Notes for Intersection theory seminar (Fall 2018)

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Abstract

These are my live-texed notes for (some subset of) the Fall 2018 student reading seminar on intersection theory. Let me know when you find errors or typos. I'm sure there are plenty.

Contents

1 Sep 13 (Song): Rational equivalence

Sorry, no notes!

2 Sep 20 (Noah): Divisors

Today we will learn how to intersect with Cartier divisors D, i.e. we will define an operation

$$D\colon A_k X \to A_{k-1}(X \cap |D|).$$

We will define it on the level of cycles $Z_k X$, and showing it descends to $A_k X$ will be the hard part.

Definition 2.1. A **Cartier divisor** on a variety X is an element of $\Gamma(X, K_X^*/\mathcal{O}_X^*)$. Write $\mathrm{Div}(X)$ for the group of Cartier divisors. The **support** of a Cartier divisor D is

$$|D| := \bigcup \{Z \subset X : \text{local equation of } D \text{ in } Z \text{ is not a unit}\}.$$

Example 2.2. Take the cusp $y^2 - x^3$. The rational function $(y-x)/(y+x)$ is a Cartier divisor.

1. The support of the associated Weil divisor is $[(1,1)] - [(1,-1)]$. In particular, it does not contain the cusp point, because both the numerator and the denominator vanish to the same order there.

2. The support of the Cartier divisor includes the cusp!

2.1 Pseudodivisors

Definition 2.3. Cartier divisors do not necessarily pull back nicely. So we define a **pseudodivisor** on a scheme X as a triple $(\mathcal{L}, Z, s)$ where:

1. $\mathcal{L} \in \operatorname{Pic}(X)$ is a line bundle on X;
2. $Z \subset X$ closed and we think of it as the **support** of the pseudodivisor;
3. s is nowhere vanishing outside Z, i.e. $\mathcal{L}|_{X\setminus Z} \xrightarrow{s} \mathcal{O}_{X\setminus Z}$.

Remark. Let $f\colon V \to X$ be a morphism of varieties. If $f(Y) \not\subset |D|$, we can define f^*D for a Cartier divisor D just by restricting equations. Otherwise no Cartier divisor pullback is defined.

Example 2.4. Given a Cartier divisor D, we get a pseudodivisor

$$(\mathcal{O}_X(D), |D|, 1).$$

Here $|D|$ is the support of the Cartier divisor.

Proposition 2.5. *Let X be a variety. Then any pseudodivisor on X is represented by a Cartier divisor D.*

1. *If $Z = X$, then D is unique up to linear equivalence.*
2. *If $Z \subset X$, then D is unique.*

Definition 2.6 (Operations on pseudodivisors)**.** We define some operations.

1. (Pullback) Let $f\colon Y \to X$ and $D = (\mathcal{L}, |D|, s)$ on X. Then
$$f^*D \coloneqq (f^*\mathcal{L}, f^{-1}(|D|), f^*s).$$
2. (Sum) Given $D_1 = (\mathcal{L}_1, |D_1|, s_1)$ and $D_2 = (\mathcal{L}_2, |D_2|, s_2)$, define
$$D_1 + D_2 \coloneqq (\mathcal{L}_1 \otimes \mathcal{L}_2, |D_1| \cup |D_2|, s_1 \otimes s_2).$$
Consequently, $-D \coloneqq (\mathcal{L}, |D|, s^{-1})$.
3. (Induced cycle class) There is a map $D \mapsto [D] \in A_{n-1}(|D|)$ given by taking any Cartier divisor representing D and using the map
$$\operatorname{Div}(X) \to A_{n-1}(X).$$

2.2 Intersection product

Definition 2.7 (Intersecting with Cartier divisors)**.** Let X be a scheme, D be a pseudodivisor on X, and let $j\colon V \hookrightarrow X$ be a k-dimensional subvariety of X. Define the **intersection product**

$$D \cdot [V] \coloneqq [j^*D] \in A_{k-1}(V \cap |D|).$$

Note that there is no way to define this on the level of cycles.

