F19 enumerative geometry seminar notes

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December 10, 2019

1 Xujia (Sep 17): Lifting cobordisms and Kontsevich-type recursions for counts of real curves

Let (X, ω) be a symplectic manifold of dimension $2n$, and let J be an almost-complex structure. Fix $B \in H_2(X)$ and take closed submanifolds $H_1, \ldots, H_\ell \subset X$ such that $2(c_1(X)B + n - 3 + \ell) = \sum_i \operatorname{codim}_{\mathbb{R}} H_i$. Then we can ask about the number of J-holomorphic curves of degree B and genus 0 passing through the H_i. These are usual genus-0 Gromov–Witten invariants

$$\langle [H_1], \ldots, [H_\ell] \rangle_B.$$

This quantity is independent of the choice of (generic) J.

More precisely, define the moduli space

$$\mathcal{M}_\ell(B) := \{u \colon \mathbb{P}^1 \to X : J\text{-holomorphic}, \, [u] = B, \, z_1, \ldots, z_\ell \in \mathbb{P}^1\} / \operatorname{Aut}(\mathbb{P}^1).$$

For today we can think of this as a smooth manifold. It has a natural Gromov compactification

$$\overline{\mathcal{M}}_\ell(B) := \{u \colon C \to X : C \text{ nodal curve}, \, J\text{-holomorphic}, \, [u] = B, \, z_1, \ldots, z_\ell \in C\} / \operatorname{Aut}(C).$$

For today we think of this as a compact smooth manifold. There are evaluation maps

$$\begin{aligned} \overline{\mathcal{M}}_\ell(B) &\xrightarrow{\text{ev}} X^\ell \\ [(u, z_1, \ldots, z_\ell)] &\mapsto (u(z_1), \ldots, u(z_\ell)). \end{aligned}$$

To compute these Gromov–Witten invariants, we use the WDVV relations (Kontsevich '92, Ruan–Tian '93). These say that

$$\#D(H_1, H_2 | H_3, H_4) = \#D(H_1, H_3 | H_2, H_4)$$

where $D(H_1, H_2 | H_3, H_4)$ is the divisor consisting of curves with a single node, with one component passing through H_1, H_2 and another passing through H_3, H_4. To see this, recall that

$$\overline{\mathcal{M}}_{0,4} = \{(z_1, z_2, z_3, z_4) \in \mathbb{P}^1\} / \operatorname{Aut}(\mathbb{P}^1) \cong \mathbb{P}^1$$

given by the cross-ratio. Consider the points $\sigma_0, \sigma_\infty \in \overline{\mathcal{M}}_{0,4}$ corresponding to $0, \infty \in \mathbb{P}^1$. They are linearly equivalent divisors, i.e.

$$[\sigma_0] = [\sigma_\infty] \in H_0(\overline{\mathcal{M}}_{0,4}).$$

Any moduli $\overline{\mathcal{M}}_\ell(B)$ has a forgetful map

$$f \colon \overline{\mathcal{M}}_\ell(B) \to \overline{\mathcal{M}}_{0,4}$$

given by forgetting the map and all but the first 4 marked points. Hence this relation on $\overline{\mathcal{M}}_{0,4}$ pulls back via f:

$$[f^{-1}(\sigma_0)] = [f^{-1}(\sigma_\infty)] \in H_*(\overline{\mathcal{M}}_\ell(B)).$$

This is exactly the WDVV relation.

Today we will discuss a similar story for the real case. Let (X, ω, ϕ) be a **real symplectic manifold** of dimension $2n$. This means (X, ω) is a symplectic manifold and $\phi\colon X \to X$ is an anti-symplectic involution, i.e.

$$\phi^2 = \mathrm{id}, \quad \phi^*\omega = -\omega.$$

We usually refer to ϕ as the **conjugation**, because typical examples include $\mathbb{P}^n$ with ϕ as the actual conjugation map. Note that the almost holomorphic structure J behaves as

$$\phi^* J = -J.$$

Let X^ϕ denote the fixed locus of ϕ. We take $B \in H_2(X)$ and $p_1, \ldots, p_k \in X^\phi$ and $H_1, \ldots, H_\ell \subset X$. Now we can count **real rational curves** $C \subset X$, namely curves with $\phi(C) = C$. (In general, "real" means ϕ-invariant.) Now if we impose the usual dimension restriction $c_1(X)B + n - 3 = k(n-1) + \sum_i(\operatorname{codim}_{\mathbb{R}} H_i - 2)$, then we can count curves like we did in the complex case. However this is not in general an invariant, because in the real setting we don't have Bézout's theorem and so on.

