

*where $p = q \cap R$, is a dimension function on* $\operatorname{Spec}(S)$.

*Proof.* A situation of the form

$$\begin{array}{ccccc} S & & q' & \longleftarrow & q \\ \uparrow & & \downarrow & & \downarrow \\ R & & p' & \longleftarrow & p \end{array}$$

gives by the formula that

$$\delta_S(q) - \delta_S(q') = \delta(p) - \delta(p') + \operatorname{trdeg}_{\kappa(p)}(\kappa(q)) - \operatorname{trdeg}_{\kappa(p')}(\kappa(q')).$$

The first two terms on the right hand side give $\dim((R/p)_{p'})$. Applying the dimension formula to the diagram gives

$$\dim((S/q)_{q'}) = \dim((R/p)_{p'}) + \operatorname{trdeg}_{R/p}(S/q) - \operatorname{trdeg}_{\kappa(q'/q)}(\kappa(p'/p))$$

But $\operatorname{trdeg}_{R/p}(S/q) = \operatorname{trdeg}_{\kappa(p)}(\kappa(q))$ and $\operatorname{trdeg}_{\kappa(q'/q)}(\kappa(p'/p)) = \operatorname{trdeg}_{\kappa(q')}(\kappa(p'))$. So we have obtained the last two terms in the right hand side. Hence

$$\delta_S(q) - \delta_S(q') = \dim((S/q)_{q'}).$$

Now use that $\operatorname{Spec}((S/q)_{q'})$ is primes in $S$ between $q'$ and $q$. So we get a well-defined global function $\delta_S$. $\square$

*Remark.* If we knew that $\mathfrak{m} \in k[x_1, \dots, x_n]$ maximal implies $\kappa(\mathfrak{m})$ is a finite extension of $k$, then we could conclude that for any maximal chain $q = q_0 \subset q_1 \subset \cdots \subset q_n = \mathfrak{m}$, we have $n = \delta(q)$. We will prove this.

**Example 1.14.8.** This statement would tell us that if $\mathfrak{m} \subset \mathbb{Z}[x_1, \dots, x_n]$ is maximal, then $\mathfrak{m}$ contains $p$ for some $p \in \mathbb{Z}$, and $\kappa(\mathfrak{m}) \supset \mathbb{F}_p$ is finite. On the other hand, for $R = \mathbb{C}[[t]]$ or any other DVR, we might conjecture the same nice behavior, but $q = (xt - 1) \subset \mathbb{C}[[t]][x]$ is a maximal ideal. Compute that $\kappa(q) = \mathbb{C}((t))$, so

$$\delta(q) = \delta(q \cap \mathbb{C}[[t]]) + \operatorname{trdeg}_{\mathbb{C}((t))}(\mathbb{C}((t))) = 1 + 0 = 1,$$

even though $q$ is a closed point!

## 1.15 Chevalley's theorem

We will do Chevalley's theorem in the Noetherian case.

**Definition 1.15.1.** If $X$ is a Noetherian topological space, then a subset $E \subset X$ is called **constructible** if and only if $E$ is a finite union of locally closed subsets.

*Remark.* Warning: for general $X$, the definition of constructible is different.

**Theorem 1.15.2.** *Let $R \to S$ be finite-type, and $R$ (and therefore $S$) Noetherian. Then:*

1. *the image of* $\operatorname{Spec} S$ *in* $\operatorname{Spec} R$ *is constructible;*
2. *more generally, the image of a constructible subset in* $\operatorname{Spec} S$ *is constructible in* $\operatorname{Spec} R$.

**Example 1.15.3.** Consider $\mathbb{C}[x, y] \to \mathbb{C}[x, u]$ given by $(x, y) \mapsto (x, xu)$. The scheme-theoretic image is missing the line $x = 0$ except at $(x, y) = (0, 0)$, i.e. the image is $D(x) \cup \{(x, y)\}$, which is indeed constructible.

*Proof sketch.* Let $k = \bar{k}$. The proof uses elimination theory. The idea is as follows: given polynomials $f_1, \dots, f_r \in k[x_1, \dots, x_n, y_1, \dots, y_m]$, we want to find

$$\{x \in k^n : \exists y \in k^m \text{ s.t. } f(x, y) = 0\}.$$

A special case is $m = 1$ and $r = 2$. Write

$$f_1(x, y) = a_d(x)y^d + \cdots + a_0, \quad f_2(x, y) = b_e(x)y^e + \cdots + b_0.$$

Their **resultant** $P(x) = \operatorname{Res}_y(f_1, f_2)$ is a polynomial in the $a$ and $b$ such that $P(x) = 0$ iff $f_1, f_2$ have a common root in $\bar{k}$, or $a_d(x) = 0$ or $b_e(x) = 0$. Hence we are done by induction. □

## 1.16 Jacobson spaces and Jacobson rings

**Definition 1.16.1.** Let $X$ be a topological space, and $X_0 \subset X$ be the set of closed points. We say $X$ is **Jacobson** if for any $Z \subset X$ closed, $Z = \overline{Z \cap X_0}$. Equivalently, any non-empty locally closed subset meets $X_0$.

**Example 1.16.2.** A one-point space like $\operatorname{Spec} k$ is Jacobson. But $\operatorname{Spec} \mathbb{C}[[t]]$ is not Jacobson. $\operatorname{Spec} \mathbb{Z}$ is Jacobson just because we know what every closed subset in $\operatorname{Spec} \mathbb{Z}$ is. Also, $\operatorname{Spec}(\prod_{n \in \mathbb{N}} \mathbb{F}_2)$ is, but this space is complicated.

**Lemma 1.16.3.** *Let $X$ be a Jacobson topological space, and $T \subset X$ be a subset. Assume $T$ is closed or open or locally closed or a union of locally closed. Then $T$ is Jacobson and $T_0$, the set of closed points of $T$, is equal to $T \cap X_0$.*

**Definition 1.16.4.** A ring $R$ is **Jacobson** if $\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$ ranging over $\mathfrak{m}$ maximal. This is like saying there are enough closed points, since we already know $\sqrt{I} = \bigcap_{p \supset I} p$ ranging over primes $p$.

**Lemma 1.16.5.** *Let $R$ be any ring, and $I_\lambda$ be radical ideals. Then*

$$\overline{\bigcup_{\lambda \in \Lambda} V(I_\lambda)} = V(\bigcap_{\lambda \in \Lambda} I_\lambda).$$

**Example 1.16.6.** This is not true for $I_\lambda$ not radical: take $R = \mathbb{C}[x]$ and $I_n = (x^n)$.

*Proof.* Let $J$ be the radical ideal corresponding to the closure $\overline{\bigcup_\lambda V(I_\lambda)}$. If $f \in \bigcap_\lambda I_\lambda$, then clearly $V(I_\lambda) \subset V(f)$, so $f \in J$. Conversely, if $f \notin \bigcap_\lambda I_\lambda$, then $f \notin I_\lambda$ for some $\lambda$. Because $I_\lambda$ is radical, we know $I_\lambda = \bigcap_{p \supset I_\lambda} p$, so there exists $p \supset I_\lambda$ with $f \notin p$. Hence $f \notin J$. It follows that the radical ideal corresponding to $V(\bigcap_\lambda I_\lambda)$ is $J$, and the desired equality follows. ☐

**Lemma 1.16.7.** *$R$ is Jacobson if and only if* $\operatorname{Spec} R$ *is Jacobson.*

*Proof.* Suppose $R$ is Jacobson, and let $V(I) \subset \operatorname{Spec} R$ be closed. Then $\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$. By the previous lemma, $V(I) = V(\sqrt{I}) = \overline{\bigcup_{\mathfrak{m} \supset I} V(\mathfrak{m})}$. But all these $\mathfrak{m}$ are closed points, and this is just the closure of $\{x \in V(I) : x \text{ closed in } \operatorname{Spec} R\}$. Hence $\operatorname{Spec} R$ is Jacobson.