Proposition 2.8. *Let X be a scheme. Then, on the smallest closed sets where these statements make sense:*

1. *(linear in cycles) if $\alpha, \alpha' \in Z_kX$ and D is a pseudodivisor,*
$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha'$$
in $A_{k-1}((|\alpha| \cup |\alpha'|) \cap |D|)$;

2. *(linear in pseudodivisors) similarly,*

$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha$$

in $A_{k-1}(|D + D'| \cap |\alpha|)$*;*

3. *(projection formula) if* $f\colon X' \to X$ *is proper and* $\alpha \in Z_k X'$*, and* g *is the induced map* $f^{-1}(|D| \cap |\alpha|) \to |D| \cap |\alpha|$*, then*
$$g * (f^* D \cdot \alpha) = D \cdot f_* \alpha;$$

4. *(pullback formula) if* D *is a pseudodivisor on* X *and* $f\colon X' \to X$ *is flat of relative dimension* m*, then*

$$f^* D \cdot f^* \alpha = f^*(D \cdot \alpha);$$

5. *(linear equivalence) if* D *on* X *is such that* $\mathcal{O}_X(D) \cong \mathcal{O}_X$*, then*

$$D \cdot \alpha = 0.$$

Theorem 2.9 (Commutativity)**.** *Let* X *be a variety of dimension* n*, and* D, D' *be Cartier divisors on* X*. Then*
$$D \cdot [D'] = D' \cdot [D]$$
in $A_{n-2}(|D| \cap |D'|)$*.*

Proof. We do this in three (technically four) cases.

1. (Trivial) When $D = D'$, this is obvious.

2. (Algebra) Suppose D, D' are both effective and they intersect properly, i.e. components of the intersection have dimension $\leq n-2$. Take a codimension 2 component $W \subset D \cap D'$. The local ring $A \coloneqq \mathcal{O}_{X,W}$ has a bunch of height 1 primes, which correspond to codimension 1 subvarieties of X containing W. Some of them are components of D, and some are components of D'. Let $a, a' \in A$ be local equations for D and D'. Take one of these codimension 1 components $V \subset X$. Recall that the coefficient of $[V]$ in $[D']$ is
$$\mathrm{ord}_V(a') = \mathrm{length}(A_{\mathfrak{p}}/a' A_{\mathfrak{p}}).$$
The coefficient of $[W]$ in $D \cdot [V]$ is $\mathrm{length}(A/(\mathfrak{p} + aA))$. Putting this together, the coefficient of $[W]$ in $D \cdot [D']$ is
$$\sum_{\mathfrak{p}\text{ height 1}} \mathrm{length}(A_{\mathfrak{p}}/a' A_{\mathfrak{p}})\,\mathrm{length}(A/(\mathfrak{p} + aA)) =: e(a, A/a'A).$$
By symmetry, the coefficient of $[W]$ in $D' \cdot [D]$ is $e(a', A/aA)$. Algebra fact: $e(a', A/aA) = e(a, A/a'A)$.

3. (Blowups) Suppose D and D' do not intersect properly. Then the idea is to do a sequence of blow-ups until they do. We need an invariant of the blow-up procedure so that we know it terminates. Define the **excess**
$$\epsilon(D, D') \coloneqq \max_{\substack{V \subset X \\ \text{codim } 1}} (\mathrm{ord}_V D)(\mathrm{ord}_V D').$$
Note that if D, D' intersect properly, there are no such V contributing to this sum. Write $D \cap D' \coloneqq D \times_X D'$, and let
$$\pi\colon \tilde{X} \to X, \quad \tilde{X} \coloneqq \mathrm{Bl}_{D \times_X D'} X$$
be the blow-up along the intersection, with exceptional divisor E. We know
$$\pi^* D = E + C, \quad \pi^* D' = E + C'$$
for effective Cartier divisors C, C' on $\tilde{X}$. By the lemma below, we can pass from $D \cdot [D']$ to $C \cdot [E]$ or $C' \cdot [E]$, and it remains to induct on $\epsilon(D, D')$ until we hit the second case.