Theorem 1.1 (Welschinger '03, '05, Solomon '06)**.** *If $n = 2$, or $n = 3$ and X^ϕ is oriented, then the number of degree B real rational curves passing through $H_1, \ldots, H_\ell, p_1, \ldots, p_k$, counted with appropriate signs, is an invariant of J, p_i, and $H_i \in [H_i] \in H_*(X - X^\phi)$.*

Now we can ask: are there WDVV relations in this real case?

Theorem 1.2 (Xujia Chen '18)**.** *In $n = 2$, the relations for Welschinger's invariants proposed by Solomon '07 hold.*

Theorem 1.3 (Chen–Zinger '19)**.** *In $n = 3$, similar relations hold if (X, ω, ϕ) has some symmetry, e.g. $\mathbb{P}^3$ with real hyperplane reflection.*

In the case of $\mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}^1$ these WDVV relations reduce to formulas in Alcalado's '17 thesis. For e.g. $\mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}^1, (\mathbb{P}^1)^3$, and real blow-ups of $\mathbb{P}^2$, we get complete recursion formulas.

Idea of proof. The usual conjugation map $\operatorname{conj}\colon \mathbb{P}^1 \to \mathbb{P}^1$ sends $z \to \overline{z}$, with fixed locus $\mathbb{RP}^1$. The real moduli is

$$\mathbb{R}\mathcal{M}_{k,\ell}(B) \coloneqq \left\{ u\colon \mathbb{P}^1 \to X : \begin{array}{c} J\text{-hol },\, [u] = B,\, u \circ \operatorname{conj} = \phi \circ u, \\ x_1, \ldots x_k \in \mathbb{RP}^1,\, z_1^\pm, \ldots, z_\ell^\pm \in \mathbb{P}^1,\, z_i^- = \operatorname{conj}(z_i^+) \end{array} \right\} / \operatorname{Aut}(\mathbb{P}^1).$$

This is, again, not compact, but it has a Gromov compactification $\mathbb{R}\overline{\mathcal{M}}_{k,\ell}(B)$ by replacing $\mathbb{P}^1$ with nodal curves.

We can lift relations from $\mathbb{R}\overline{\mathcal{M}}_{1,2}$ and $\mathbb{R}\overline{\mathcal{M}}_{0,3}$. However the problem is that the spaces

$$\mathbb{R}\overline{\mathcal{M}}_{k,\ell}(B), \quad X^\phi, \quad \mathbb{R}\overline{\mathcal{M}}_{1,2}$$

may not be orientable. In Solomon's thesis, he showed that $\operatorname{ev}|_{\mathbb{R}\mathcal{M}_{k,\ell}(B)}$ is *relatively* orientable, i.e. pullback of the first Stiefel–Whitney class of the target equals that of the domain. The relative orientation extends through some codimension-1 strata, but not all of them.

Let $\Gamma \subset \mathbb{R}\overline{\mathcal{M}}_{1,2}$ (or $(0,3)$) consist of curves such that $z_2^\pm$ or $z_3^\pm$ coincide with $z_1^\pm$. Georgieva–Zinger '13 shows that Γ bounds in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$. Take $Y \subset \mathbb{R}\overline{\mathcal{M}}_{1,2}$ such that $\partial Y = \Gamma$ and

$$\mathbb{R}\overline{\mathcal{M}}_{k,\ell}(B) \xrightarrow{\operatorname{ev} \times f} (X^\phi)^k \times X^\ell \times \mathbb{R}\overline{\mathcal{M}}_{1,2} \hookleftarrow (p_1 \times \cdots \times H_\ell) \times Y.$$

Let C denote the constraints $p_1 \times \cdots \times H_\ell$. Then

$$\mathbb{R}\overline{\mathcal{M}}_{k,\ell}(B) \cdot (C \times \Gamma) = \pm 2(\text{bad strata}) \cdot (C \times Y). \tag{1}$$

This comes from cutting $\mathbb{R}\overline{\mathcal{M}}$ open along the bad strata. Call the resulting space $\mathbb{R}\hat{\mathcal{M}}$. It is relatively orientable now. Then

$$\partial(\mathbb{R}\hat{\mathcal{M}} \cdot (C \times Y)) = (\partial\mathbb{R}\hat{\mathcal{M}}) \cdot (C \times Y) \pm \mathbb{R}\hat{\mathcal{M}} \cdot \partial(C \times Y).$$

The lhs is 0, and the rhs gives the desired formula (1).