Suppose $\operatorname{Spec} R$ is Jacobson, and let $I \subset R$ be an ideal. Then $V(I) = \overline{\{x \in V(I) : x \text{ closed in } \operatorname{Spec} R\}}$. By the lemma, this is equal to $V(\bigcap_{\mathfrak{m} \supset I} \mathfrak{m})$. Hence $\sqrt{I} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$, and $R$ is Jacobson. ☐

## 1.17 Nullstellensatz

**Lemma 1.17.1.** *Let $f \colon X \to Y$ is a continuous map of Noetherian topological spaces. If Chevalley's theorem holds, $Y$ is Jacobson, and the fibers $X_y \coloneqq f^{-1}(y)$ are Jacobson, then*

1. *$X$ is Jacobson, and*
2. *$f$ maps closed points to closed points.*

*Proof.* Let $T \subset X$ be a non-empty locally closed subset. To prove (a), it suffices to find a point $t \in T$ which is closed in $X$. By Chevalley's theorem, $f(T)$ is constructible. Because $Y$ is Jacobson, there exists $y \in f(T)$ closed in $Y$. The fiber $X_y$ is closed in $X$, and $X_y \cap T$ is locally closed in $X_y$. By assumption, there exists $x \in X_y \cap T$ which is closed in $X_y$. But $X_y$ is closed in $X$, so $x$ is actually closed in $X$.

To get (b), suppose $x \in X$ is closed. Apply the previous argument to $T = \{x\}$ to get $y = f(x) \in f(T)$ closed in $Y$. ☐

*Remark.* Let $A \to B$ be a ring map. Then the fiber of $\operatorname{Spec} B \to \operatorname{Spec} A$ over $p$ is $\operatorname{Spec}(B_p/pB_p)$ as a topological space. This comes from considering the diagram

$$\begin{array}{ccccc} B & \longrightarrow & B_p & \longrightarrow & B_p/pB_p \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A_p & \longrightarrow & \kappa(p). \end{array}$$

In particular, if $B = A[x]$, then $B_p = A_p[x]$, and $B_p/pB_p = \kappa(p)[x]$, which is a polynomial ring over a field, and therefore Jacobson.

**Lemma 1.17.2** (Nullstellensatz)**.** *Let $R$ be Noetherian and Jacobson. Then:*

1. *$R[x]$ is Jacobson;*
2. *for any maximal ideal $\tilde{\mathfrak{m}}$ in $R[x]$,*

(a) $\mathfrak{m} = R \cap \tilde{\mathfrak{m}}$ *is maximal, and*

(b) $\kappa(\tilde{\mathfrak{m}})/\kappa(\mathfrak{m})$ *is finite.*

*Proof.* Apply the previous lemma to $\operatorname{Spec}(R[x]) \to \operatorname{Spec}(R)$. This map is indeed a continuous map of Noetherian spaces, and $\operatorname{Spec}(R)$ is Jacobson. The fibers are Jacobson by the preceding remark. □

**Corollary 1.17.3.** *Any finite type algebra $B$ over a Noetherian Jacobson ring $A$ is a Noetherian Jacobson ring. Moreover,* $\operatorname{Spec} B \to \operatorname{Spec} A$ *maps closed points to closed points and induces finite residue field extensions at those closed points.*

**Example 1.17.4.** Many algebras are Noetherian Jacobson rings: fields, $\mathbb{Z}$, and any finite-type algebra over these. Also, $A_f$ if $A$ is local Noetherian and $f \in \mathfrak{m}$ works.

## 1.18 Noether normalization

**Theorem 1.18.1** (Noether normalization)**.** *If $k$ is a field and $A$ a finite type $k$-algebra, then there exists a finite injective map $k[x_1, \ldots, x_d] \to A$ of $k$-algebras.*

*Remark.* A good exercise, using the tools we have so far, is to show that $d = \dim A$.

**Lemma 1.18.2.** *Let $A$ be a finite type $k$-algebra, and $p \subset A$ be prime. Let $X = \operatorname{Spec} A$ and $x = p$. Then*

$$\dim_x X = \dim A_p + \operatorname{trdeg}_k \kappa(p),$$

*where* $\dim_x X \coloneqq \inf\{\dim U : U \subset X \text{ open}, x \in U\}$.

# Chapter 2

# Regular and smooth rings

First goal: the localization $R_p$ of a regular local ring $R$ at a prime $p$ is also a regular local ring. Think of "regular" as being synonymous with "non-singular," so that smooth (over a field) implies regular, but it won't always be the case that regular implies smooth.

## 2.1 Colimits

**Definition 2.1.1.** A **pre-ordered set** $I$ is a set equipped a **pre-order**, i.e. a transitive and reflexive (but not necessarily symmetric) relation.

**Definition 2.1.2.** Let $\mathcal{C}$ be a category. A **system over** $I$ in $\mathcal{C}$ is given by $(X_i, \varphi_{ij})$, where

1. $X_i$ for $i \in I$ are objects in $\mathcal{C}$, and
2. $\varphi_{ij} \colon X_i \to X_j$ for $i \leq j$ in $I$ are morphisms in $\mathcal{C}$,

such that $\varphi_{ii} = \mathrm{id}_{X_i}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ for all $i \leq j \leq k$.

**Definition 2.1.3.** The **colimit** of $(X_i, \varphi_{ij})$, denoted $\mathrm{colim}_{i\in I} X_i$, is an object of $\mathcal{C}$ with morphisms

$$p_i \colon X_i \to \mathrm{colim}_{i\in I} X_i$$

such that for all $i \leq j$, we have $p_j \circ \varphi_{ij} = p_i$, and $(\mathrm{colim}_{i\in I} X_i, p_i)$ has the corresponding universal property.

**Example 2.1.4.** In the category $\mathsf{Set}$ of sets, the colimit is

$$\mathrm{colim}_{i\in I} X_i = \coprod_{i\in I} X_i / \sim$$

where $\sim$ is the equivalence relation generated by $x_i \sim \varphi_{ij}(x_i)$.

**Example 2.1.5.** In $\mathsf{AbGrp}$, the category of abelian groups, the colimit is

$$\mathrm{colim}_{i\in I} X_i = (\bigoplus_{i\in I} X_i)/S$$

where $S$ is the subgroup generated by $x_i - \varphi_{ij}(x_i)$.

**Example 2.1.6.** The category of finite abelian groups does not have colimits, because infinite direct sums do not exist within the category.

**Definition 2.1.7.** A pre-ordered set $I$ is called a **directed set** if

1. $I \neq \emptyset$, and
2. for all $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

A system over a directed set is called a **directed system**. A colimit over a directed set is called a **directed colimit**, or a **filtered colimit**.

**Example 2.1.8.** If $I$ is a directed set, then in $\mathsf{Set}$, elements $x_i \in X_i$ and $x_j \in X_j$ determine the same element in $\operatorname{colim}_{i \in I} X_i$ iff there exists $k \geq i, j$ such that $\varphi_{ik}(x_i) = \varphi_{jk}(x_j)$.

**Proposition 2.1.9.** *If $(X_i, \varphi_{ij})$ is a system of groups, abelian groups, rings, modules, or Lie algebras over a directed set, then $\operatorname{colim} X_i$ is the colimit in $\mathsf{Set}$ with the relevant induced algebraic operations.*

**Example 2.1.10.** Let $R$ be a ring, and $I$ be the set of its finite type $\mathbb{Z}$-subalgebras. Given $i \in I$, write $R_i \subset R$, and define $i \leq j$ by $R_i \subset R_j$. Then $\operatorname{colim} R_i = R$, since $I$ is indeed a directed set. This example shows that to prove any statement about commutative algebras that pass through colimits, it suffices to prove the statement for finite type algebras. This technique is called **absolute Noetherian reduction**.

**Example 2.1.11.** Let $S \subset R$ be a multiplicative subset. Let $I = S$, where $i \in I$ corresponds to $f_i \in S$, and $i \leq i'$ iff there exists $f \in S$ such that $f_{i'} = f f_i$. This makes $S$ into a directed set. Then

$$S^{-1}R = \operatorname{colim}_{f \in S} R_f.$$

**Example 2.1.12.** Let $I$ be a directed set. Let $(\mathcal{C}_i, F_{ij})$ be a system of categories over $I$. Then set $\operatorname{colim} \mathcal{C}_i$ to be the category with

$$\operatorname{Ob}(\operatorname{colim}_{i \in I} \mathcal{C}_i) = \operatorname{colim}_{i \in I} \operatorname{Ob}(\mathcal{C}_i), \quad \operatorname{Mor}(\operatorname{colim}_{i \in I} \mathcal{C}_i) = \operatorname{colim}_{i \in I} \operatorname{Mor}(\mathcal{C}_i).$$

More precisely, given $x_a \in \operatorname{Ob}(\mathcal{C}_a)$ and $x_b \in \operatorname{Ob}(\mathcal{C}_b)$, we are setting

$$\operatorname{Mor}_{\operatorname{colim} \mathcal{C}_i}(x_a, x_b) = \operatorname{colim}_{c \geq a, b} \operatorname{Mor}_{\mathcal{C}_c}(F_{ac}(x_a), F_{bc}(x_b))$$

where here $x_a$ and $x_b$ on the left hand side represent their classes in $\operatorname{colim} \mathcal{C}_i$.