4. (General case) Extend to case of non-effective Cartier divisors by linearity. ☐

Lemma 2.10. *In the blow-up situation of the theorem:*

1. *C and C' are disjoint;*
2. *If $\epsilon(D, D') > 0$, then $\epsilon(C, E)$ and $\epsilon(C', E)$ are both smaller.*

Proof. For (1), pass to the local picture, i.e. let $X = \operatorname{Spec} A$ and a, a' be local equations for D, D'. Then if $I := (a, a')$,

$$\tilde{X} = \operatorname{Proj} \bigoplus I^n \hookrightarrow \mathbb{P}^1 \times X$$

where the embedding is given by a, a' (in degree 1). From this picture, it is clear that C lies over $0 \in \mathbb{P}^1$ and C' lies over $\infty \in \mathbb{P}^1$ and are therefore disjoint.

For (2), take $\tilde{V} \subset \tilde{X}$ of codimension 1 in $C \cap E$ or $C' \cap E$. Since π is an isomorphism outside of $D \cap D'$, we know $\pi(\tilde{V}) \subset X$ is a codimension 1 subvariety of X contained in $D \cap D'$. By the projection formula,

$$\pi_*[E + C] = \pi_*[\pi^* D \cdot \tilde{X}] = D \cdot \pi_*[\tilde{X}] = [D].$$

Hence $\operatorname{ord}_V \geq \operatorname{ord}_{\tilde{V}} E + \operatorname{ord}_{\tilde{V}} C$ (since there may be contributions from other components). But we can choose $\tilde{V}$ so that

$$\epsilon(C, E) = \operatorname{ord}_{\tilde{V}} C + \operatorname{ord}_{\tilde{V}} E.$$

Then we have the sequence of inequalities

$$\begin{aligned}
\epsilon(D, D') &\geq \operatorname{ord}_V D + \operatorname{ord}_V D' \\
&\geq (\operatorname{ord}_{\tilde{V}} E + \operatorname{ord}_{\tilde{V}} C)(\operatorname{ord}_{\tilde{V}} E + \operatorname{ord}_{\tilde{V}} C') \\
&\geq (\operatorname{ord}_{\tilde{V}} E)^2 + \epsilon(C, E) + 0.
\end{aligned}$$

☐

Corollary 2.11. *Given D a pseudodivisor on a scheme X and $\alpha \in Z_k X$ equivalent to 0, then $D \cdot \alpha = 0$.*

Proof. In this case, $\alpha = \operatorname{div}(r)$ for $r \in K(V)^*$ where $V \subset X$ is a dimension $k+1$ subvariety in X. Then we can replace X by V using proper pushforward, i.e. compute

$$D \cdot [\operatorname{div}(r)] \in A_{k-1}(V).$$

By commutativity, this is the same as $\operatorname{div}(r) \cdot [D]$. A previous lemma said this is zero. ☐

Remark. This corollary allows the intersection product to descend to A_k from Z_k. Hence we can now intersect with Cartier divisors in A_k.

Corollary 2.12. *$D \cdot D' \cdot \alpha = D' \cdot D \cdot \alpha$.*

3 Sep 27 (Alex): Chern classes

Let X be a scheme and $L \to X$ be a line bundle. Let V be a k-dimensional subvariety.

Definition 3.1. Let $L|_V = \mathcal{O}_V(C)$ for some Cartier divisor C. Take the associated Weil divisor $[C]$, and define

$$c_1(L) \cap [V] := [C].$$

Note that this is exactly what we defined last time.

Proposition 3.2. *Properties of first Chern classes:*

1. *$c_1(L) \cap -$ is well-defined;*

2. $c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$;

3. *if* $f\colon X' \to X$ *is proper, then*
$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*\alpha;$$

4. *if* $f\colon X' \to X$ *is flat, then*
$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha);$$

5. $c_1(L \otimes L') = c_1(L) + c_1(L')$ *and hence* $c_1(L^\vee) = -c_1(L)$.