**Lemma 2.1.13.** *Let $I$ be a directed set, and $A_i$ a system of rings over $I$. Let $A = \operatorname{colim} A_i$. Then the colimit over $I$ of the category of finitely presented $A_i$-modules $\mathsf{Mod}_{A_i}^{fp}$ is (equivalent to) the category of finitely presented $A$-modules $\mathsf{Mod}_A^{fp}$.*

*Proof.* Show that the induced functor from $\operatorname{colim} \mathsf{Mod}_{A_i}^{\mathrm{fp}}$ is essentially surjective and fully faithful. We show essential surjectivity.

1. (Essentially surjective for objects) Let $M \in \operatorname{Ob}(\mathsf{Mod}_A^{\mathrm{fp}})$. Choose a presentation $A^{\oplus m} \xrightarrow{T} A^{\oplus n} \to M \to 0$. For $i \in I$ sufficiently large, we can find a matrix $T_i \in \operatorname{Mat}(m \times n, A_i)$ whose image in $\operatorname{Mat}(m \times n, A)$ is $T$. Set $M_i \colon \operatorname{coker}(A_i^{\oplus m} \xrightarrow{T_i} A_i^{\oplus n})$. Then we see that $M_i \otimes_{A_i} A \cong M$. So $M$ is in the essential image.
2. (Essentially surjective for morphisms) For $i, j \in I$, let $M_i \in \mathsf{Mod}_{A_i}^{\mathrm{fp}}$ and $M_j \in \mathsf{Mod}_{A_j}^{\mathrm{fp}}$ and $\varphi \colon M_i \otimes_{A_i} A \to M_j \otimes_{A_j} A$. Goal: find $k \geq i, j$ such that $\varphi_k \otimes \mathrm{id}_A \colon M_i \otimes_{A_i} A_k \otimes_{A_k} A \to M_j \otimes_{A_j} A_k \otimes_{A_k} A$, which is canonically identified with $\varphi$. The obvious argument works. □

**Corollary 2.1.14.** *Let $R$ be a ring, $M$ a finitely presented $R$-module, and $p$ prime such that $M_p \cong R_p^{\oplus n}$. Then there exists $f \in R$, with $f \notin p$, such that $M_f \cong R_f^{\oplus n}$.*

*Proof.* $R_p = \operatorname{colim}_{f \in R \setminus p} R_f$. □

## 2.2 Projective modules

**Definition 2.2.1.** Let $R$ be a ring. An $R$-module $P$ is **projective** if and only if the functor

$$\mathrm{Hom}_R(P, -)\colon \mathsf{Mod}_R \to \mathsf{Mod}_R$$

is exact.

*Remark.* In any abelian category $\mathcal{A}$, for any object $X \in \mathcal{A}$, the functor $\mathrm{Hom}_{\mathcal{A}}(X, -)\colon \mathcal{A} \to \mathsf{AbGrp}$ is always left-exact: given a short exact sequence $0 \to Y_1 \to Y_2 \to Y_3 \to 0$ in $\mathcal{A}$,

$$0 \to \mathrm{Hom}_{\mathcal{A}}(X, Y_1) \to \mathrm{Hom}_{\mathcal{A}}(X, Y_2) \to \mathrm{Hom}_{\mathcal{A}}(X, Y_3)$$

is exact. So the definition is imposing that $\mathrm{Hom}_R(P, -)$ is also right exact. This is equivalent to the usual definition: given a surjection $Y_2 \to Y_3$ and a map $P \to Y_3$, it should always lifts to $P \to Y_2$.

**Example 2.2.2.** Any free module is projective. This comes from $\mathrm{Hom}_R(\bigoplus_{i\in I} R, -) = \prod_{i\in I} \mathrm{Hom}_R(R, -)$.

**Lemma 2.2.3.** *Let $R$ be a ring and $P$ an $R$-module. The following are equivalent:*

1. *$P$ is projective;*
2. *$P$ is a summand of a free module;*
3. *$\mathrm{Ext}_1^R(P, M) = 0$ for all $R$-modules $M$.*

*Proof.* (1) $\implies$ (2) Suppose $P$ is projective. Choose a surjection $\pi\colon F \to P$ with $F$ free. Then lift $P \to P$ to $i\colon P \to F$. So $F = i(P) \oplus \ker(\pi)$.

(2) $\implies$ (3) Suppose $P \oplus Q = F$ is free. Take the free resolution

$$\cdots \xrightarrow{b} F \xrightarrow{a} F \xrightarrow{b} F \to P \to 0$$

where $a$ and $b$ are the projections onto $P$ and $Q$ respectively. Then we get

$$\mathrm{Hom}_R(F, M) \xrightarrow{b^t} \mathrm{Hom}_R(F, M) \xrightarrow{a^t} \mathrm{Hom}_R(F, M) \xrightarrow{b^t} \mathrm{Hom}_R(F, M) \xrightarrow{a^t} \cdots.$$

But $a^t$ and $b^t$ are still both projections, and $a^t + b^t = \mathrm{id}$ by taking the transpose of $a + b = \mathrm{id}$. Hence this sequence is still exact (except on the first term). In particular, $\mathrm{Ext}^1_R(P, M) = 0$.

(3) $\implies$ (1) Assume $\mathrm{Ext}_1^R(P, M) = 0$ for all $R$-modules $M$. A short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ gives a long exact sequence

$$0 \to \mathrm{Hom}(P, M_1) \to \mathrm{Hom}(P, M_2) \to \mathrm{Hom}(P, M_3) \to \mathrm{Ext}^1(P, M_1) = 0 \to \cdots$$

so the functor $\mathrm{Hom}(P, -)$ is right exact. □

**Corollary 2.2.4.** *Direct sums of projectives are projective.*

**Definition 2.2.5.** Let $R$ be a ring and $M$ be an $R$-module.

1. We say $M$ is **locally free** if there is a cover of $\operatorname{Spec} R$ by principal opens $D(f_i)$ such that $M_{f_i}$ is a free $R_{f_i}$-module for all $i$.
2. We say $M$ is **finite locally free** if in addition $M_{f_i}$ is actually a finite $R_{f_i}$-module for all $i$.
3. We say $M$ is **finite locally free of rank** $r$ if $M_{f_i} = (R_{f_i})^{\oplus r}$ for all $i$.

**Definition 2.2.6.** An $R$-module $M$ is **finitely presented** iff there exists an exact sequence

$$R^{\oplus m} \to R^{\oplus n} \to M \to 0$$

of $R$-modules.

*Remark.* If $R$ is Noetherian, then being finitely presented is equivalent to being finite.

**Lemma 2.2.7.** *Let $R$ be a ring and $M$ an $R$-module. Take $f_i \in R$ for $i \in I$.*

1. $\operatorname{Spec}(R) = \bigcup D(f_i)$ *if and only if* $(f_i : i \in I) = R$.
2. *If (a) holds, $M$ is a finite $R$-module if and only if $M_{f_i}$ is a finite $R_{f_i}$-module for all $i$.*
3. *If (a) holds, $M$ is a finitely presented $R$-module if and only if $M_{f_i}$ is a finitely presented $R_{f_i}$-module for all $i$.*

**Example 2.2.8.** Let $R = \mathbb{Q}[x] \subset \mathbb{Q}(x)$ and $M = \sum_{\alpha \in \mathbb{Q}} 1/(x-\alpha)\mathbb{Q}[x]$. This is infinitely generated by $1/(x-\alpha)$ for all $\alpha \in \mathbb{Q}$. But localizing at a prime $p \subset R$, it is true that $M_p \cong R_p$. So the preceding definition and lemma would be different if we localized over primes $p$ instead of over principal opens $D(f_i)$.

**Definition 2.2.9.** An $R$-module $M$ is **flat over** $R$ iff the functor $M \otimes_R -$ is exact.

**Lemma 2.2.10.** *Let $R$ be a ring and $M$ an $R$-module. The following are equivalent:*

1. *$M$ is finitely presented and flat;*
2. *$M$ is finite projective (i.e. finite and projective);*
3. *$M$ is a direct summand of a finite free module;*
4. *$M$ is finitely presented and $M_p$ is free for all $p \in \operatorname{Spec} R$;*
5. *$M$ is finitely presented and $M_\mathfrak{m}$ is free for all $\mathfrak{m} \in \operatorname{Spec} R$ maximal;*
6. *$M$ is finite and locally free;*
7. *$M$ is finite locally free;*
8. *$M$ is finite and $M_p$ is free for all $p \in \operatorname{Spec} R$ and the function $\rho \colon \operatorname{Spec} R \to \mathbb{Z}$ given by $p \mapsto \operatorname{rank}(M_p)$ is locally constant in the Zariski topology.*

*Proof.* We're skipping (1).

(2) $\implies$ (3) Pick a surjection $R^{\oplus n} \to M$. Because $M$ is projective, we get a splitting, and therefore $M$ is a summand of a finite free module

(3) $\implies$ (4) A summand of a finitely presented module is finitely presented. Over a local ring, a summand of a finite free module is finite free. (Projective implies free over local rings, by using Nakayama: if $R^{\oplus n} = M \oplus N$, then $(R/\mathfrak{m})^{\oplus n} = M/\mathfrak{m}M \oplus N/\mathfrak{m}N$ so that we can pick bases and lift.)

(4) $\implies$ (5) Trivial.

(5) $\implies$ (6) Clearly $M_\mathfrak{m}$ is finite free, so by Corollary 2.1.14, there exists $f \in R$ and $f \notin \mathfrak{m}$ such that $M_f$ is finite free. So for every closed point $x \in \operatorname{Spec} R$, we have $f \in R$ such that $x \in D(f)$ and $M_f$ is finite free. Because of the topology of $\operatorname{Spec} R$, we know these $D(f)$ cover $\operatorname{Spec} R$.

(6) $\implies$ (7) Finite and free implies finite free.

(7) $\implies$ (8) Obvious, except we have to show that finite locally free implies finite. (We don't actually need freeness.)

(8) $\implies$ (7) Pick $\mathfrak{m} \subset R$ maximal. Choose $x_1, \ldots, x_n \in M$ which map to a basis of $M/\mathfrak{m}M = M_\mathfrak{m}/\mathfrak{m}M_\mathfrak{m}$, so that $n = \rho(\mathfrak{m})$. By Nakayama, the map $R^{\oplus n} \to M$ given by $(r_1, \ldots, r_n) \mapsto r_1x_1 + \cdots + r_nx_n$ is surjective, and there is an $f \in R \setminus \mathfrak{m}$ such that the induced map is surjective. By assumption, there exists $g \in R \setminus \mathfrak{m}$

such that $\rho|_{D(g)}$ is constant with value $n$. Then $R_{fg}^{\oplus n} \to M_{fg}$ is surjective and for all primes $p \subset R_{fg}$, we have $(M_{fg})_p \cong (R_{fg})_p^{\oplus n}$. Hence $M$ is finite locally free.

(7) $\implies$ (2) It is enough to show $M$ is projective, since it is already finite by hypothesis. Let $N \to N'$ be a surjective map. We want to show $\psi\colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, N')$ is surjective. Pick $f_1, \dots, f_n \in R$ such that $R = (f_1, \dots, f_n)$ and $M_{f_i} \cong R_{f_i}^{\oplus n_i}$ for every $i$. This implies $M$ is finitely presented. But then $\operatorname{Hom}_R(M, N)_{f_i} = \operatorname{Hom}_{R_{f_i}}(M_{f_i}, N_{f_i})$ (which requires $M$ finitely presented). Then $M_{f_i}$ finite free implies $\psi_{f_i}$ surjective. Hence $\psi$ is surjective. ☐

**Example 2.2.11.** Let $R = C^\infty(\mathbb{R})$ and take the ideal $I$ to consist of functions vanishing in a neighborhood (classical topology) of 0. Let $M = R/I$. It is finite and $M_p$ is either $R_p$ or (0) for all $p \in \operatorname{Spec} R$, but $M$ is not projective.

This is because $I$ is not finitely generated, so $M = R/I$ is not finitely presented. In fact, $M \cong R_{\mathfrak{m}}$, the maximal ideal given by the kernel of the evaluation at 0 map. From this we get $M_p$ being either $R_p$ or (0).

## 2.3 What makes a complex exact?

Fix an exact complex

$$0 \to R^{\oplus n_e} \xrightarrow{\varphi_e} R^{\oplus n_{e-1}} \xrightarrow{\varphi_{e-1}} \cdots \to R^{\oplus n_1} \xrightarrow{\varphi_1} R^{\oplus n_0}$$

An important thing to notice about this complex is that it starts somewhere.

**Lemma 2.3.1.** *Suppose $R$ is local with maximal ideal $\mathfrak{m}$ and for some $1 \le i \le e$, some matrix coefficient of $\varphi_i$ is not in $\mathfrak{m}$. Then the complex is isomorphic to a complex*

$$0 \to R^{\oplus n_e} \to \cdots \to R^{\oplus n_{i+1}} \to R^{\oplus n_i - 1} \to R^{\oplus n_{i-1} - 1} \to R^{\oplus n_{i-2}} \to \cdots \to R^{n_0}$$

*direct sum*

$$0 \to 0 \to \cdots \to 0 \to R \xrightarrow{1} R \to 0 \to \cdots \to 0.$$

**Definition 2.3.2** (Only for today)**.** Suppose that $\varphi\colon R^{\oplus m} \to R^{\oplus n}$ is an $R$-linear map.

1. The **rank of** $\varphi$ is the maximum $r$ such that $\wedge^r \varphi\colon \wedge^r R^{\oplus m} \to \wedge^r R^{\oplus n}$ is non-zero.
2. $I(\varphi)$ is the ideal generated by $r \times r$ minors of $\varphi$ where $r$ is the rank of $\varphi$.

**Lemma 2.3.3.** *If our complex is trivial, i.e. isomorphic to a direct sum of complexes of the form $0 \to \cdots \to 0 \to R \xrightarrow{1} R \to 0 \cdots \to 0$, then*

1. *$\varphi_i$ has rank $r_i \coloneqq n_i - n_{i+1} + \cdots + (-1)^{e-i} n_e$;*
2. *for all $1 \le i \le e$, the rank of $\varphi_{i+1} + \varphi_i = n_i$;*
3. *$I(\varphi_i) = R$.*

**Lemma 2.3.4.** *Let $R$ be local Noetherian with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m} \in \operatorname{Ass}(R)$, i.e. $\operatorname{depth}(R) = 0$. Suppose our complex is exact. Then our complex is trivial.*

*Proof.* We may assume all matrix coefficients are in $\mathfrak{m}$, because otherwise we can remove a trivial summand from the complex and continue inductively. Pick $z \in R$ non-zero with $\mathfrak{m} z = 0$, since $\mathfrak{m} \in \operatorname{Ass}(R)$. Then $zv \in \ker \varphi_e$ is non-zero, where $v$ is some basis vector. This is a contradiction unless $n_e = 0$. ☐

**Lemma 2.3.5.** *If the complex is exact and $x \in R$ is a non-zerodivisor, then*

$$0 \to (R/xR)^{n_e} \xrightarrow{\bar{\varphi}_e} (R/xR)^{n_{e-1}} \xrightarrow{\bar{\varphi}_{e_1}} \cdots \xrightarrow{\bar{\varphi}_2} (R/xR)^{n_1}$$

*is exact.*

*Proof.* There is a short exact sequence of complexes

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & R^{n_e} & \longrightarrow & R^{n_{e-1}} & \longrightarrow & \cdots & \longrightarrow & R^{n_1} & \longrightarrow & R^{n_0} \\
 & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} \\
0 & \longrightarrow & R^{n_e} & \longrightarrow & R^{n_{e-1}} & \longrightarrow & \cdots & \longrightarrow & R^{n_1} & \longrightarrow & R^{n_0} \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & (R/xR)^{n_e} & \longrightarrow & (R/xR)^{n_{e-1}} & \longrightarrow & \cdots & \longrightarrow & (R/xR)^{n_1} & \longrightarrow & (R/xR)^{n_0} \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & & & 0 & & 0
\end{array}$$

from which we get a long exact sequence of cohomology

$$H^e(R^{n_e}) \to H^e(R^{n_e}) \to H^e((R/xR)^{n_e}) \to H^{e-1}(R^{n_{e-1}}) \to \cdots \to H^1((R/xR)^{n_1}) \to H^0(R^{n_0}) \to H^0(R^{n_0}).$$