Proposition 3.3. *Given a fiber square*
$$\begin{array}{ccc} X' & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$
where $\tilde{f}$ *is proper and* $\tilde{g}$ *is flat, we have*
$$\tilde{f}_*\tilde{g}^*\alpha = g^*f_*\alpha.$$

Definition 3.4 (Segre classes)**.** Setup: $E \to X$ be a vector bundle of rank $r \coloneqq e + 1$. Then we have the tautological bundle
$$\begin{array}{ccccc} \mathcal{O}_E(-1) & \longrightarrow & p^*E & \longrightarrow & E \\ & & \downarrow & & \downarrow{\scriptstyle p} \\ & & \mathbb{P}(E) & \longrightarrow & X. \end{array}$$

Define the **Segre classes**
$$s_i(E) \cap -\colon A_k(X) \to A_{k-i}(X), \quad \alpha \mapsto p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^*\alpha).$$

Proposition 3.5. *The Segre class operations* $s_i(E) \cap -$ *inherit all the properties of the first Chern class* $c_1(E) \cap -$.

Proof. We prove the properties $s_i(E) \cap \alpha = 0$ for $i < 0$, and that $s_0(E) \cap -$ is the identity. Wlog make the reductions that $\alpha = [V]$ is represented by a subvariety of dimension k. By projection formula, $V = X$. We have $A_{k-i}X = 0$ for $i < 0$. Similarly,
$$s_0(E) \cap - = p_*(c_1(\mathcal{O}_E(1))^e \cap p^*[X]) = m[X]$$
and we want to show $m = 1$. By pullback, reduce to the local case $\mathbb{P}(E) = U \times \mathbb{P}^e$ where $i\colon U \hookrightarrow X$ is an open immersion. Note that $\mathcal{O}_E(1)$ has a section whose zero locus is $\mathbb{P}^{e-1} \subset \mathbb{P}^e$. Hence
$$c_1(\mathcal{O}(1)) \cap [U \times \mathbb{P}^e] = [U \times \mathbb{P}^{e-1}]$$
and applying this e times we obtain $[U]$. Hence $m = 1$. ☐

Corollary 3.6. *The map* $p^*\colon A_kX \to A_{k+e}\mathbb{P}(E)$ *is injective.*

Proof. Define $\phi(\beta) \coloneqq p_*(c_1(\mathcal{O}_E(1))^e \cap \beta)$. Then
$$\phi(p^*\alpha) = s_0(E) \cap \alpha = \alpha.$$
Hence ϕ is an explicit inverse to p^*. ☐

Definition 3.7. Let $E \to X$ be a vector bundle. The **Segre polynomial** of E is the formal power series

$$s_t(E) \coloneqq \sum s_i(E)t^i.$$

Define the **Chern polynomial** of E to be the formal power series given as the inverse, i.e. it satisfies

$$c_t(E)s_t(E) = 1.$$

Remark. Sanity check: we need to make sure c_1 agrees with our previous definition. Namely, $c_1(L) = -s_1(L)$ and we check

$$c_1(L) \cap \alpha = -s_1(L) \cap \alpha = -p_*(\mathcal{O}_L(1) \cap p^*\alpha) = -c_1(L^\vee) \cap \alpha = c_1(L) \cap \alpha.$$

This is because $\mathbb{P}(L) = X$ and hence $p\colon \mathbb{P}(L) \to X$ is an isomorphism.

Theorem 3.8. *Properties of Chern classes:*

1. $c_i(E) = 0$ *for* $i > \operatorname{rank} E$*;*
2. *they are commutative;*
3. $f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha$*;*
4. $c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$*;*
5. *given a short exact sequence* $0 \to E \to F \to G \to 0$*,*

$$c_t(F) = c_t(E)c_t(G);$$

6. *if* $L \to X$ *is a line bundle* $L = \mathcal{O}(D)$*, then* $c_1(L) \cap [X] = [D]$*.*

Proof. We already know (2), (3), (4), and (6), because we inherit those properties from Segre classes. The rest follow from the splitting construction below. ☐

Lemma 3.9 (Splitting construction)**.** *Given* $E \to X$*, there exists a flat morphism* $f\colon X' \to X$ *such that:*