□

**Lemma 2.3.6** (Acyclicity)**.** *Let $R$ be local Noetherian and $0 \to M_e \to M_{e-1} \to \cdots \to M_0$ a complex of finite $R$-modules. Let $i_0$ be the largest index such that the complex is not exact at $M_{i_0}$. Assume* $\operatorname{depth} M_i \geq i$ *for all $i$. Then the depth of the cohomology at $M_{i_0}$ is at least 1, provided $i_0 > 0$.*

*Proof.* Break the complex into short exact sequences. Then the cohomology at $i_0 = e$ is a submodule of $M_e$ and $e \geq 1$ so depth $\geq 1$ (since depth is inherited by submodules). If $i_0 = e - 1$, then

$$0 \to M_e \to M_{e-1} \to M_{e-1}/M_e \to 0$$

is exact, and the cohomology at $e - 1$ is a submodule of $M_{e-1}/M_e$. By a previous lemma,

$$\operatorname{depth}(M_{e-1}/M_e) \geq \min(\operatorname{depth}(M_{e-1}), \operatorname{depth}(M_e) - 1) \geq e - 1$$

so we are done. The idea is the same for general $i_0$. □

**Proposition 2.3.7.** *Let $R$ be local Noetherian. The complex is exact at $R^{n_e}, \ldots, R^{n_1}$ iff for all $1 \leq i \leq e$, we have:*

1. $\operatorname{rank}(\varphi_i) = r_i$*, the "expected rank";*
2. *$I(\varphi_i)$ is either $R$ or contains a regular sequence of length $i$.*

*Proof.* We may assume all coefficients of all maps lie in the maximal ideal $\mathfrak{m}$ (small exercise). Assume (1) and (2) hold for all $i$. Then in particular, $\operatorname{depth}(R) \geq e$. If there is some non-zero cohomology in degree $i_0$ for $i_0 > 0$, then it has depth $\geq 1$ by the acyclicity lemma. So its support has dimension at least 1. So we can find a prime $p \subset R$ with $p \neq \mathfrak{m}$ such that $0 \to R_p^{n_e} \to R_p^{n_{e-1}} \to \cdots \to R_p^{n_0}$ still has non-zero cohomology at $i_0$. Now we must check that (1) and (2) still hold for this new complex (omitted). Then we are done by induction on the dimension.

Conversely, assume exactness at $n_e, \ldots, n_1$. Let $q \in \operatorname{Ass}(R)$, and consider the complex over $R_q$. Note that $R_q$ is a local ring of depth 0. By a previous lemma, $\operatorname{rank} \varphi_{i,q} = r_i$ and $I(\varphi_{i,q}) = I(\varphi_i)_q = R_q$. This means

$$I(\varphi_1) \cdots I(\varphi_e) \not\subset q.$$

But this is for every $q \in \operatorname{Ass}(R)$, so by prime avoidance we can find a non-zerodivisor $x \in I(\varphi_1) \cdots I(\varphi_e)$. Then use induction to shorten the complex by one element, by looking at

$$0 \to (R/xR)^{n_e} \to \cdots \to (R/xR)^{n_1}.$$

□

## 2.4 Regular local rings

**Lemma 2.4.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring. Then the graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to $\kappa[X_1, \ldots, X_d]$ with $d = \dim R$.*

*Proof.* Let $x_1, \ldots, x_d \in \mathfrak{m}$ be a minimal set of generators, so $d = \dim R$. Then we get a surjection of graded rings

$$\kappa[X_1, \ldots, X_d] \to \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}, \quad F(X_1, \ldots, X_d) \mapsto F(x_1, \ldots, x_d) \in \mathfrak{m}^{\deg F}/\mathfrak{m}^{\deg F+1}.$$

We know $\dim \kappa[x_1, \ldots, x_d]_n = \binom{n+d-1}{d-1}$, which is of degree $d-1$. If there were a kernel of this surjection, then the degree of $n \mapsto \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$ would have degree $< d-1$. □

**Lemma 2.4.2.** *Any regular local ring is a domain.*

*Proof.* By a previous lemma, $\bigcap_n \mathfrak{m}^n = 0$ and we know $\mathrm{Gr}_m(R)$ is a domain. Hence it follows that $R$ is a domain: if $f, g \in R$ are non-zero, let $n$ and $m$ be the maximal integers such that $f \in \mathfrak{m}^n$ and $g \in \mathfrak{m}^m$, so that $fg \neq 0 \in \mathfrak{m}^{n+m}$. □

**Lemma 2.4.3.** *Let $R$ be regular local, and $x_1, \ldots, x_d$ be a minimal set of generators of the $\mathfrak{m}$. Then $x_1, \ldots, x_d$ is a regular sequence and $R/(x_1, \ldots, x_c)$ is a regular local ring of dimension $d-c$. In particular, $R$ is CM.*

*Proof.* By the previous lemma, we know $x_1$ is a non-zerodivisor. Let $R_1 \coloneqq R/\mathfrak{m}_1$, with maximal ideal $\mathfrak{m}_1 \coloneqq \mathfrak{m}/(x_1)$. Then $\dim R_1 = d-1$, and therefore $R_1$ is regular with $\bar{x}_2, \ldots, \bar{x}_n$ a regular system of generators of $\mathfrak{m}_1$. Now induct. □

**Lemma 2.4.4.** *Let $R$ be Noetherian local and $M$ a finite $R$-module. Assume $x \in R$ is a non-zerodivisor on $M$, and $M/xM$ is free over $R/xR$. Then $M$ is free over $R$.*

*Proof.* Pick $m_1, \ldots, m_r \in M$ mapping to a basis of $M/xM$. Nakayama says they generate $M$. If some $\sum_i a_i m_i = 0$ is a relation in $M$, then $x \mid a_i$ for all $i$ (since $\sum_i \bar{a}_i \bar{m}_i = 0$ implies $\bar{a}_i = 0$ in $M/xM$), so $a_i = xb_i$ for some $b_i$. Hence the kernel $K$ of the surjection $R^{\oplus r} \to M$ satisfies $K = xK$. By Nakayama again, $K = 0$. □

**Definition 2.4.5.** A module $M$ is **maximal Cohen–Macaulay** (MCM) if it is CM and $\operatorname{depth} M = \dim R$. (We know from it being CM that $\operatorname{depth} M = \dim \operatorname{supp} M$, so this says $\dim \operatorname{supp} M = \dim R$.)

**Lemma 2.4.6.** *Let $R$ be regular local. Any MCM-module is free.*

*Proof.* Pick $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then by a previous lemma, $x$ is a non-zerodivisor on $M$ (here we're using that $M$ is MCM). Then $M/xM$ has depth exactly one less than $M$, and it lives over the regular local ring $R/xR$, which also has dimension exactly one less than $R$. By induction on depth, $M/xM$ is free. By the previous lemma, $M$ is free. □

**Lemma 2.4.7.** *Let $R$ be Noetherian local, and $x \in \mathfrak{m}$ a non-zerodivisor such that $R/xR$ is regular. Then $R$ is regular.*

*Proof.* Let $R_1 \coloneqq R/xR$ and $\mathfrak{m}_1 = \mathfrak{m}/(x)$. We know $\dim R = \dim R_1 + 1$ and $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathfrak{m}_1/\mathfrak{m}_1^2 + 1$ or $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathfrak{m}_1/\mathfrak{m}_1^2$. But we know $\dim R \leq \dim \mathfrak{m}/\mathfrak{m}^2$, with equality iff $R$ is regular. So $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathfrak{m}_1/\mathfrak{m}_1^2$ cannot happen. It follows that $R$ is regular. □

**Example 2.4.8.** Let $k = F_p(t)$ and $R = (k[x,y]/(x^2 - y^p + t))_{(x, y^p - t)}$. Then the morphism $\operatorname{Spec} k[x,y]/(x^2 - y^p + t) \to \operatorname{Spec} k$ is not smooth at the point $(x)$ (by the Jacobian criterion). However the ring is indeed a regular local ring of dimension 1. So regular does not necessarily imply smooth.

## 2.5 Projective and global dimension

**Definition 2.5.1.** Let $R$ be a ring and $M$ an $R$-module. We say $M$ has **finite projective dimension** if it has a finite length resolution by projective modules. The minimal length of such a resolution is the **projective dimension** $\mathrm{pd}_R(M)$.

**Definition 2.5.2.** Let $R$ be a ring. We say $R$ has **finite global dimension** if there exists an $n \in \mathbb{Z}$ such that $\mathrm{pd}_R(M) \leq n$ for all $R$-modules $M$. The smallest such $n$ is the **global dimension** of $R$.

**Lemma 2.5.3** (Schanul's lemma)**.** *Let $R$ be a ring and $M$ an $R$-module. Suppose there are short exact sequences*

$$0 \to K \to P_1 \to M \to 0, \quad 0 \to L \to P_2 \to M \to 0$$

*with $P_i$ projective. Then $K \oplus P_2 \cong L \oplus P_1$.*

*Proof.* Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_2 & = & P_2 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & P_1 \oplus P_2 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & P_1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

arising from the snake lemma. By projectivity of $P_2$, the first column splits. But it is an exact sequence, so $N \cong P_2 \oplus K$. By symmetry we are done. □

**Lemma 2.5.4.** *Let $R$ be a ring and $M$ an $R$-module. Let $\mathrm{pd}_R(M) = d$. Suppose we have a resolution*

$$F_e \to F_{e-1} \to \cdots \to F_0 \to M \to 0$$

*with $F_i$ projective, and $e \geq d - 1$. Then $\ker(F_e \to F_{e-1})$ (or $\ker F_0 \to M$ if $d = 0$) is projective.*

*Proof.* Johan: "look it up." It is essentially induction using the previous lemma. □

**Lemma 2.5.5.** *Let $R$ be a ring and $M$ an $R$-module. Let $d \geq 0$. The following are equivalent:*

1. *$M$ has projective dimension $\leq d$;*
2. *there exists a resolution $0 \to P_d \to P_{d_1} \to \cdots \to P_0 \to M \to 0$ with $P_i$ projective;*
3. *for some resolution $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ with $P_i$ projective, $\ker(P_{d-1} \to P_{d-2})$ is projective;*
4. *for any resolution $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ with $P_i$ projective, $\ker(P_{d-1} \to P_{d-2})$ is projective.*

*If $R$ is local, then these are also equivalent to:*

5. *there exists a resolution $0 \to P_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0$ with $P_i$ free.*

*If $R$ is Noetherian (but not necessarily local) and $M$ is finite, then (1) — (4) are also equivalent to:*

6. *there exists a resolution $0 \to P_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0$ with $P_i$ finite projective.*

**Lemma 2.5.6.** *Let $R$ be a ring and $M$ an $R$-module. Let $n \geq 0$. The llowing are equivalent:*

1. $\operatorname{pd}_R(M) \leq n$;

2. $\operatorname{Ext}^i_R(M, N) = 0$ *for all* $i \geq n+1$ *and for all* $N$;

3. $\operatorname{Ext}^{n+1}_R(M, N) = 0$ *for all* $N$.

*Proof.* (1) $\implies$ (2) By the previous lemma, there exists a resolution $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ with $P_i$ projective. It is a fact that we are allowed to compute $\operatorname{Ext}^i_R$ using projective resolutions. Hence $\operatorname{Ext}^i_R(M, -) = 0$ for all $i \geq n+1$.

(2) $\implies$ (3) Trivial.

(3) $\implies$ (1) Use a dimension shifting argument. If $n = 0$, then $M$ is projective. If $n > 0$, choose a free module $F$ and a surjection $F \to M$ with kernel $K$. By the long exact sequence associated to $0 \to K \to F \to M \to 0$, we get $\operatorname{Ext}^n_R(K, N) = 0$. By induction, $\operatorname{pd}_R(K) \leq n-1$. So of course $\operatorname{pd}_R(M) \leq n$. □

**Corollary 2.5.7.** *Let* $0 \to M' \to M \to M'' \to 0$ *be short exact. Then:*

1. $\operatorname{pd}_R(M) \leq n$ *and* $\operatorname{pd}_R(M'') \leq n+1$ *implies* $\operatorname{pd}_R(M') \leq n$;

2. $\operatorname{pd}_R(M') \leq n$ *and* $\operatorname{pd}_R(M'') \leq n$ *implies* $\operatorname{pd}_R(M) \leq n$;

3. $\operatorname{pd}_R(M') \leq n$ *and* $\operatorname{pd}_R(M) \leq n+1$ *implies* $\operatorname{pd}_R(M') \leq n+1$.

**Lemma 2.5.8.** *Given a ring* $R$ *and an integer* $n \geq 0$*, the following are equivalent:*

1. *$R$ has global dimension at most $n$;*

2. *every finite $R$-module $M$ has* $\operatorname{pd}_R(M) \leq n$;

3. *every cyclic $R$-module $M$ has* $\operatorname{pd}_R(M) \leq n$.

*Proof.* (3) $\implies$ (1) Let $M$ be an $R$-module and $E \subset M$ be a set of generators for $M$. Choose a well-ordering on $E$. For $e \in E$, let $M_e \subset M$ be the submodule generated by all $e' \in E$ with $e' < e$. Then $M = \bigcup_{e \in E} M_e$ and for each $e$, the quotient $M_e / \bigcup_{e' < e} M_{e'}$ is cyclic (or zero). So these quotients have $\operatorname{pd}_R \leq n$.

Let $n = 0$. By transfinite induction we will show $M$ is projective. Namely, for each $e$, if we let $P_e := M_e / \bigcup_{e' < e} M_{e'}$, there is a splitting $M_e = \bigcup_{e' < e} M_{e'} \oplus P_e$. It follows that $M_e = \bigoplus_{e' \leq e} P_{e'}$.

For $n > 0$, read the rest of the proof on the Stacks Project. Johan: "I apologize, it's just too annoying." □

**Lemma 2.5.9.** *Let $R$ be a ring, $M$ an $R$-module, and $S \subset R$ a multiplicative subset. Then:*

1. *if* $\operatorname{pd}_R M \leq n$*, then* $\operatorname{pd}_{S^{-1}R} S^{-1}M \leq n$;

2. *if $R$ has global dimension $\leq n$, then $S^{-1}R$ has global dimension $\leq n$.*

*Proof.* The projective dimension $\operatorname{pd}_R(M) \leq n$ iff there exists a projective resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$. Localization is exact, so $0 \to S^{-1}P_n \to \cdots \to S^{-1}P_0 \to S^{-1}M \to 0$ is a projective resolution.

Now take any $S^{-1}R$-module $M$ and view it as an $R$-module. Then $\operatorname{pd}_R M \leq n$, and so $\operatorname{pd}_{S^{-1}R}(S^{-1}M) \leq n$. But $S^{-1}M = M$ since $M$ is already an $S^{-1}R$-module. □

**Theorem 2.5.10.** *A regular local ring has finite global dimension.*

*Proof.* By the previous lemma, it suffices to find a universally bounded resolution of any finite module $M$ over $R$. We will do this by induction on $\operatorname{depth}_R M$. If $M = 0$, then the depth is infinite, and in this case the theorem clearly holds. If $M \neq 0$, then $0 \leq \operatorname{depth}_R M \leq \dim R$. If $\operatorname{depth}_R M = \dim R$, then $M$ is MCM. We know that over a regular local ring, an MCM module is free. Now assume $\operatorname{depth} M < \dim R$. Choose a short exact sequence $0 \to K \to R^{\oplus n} \to M \to 0$. Then

$$\operatorname{depth}_R(K) \geq \min(\operatorname{depth}_R(R^{\oplus n}), \operatorname{depth}(M)+1) = \min(\dim(R), \operatorname{depth}_R(M)+1) > \operatorname{depth}_R(M).$$

The induction hypothesis applies to $K$, so $K$ has a universally bounded free resolution. Therefore so does $M$, except with length one greater. □

*Remark.* More strongly, this argument actually proves $\operatorname{pd}_R(M) + \operatorname{depth}_R(M) = \dim R$ for $R$ a regular local ring and $M$ a finite $R$-module.

## 2.6 Dimension of regular local rings

**Lemma 2.6.1.** *Let* $(R, \mathfrak{m}, \kappa)$ *be Noetherian local. Then* $\operatorname{pd}_R(\kappa) \geq \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$.