1. $f^*\colon A_*X \to A_*X'$ *is injective;*
2. *there is a filtration*

$$f^*E = E_{e+1} \supset E_e \supset \cdots \supset E_1 \supset E_0$$

with line bundles $L_i \coloneqq E_i/E_{i-1}$*.*

We call f *a splitting of* E*.*

Proof. We use $f\colon \mathbb{P}(E) \to X$; we already know $f^*\colon A_*X \to A_*\mathbb{P}(E)$ is injective. There is also a nice line subbundle $\mathcal{O}_E(-1) \subset f^*E$. Mod out by it to reduce the rank by one, and induct. ☐

Proposition 3.10 (Fulton lemma 3.2)**.** *The splitting construction gives that*

$$c_t(E) = \prod_{i=1}^{r}(1 + c_1(L_i)t),$$

on the X' *given by the splitting construction. (There is an implicit* f^* *on the lhs.)*

Proof. Look at $f\colon \mathbb{P}(E) \to X$. Then $\mathcal{O}_E(-1) \subset f^*E$ as bundles on $\mathbb{P}(E)$. Twist by $\mathcal{O}_E(1)$, so that $f^*E \otimes \mathcal{O}_E(1)$ has a non-zero section s. Claim:

$$\prod_{i=1}^{r} c_1(f^*L_i \otimes \mathcal{O}(1)) = 0.$$

Do this inductively. The section s induces a section $\tilde{s}$ of $f^*L_i \otimes \mathcal{O}(1)$. Let Y_r be the zero scheme of $\tilde{s}$. We have a pseudodivisor D_r. If $j\colon Y_r \to X$ is the inclusion, then

$$c_1(f^*L_r \otimes \mathcal{O}(1)) = j_*(D \cap \alpha).$$

Applying properties of the first Chern class,

$$\prod_{i=1}^{r} c_1(p^*L_i \otimes \mathcal{O}(1)) \cap \alpha = j_*(\prod_{i=1}^{r} c_1(j^*(p^*L_i \otimes \mathcal{O}(1))) \cap (D_r \cdot \alpha)).$$

For the remainder of the proof, suppose $\operatorname{rank} E = 2$. Let $\xi \coloneqq c_1(\mathcal{O}_E(1))$. Then

$$c_1(f^*L_i \otimes \mathcal{O}(1)) = c_1(p^*L_i) + c_1(\mathcal{O}(1)) =: c_1^i + \xi.$$

By our previous computation,

$$0 = c_1^1 c_1^2 + c_1^1 \xi + \xi c_1^2 + \xi^2.$$

Hence for all $\alpha \in A_*X$ and for all i, we have

$$f_*(\xi^{1+i} \cap f^*\alpha) + f_*(\xi^{i-1} c_1^1 c_1^2 \cap f^*\alpha) + f_*(\xi^i (c_1^1 + c_1^2) \cap f^*\alpha) = 0.$$

Now rewrite in terms of Segre classes and compute that these relations give us

$$(1 + (c_1^1 + c_1^2)t + c_1^1 c_1^2 t)(1 + s_1(E)t + s_2(E)t^2 + \cdots) = 1.$$

Since inverses of power series is unique, it follows that $c_t(E) = \prod_{i=1}^{2}(1+c_1(L_i)t)$. (For details, see Fulton.) $\square$

Definition 3.11. Let X be a scheme and $\pi\colon E \to X$ be a vector bundle of rank $r = e+1$. Let $p\colon \mathbb{P}(E) \to$ be the projectivization and $\mathcal{O}_E(1)$ be the dual of the tautological bundle on $\mathbb{P}(E)$. Define the canonical homomorphism

$$\Theta_E\colon \bigoplus_{i=0}^{e} A_{k-e+i}X \to A_k\mathbb{P}(E), \quad (\alpha_i) \mapsto \sum_{i=0}^{e} c_i(\mathcal{O}(1))^i \cap p^*\alpha_i.$$

Theorem 3.12 (Fulton 3.3)**.** 1. *The pullback $\pi^*\colon A_{k-r}X \to A_*E$ is an isomorphism.*

2. *The homomorphism Θ_E is also an isomorphism. For all $\beta \in A_*\mathbb{P}(E)$, it can be written as*

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i.$$