*Proof.* Let $x_1, \dots, x_n \in \mathfrak{m}$ be such that their images are a basis for $\mathfrak{m}/\mathfrak{m}^2$. Consider the **Koszul complex** $K_\bullet$ on $x_1, \dots, x_n$:

$$\wedge^n R^{\oplus n} \to \cdots \to \wedge^2(R^{\oplus n}) \to R^{\oplus n} \to R$$

with differential

$$d(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{a=1}^{i} (-1)^{a+1} x_{j_a} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_a} \wedge \cdots \to e_{j_i}$$

So if we can show any resolution of $\kappa$ is longer than this Koszul complex, we are done. Let $F_\bullet \to \kappa$ be a finite resolution by finite free, which exists because if $\operatorname{pd}_R(\kappa) = \infty$, we are done, so assume $\operatorname{pd}_R(\kappa) < \infty$ and apply a previous lemma. By another previous lemma, assume all maps in $F_\bullet$ have matrix coefficients in $\mathfrak{m}$. Hence $F_i$ maps into $\mathfrak{m}F_{i-1}$. Lift the map $\operatorname{id}\colon \kappa \to \kappa$ to a map $\alpha_\bullet$ of complexes:

$$\begin{array}{ccccccccc} K_\bullet & \longrightarrow & \cdots & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & \kappa \\ \downarrow{\scriptstyle \alpha_\bullet} & & & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} \\ F_\bullet & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & \kappa. \end{array}$$

The claim is that $\alpha_i \bmod \mathfrak{m}$ is injective. If the claim is true, then $F_n \neq 0$, so $\operatorname{pd}_R(\kappa) \geq n$ as desired.

1. For $i = 0$, note that $F_0$ must be free of rank 1. Since $K_0 \to \kappa \to \kappa$ is not the zero map, neither is $\alpha_0$ by commutativity of the last square. So $\alpha_0$ is multiplication by a unit.
2. For $i = 1$, note that $F_1 \to F_0 = R$ and $R^{\oplus n} \to R$ factor through $\mathfrak{m}$. So there is an induced diagram

$$\begin{array}{ccc} R^{\oplus n}/\mathfrak{m}R^{\oplus n} & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \\ \downarrow & & \downarrow{\scriptstyle \alpha_0 \bmod \mathfrak{m}} \\ F_1/\mathfrak{m}F_1 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \end{array}$$

   where the induced map $\alpha_0 \bmod \mathfrak{m}$ is an isomorphism by the $i = 0$ case. Hence the left arrow $\alpha_1 \bmod \mathfrak{m}$ is injective.
3. For $i \geq 1$, assume $\alpha_j \bmod \mathfrak{m}$ injective for $j < i$. Then again there is a diagram

$$\begin{array}{ccc} \wedge^i(R^{\oplus n}) & \longrightarrow & \wedge^{i-1}(R^{\oplus n}) \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i-1}} \\ F_i & \longrightarrow & F_{i-1} \end{array}$$

   where the top and bottom arrows factor through $\mathfrak{m}\wedge^{i-1}$ and $\mathfrak{m}F_{i-1}$ respectively. Tensor with $\kappa$ or mod out by $\mathfrak{m}$ to get

$$\begin{array}{ccc} \wedge^i(\kappa^{\oplus n}) & \longrightarrow & (\mathfrak{m}/\mathfrak{m}^2) \otimes_k \wedge^{i-1}(\kappa^{\oplus n}) \\ \downarrow{\scriptstyle \alpha_i \bmod \mathfrak{m}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{m}/\mathfrak{m}^2} \otimes \alpha_{i-1} \bmod \mathfrak{m}} \\ F_i/\mathfrak{m}F_i & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \otimes_\kappa F_{i-1}/\mathfrak{m}F_{i-1}. \end{array}$$

By the induction hypothesis, the right arrow is injective, so to show the left arrow is injective, it suffices to show the top arrow is injective. But the top arrow is just part of the Koszul complex: we know it is

$$e_{j_1} \wedge \cdots \wedge e_{j_i} \mapsto \sum_a (-1)^{a+1} \bar{x}_{j_a} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_a} \wedge \cdots \wedge e_{j_i},$$

which is injective. □

**Lemma 2.6.2.** *Let $(R, \mathfrak{m}, \kappa)$ be Noetherian local. Suppose $\operatorname{pd}_R \kappa = p < \infty$. Then $\dim R \geq p$.*

*Proof.* Let $0 \to F_p \to \cdots \to F_1 \to F_0 \to \kappa \to 0$ be a minimal resolution by finite free, with $F_i$ mapping into $\mathfrak{m}F_{i-1}$ for all $i \geq 1$. By "what makes a complex exact," we see $\operatorname{depth}(R) \geq p$. Hence $\dim(R) \geq p$ by a previous lemma. □

**Theorem 2.6.3.** *Let $(R, \mathfrak{m}, \kappa)$ be Noetherian local. The following are equivalent:*

1. *$R$ is regular;*
2. *the global dimension of $R$ is finite;*
3. *the projective dimension $\operatorname{pd}_R(\kappa)$ is finite.*

*Proof.* We showed (1) $\implies$ (2) already, and (2) $\implies$ (3) is trivial. So suppose $\operatorname{pd}_R(\kappa) = p < \infty$. Then $\dim(R) \geq p \geq \dim_R(\mathfrak{m}/\mathfrak{m}^2)$. Hence equality holds at each step, and $R$ is regular. □

**Corollary 2.6.4.** *$R$ is regular local implies $R_p$ is regular local.*

*Proof.* By the theorem, $R$ regular means $R$ has finite global dimension, which means $R_p$ has finite global dimension, which means $R_p$ is regular. □

**Definition 2.6.5.** A Noetherian ring is **regular** if every localization $R_p$ is regular.

*Remark.* Fact: if $R$ is regular, then $\dim(R)$ is equal to the global dimension, including the case when either is infinite.

# Chapter 3

# Completions

This material is not on the final.

## 3.1 Completions

**Definition 3.1.1.** A **topological ring** is a ring with a topology where addition and multiplication are continuous. A **topological module** (over a topological ring) is a module $M$ with a topology such that addition and the module structure map are continuous. We say $M$ is **separated** if $\bigcap_\lambda M_\lambda = \{0\}$.

**Definition 3.1.2.** A topological module is **linearly topologized** if there is a fundamental system of zero consisting of submodules.

**Definition 3.1.3.** Given $M$ a topological module and $M_\lambda$ an open submodule. Then $\hat{M} := \varprojlim_\lambda M_\lambda$ is the **completion** and has a canonical map $M \to \hat{M}$. It inherits a **limit topology** via $\ker(\hat{M} \to M/M_\lambda)$.

**Lemma 3.1.4.** *The completion is complete with respect to the limit topology.*

**Example 3.1.5.** Let $M = \mathbb{Z}$ and $\lambda \in \mathbb{N}$. Let $M_\lambda := \lambda\mathbb{Z}$. Then

$$\varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$$

where $\mathbb{Z}_p$ is the $p$-adics.

**Example 3.1.6.** Let $M_\lambda := 17^\lambda\mathbb{Z}$ for $\lambda \in \mathbb{N}$. Then $\hat{M} = \mathbb{Z}_{17}$, the 17-adics.

**Definition 3.1.7.** Let $R$ be a ring and $I \subset R$ be an ideal. The ***I*-adic topology** on $R$ is given by setting $\{I^n\}$ to be a fundamental system of neighborhoods of zero. (We skip the verification that the product is continuous.) Similarly, any $R$-module $M$ has the ***I*-adic topology** given by $\{I^n M\}$. If the canonical map $M \to \hat{M} := \varprojlim_n M/I^n M$ from $M$ to its $I$-adic completion is an isomorphism, we say $M$ is ***I*-adically complete**.

*Remark.* The $I$-adic completion is not in general $I$-adically complete. (The limit topology will not always be the same as the $I$-adic topology.) It is, however, always complete, since completions are complete.

**Lemma 3.1.8.** *Let $R$ be a ring and $I \subset R$ be an ideal. Let $\varphi\colon M \to N$ be a homomorphism of $R$-modules. If $M/IM \to N/IN$ is surjective, then $\hat{M} \to \hat{N}$ is surjective.*

*Proof.* Assume $M/IM \to N/IN$ is surjective. By Nakayama, $M/I^n M \to N/I^n N$ is surjective for all $n \geq 1$. Let

$$K_n := \{x \in M : \varphi(x) \in I^n N\}.$$

Then we get a short exact sequence

$$0 \to K_n/I^nM \to M/I^nM \to N/I^nN \to 0.$$

By the Mittag–Leffler condition for vanishing of $\varprojlim^1$, it suffices to show $K_{n+1}/I^{n+1}M \to K_n/I^nM$ is surjective. Let $x \in K_n$ and write $\varphi(x) = \sum_j z_j n_j$ where $z_j \in I^n$ and $n_j \in N$. By assumption, write $n_j = \varphi(m_j) + \sum_k z_{jk} n_{jk}$ where $m_j \in M$ and $z_{jk} \in I$ and $n_{jk} \in N$. Then

$$\varphi(x - \sum z_j m_j) = \sum_{j,k} z_j z_{jk} n_{jk} \in I^{n+1}N. \qquad \square$$

**Lemma 3.1.9.** *Let $R$ be a ring, $I \subset R$ a finitely generated ideal, and $M$ an $R$-module.*

1. *The $I$-adic completion $\hat{M}$ is $I$-adically complete.*
2. *$I^n\hat{M} = \ker(\hat{M} \to M/I^nM) = \widehat{I^nM}$ for all $n \geq 1$.*

*Proof.* Since $I$ is finitely generated, $I^n$ is finitely generated. Say $I^n = (f_1, \ldots, f_r)$. Apply the previous lemma to the surjection

$$(f_1, \ldots, f_r)\colon M^{\oplus r} \to I^nM$$

yields a surjection

$$\hat{M}^{\oplus r} \to \widehat{I^nM} = \varinjlim_{m \geq n} I^nM/I^mM.$$

But this is $\ker(\hat{M} \to M/I^nM)$. On the other hand, taking $(f_1, \ldots, f_r)\colon \hat{M}^{\oplus r} \to I^n\hat{M}$ generates $I^n\hat{M}$. Thus $\hat{M}/I^n\hat{M} = M/I^nM$. Taking limits, we are done. $\square$

**Example 3.1.10.** Let $R = k[x_1, x_2, \ldots]$ and $I = (x_1, x_2, \ldots)$. Then $\hat{R}$ is the submodule of formal power series consisting of those which have finitely many in each degree. Look at $\mathfrak{m} := \ker(\hat{R} \to k)$ given by taking the constant term. Claim: $\mathfrak{m} \neq I\hat{R}$. Then

$$f := x_1 + x_2x_3 + x_4x_5x_6 + \cdots \notin I\hat{R}$$

because if $f = x_1g_1 + \cdots + x_ng_n$ for some $g_1, \ldots, g_n \in \hat{R}$, then modding out by $(x_1, \ldots, x_n)$ sends $x_1g_1 + \cdots + x_ng_n$ to 0, but not $f$.

## 3.2 Completions of Noetherian rings

All completions are $I$-adic completions in this section.

**Lemma 3.2.1.** *Let $R$ be a ring and $I \subset R$ an ideal.*

1. *If $N \to M$ is an injective homomorphism of finite $R$-modules, then $\hat{N} \to \hat{M}$ is injective.*
2. *If $M$ finite, then $\hat{M} = M \otimes_R \hat{R}$.*

*Proof.* The kernel of the map $N/I^{n+c}N \to M/I^{n+c}M$ is $N \cap I^{n+c}M$, which by Artin–Rees is contained in $I^nN$. Hence if we look at the square

$$\begin{array}{ccc} N/I^nM & \longrightarrow & M/I^nM \\ \uparrow & & \uparrow \\ N/I^{n+c}N & \longrightarrow & M/I^{n+c}N \end{array}$$

we see that anything in the kernel is actually 0 in the inverse system.

Choose a presentation $0 \to K \to R^{\oplus t} \to M \to 0$. Then by a previous lemma,

$$R^{\oplus t} \otimes_R \hat{R} = \hat{R}^{\oplus t} = \widehat{R^{\oplus} t} \to M$$

factors through $M \otimes_R \hat{R}$ so that the map $M \otimes_R \hat{R} \to \hat{M}$ is still surjective. $\square$

*Remark.* We collect all the facts we know so far about the exactness of completion:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{K} >>> \hat{R}^{\oplus t} & \longrightarrow & \hat{M} & \longrightarrow & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & K \otimes_R \hat{R} & \longrightarrow & \hat{R}^{\oplus t} & \longrightarrow & M \otimes_R \hat{R} & \longrightarrow & 0 \end{array}$$

where the left and right vertical maps are surjective by a previous lemma, and the middle map is an isomorphism. To show the bottom sequence is exact (as desired), it suffices to show $\hat{K} = \ker(\hat{R}^{\oplus t} \to \hat{M})$.

**Lemma 3.2.2.** $\hat{K} = \ker(\hat{R}^{\oplus t} \to \hat{M})$.

*Proof.* Let $(x_n)$ be in the kernel with $x_n \in \operatorname{im}(K/I^nK \to (R/I^n)^{\oplus t})$. Using Artin–Rees, choose $c$ such that $(I^n)^{\oplus t} \cap K \subset I^{n-c}K$. For $n \geq c$, choose $y_n \in K/I^{n+c}K$ mapping to $x_{n+c}$ such that $z_n \equiv y_n \bmod I^nK$. Then

$$z_{n+1} - z_n \bmod I^nK \equiv y_{n+1} - y_n \bmod I^nK$$

and $y_{n+1} - y_n \in (I^{n+c})^{\oplus t}$ by construction. Hence $z_{n+1} - z_n \in I^nK$ by choice of $c$. So $(z_n) \in \hat{K}$ maps to $(x_n)$. ☐

*Remark.* This proof is not the best way to think about the situation. Rather, interpret Artin–Rees as saying that for finite modules, given a module with the $I$-adic topology, the induced topologies on submodules and quotients are also $I$-adic topologies.

**Lemma 3.2.3.** *Let $R$ be a Noetherian ring, and $I \subset R$ an ideal.*

1. *$R \to \hat{R}$ is flat.*
2. *The functor $M \mapsto \hat{M}$ is exact on the full subcategory of finite modules.*

*Proof.* A previous lemma says $\hat{M} = M \otimes_R \hat{R}$, so (2) $\implies$ (1). We know $M \mapsto \hat{M}$ preserves injectivity, by the previous lemma. Hence we are done. ☐

**Lemma 3.2.4.** *If $(R, \mathfrak{m})$ is a local Noetherian ring, then the complete local ring $\hat{R} = \varprojlim_n R/\mathfrak{m}^n$ is Noetherian and is (faithfully) flat over $R$.*

## 3.3 Cohen structure theorem

**Definition 3.3.1.** Let $\kappa$ be a field of characteristic $p$. A **Cohen ring for** $\kappa$ is a complete discrete valuation ring $\Lambda$ with uniformizer $p$ such that $\Lambda/p\Lambda \cong \kappa$.

**Example 3.3.2.** If $\kappa = \mathbb{F}_p$, take $\Lambda = \mathbb{Z}_p$. If $\kappa$ is perfect, take the (small) Witt ring $W(\kappa)$. For other fields, the construction is annoying.

**Lemma 3.3.3.** *A Cohen ring always exists, and is unique up to non-unique isomorphism.*

**Theorem 3.3.4** (Cohen structure theorem)**.** *Let $(R, \mathfrak{m}, \kappa)$ be a complete (with respect $\mathfrak{m}$) Noetherian local ring.*

1. *If $\operatorname{char}(\kappa) = 0$, then there exists a surjection $\kappa[[x_1, \ldots, x_n]] \to R$.*
2. *If $\operatorname{char}(\kappa) = p$ and $p = 0$ in $R$, then there exists a surjection $\kappa[[x_1, \ldots, x_n]] \to R$.*
3. *If $\operatorname{char}(\kappa) = p$ and $p \neq 0$ in $R$, then there exists a surjection $\Lambda[[x_1, \ldots, x_n]] \to R$ where $\Lambda$ is a Cohen ring for $\kappa$.*

*Proof sketch.* Pick generators $f_1, \ldots, f_r$ for $\mathfrak{m}$ and use the map $x_i \to f_i$. But how do we fit $\kappa$ into $R$? Use infinitesimal deformations to get lifts $\kappa \to R/\mathfrak{m}^n$, and then use that $R$ is complete. ☐

**Definition 3.3.5.** A complete Noetherian local ring $(R, \mathfrak{m})$ is a **complete intersection** iff

$$R \cong \Lambda[[x_1, \dots, x_n]]/(f_1, \dots, f_c)$$

where $f_1, \dots, f_c$ is a regular sequence. A Noetherian local ring $(R, \mathfrak{m})$ is a **complete intersection** iff $\hat{R}$ is a complete intersection ring.