Finally, the lifted relations (1), with splitting relations, give the desired relations between Welschinger's invariants. Splitting works as follows. A dimension count together with a good choice of Y shows that, for all bad strata contributing to the rhs of (1),

1. the first bubble is rigid,
2. the condition "cut out by Y" is the same as specifying the position of the node on the first bubble.

Hence the count of such nodal curves is exactly the count of first bubbles, and the count of second bubbles with one additional real point specifying the position of the node.

As for the lhs, when $n = 2$ the splitting is immediate. When $n = 3$, a dimension count gives two cases.

1. The real bubble is rigid. Then

$$\#(\text{nodal}) = \#(\text{first bubble}) \cdot \#(\text{second bubble with curve insertion}).$$

2. The complex bubble is rigid, and the real bubble passes through it. Then

$$\#(\text{nodal}) = \#(\text{complex bubble}) \cdot \#(\text{real bubble passing through } C_1 \sqcup \cdots \sqcup C_N).$$

   This case is why we need to assume the symmetry property in the theorem. Assume there is a $G \subset \mathrm{Aut}(X, \omega, \phi)$ such that

$$H_2(X - X^\phi)^G \xrightarrow{\sim} H_2(X).$$

   Then if we take $H_1, \ldots, H_\ell$ to be G-invariant, $C_1 \sqcup \cdots \sqcup C_N$ is also G-invariant. This way we can express the second term above in terms of usual GW invariants. □

2 Nathan (Sep 24): BHK mirror symmetry and beyond

The outline for today will be:

1. LG models and BHK mirror symmetry;
2. GW theory and LG/CY correspondence;
3. other forms of mirror symmetry.

What is a Landau–Ginzburg model? Mathematically this is called **FJRW theory**. The input is a pair (W, G).

- W is a quasi-homogeneous polynomial, meaning that

$$W(\lambda^{w_1} x_1, \ldots, \lambda^{w_N} x_N) = \lambda^d W(x_1, \ldots, x_N), \quad \gcd(w_1, \ldots, w_N, d) = 1.$$

- W is non-degenerate, meaning that there is an isolated critical point at 0 and there are no terms like $x_i x_j$. This implies there is exactly one choice of weights w_i to make homogeneity work.
- (Calabi–Yau condition) $\sum w_i = d$. This is not strictly necessary for FJRW theory, but we'll assume it for today.

- $G \subset \mathrm{Aut}(W)$, where

$$\mathrm{Aut}(W) \coloneqq \{(g_1, \dots, g_N) \in (\mathbb{Q}/\mathbb{Z})^N : W(e^{2\pi i g_1} x_1, \dots, e^{2\pi i g_N} x_N) = W(x_1, \dots, x_N)\}.$$

- (A-admissible) G must contain the *exponential grading operator* $j_W \coloneqq (w_1/d, \dots, w_N/d)$. This is a condition we *always* need for FJRW theory, since it is the LG A-model.
- (B-admissible) $\sum g_i \equiv 0 \bmod \mathbb{Z}$. This is required for a LG B-model. It is necessary for us today.

The output of FJRW theory is the following.

- A state space

$$\mathcal{H}_{W,G} = \bigoplus_{g \in G} H^{\mathrm{middle}}(\mathbb{C}^{N_g}, W_g^{+\infty}, \mathbb{C})^G$$

where N_g is the dimension of the fixed locus of g, and $W_g^{+\infty}$ is the Milnor fiber.

- A moduli space of W-curves

$$\overline{W}_{g,k} \coloneqq \{(C, p_1, \dots, p_k, \mathcal{L}_1, \dots, \mathcal{L}_N, \varphi_1, \dots, \varphi_s) : \mathrm{genus}(C) = g,\ \varphi_i \colon W_i(\mathcal{L}_1, \dots, \mathcal{L}_N) \xrightarrow{\sim} \omega_{\log}(C)\}$$

where we write $W = \sum_{i=1}^s W_i$ as a sum of monomials. This moduli has a "virtual class"

$$[\overline{W}_{g,k}]^{\mathrm{vir}} \in H_*(\overline{W}_{g,k}(W, g_1, \dots, g_n)) \otimes \prod H_{\mathrm{middle}}(\mathbb{C}^{N_g}, W_g^{+\infty}, \mathbb{Q})$$

constructed analytically, not via an obstruction theory.

- FJRW invariants

$$\langle \tau_{\ell_1}(\alpha_1), \dots, \tau_{\ell_k}(\alpha_k) \rangle_{g,k}^{W,G} \coloneqq \int_{[\overline{M}_{g,k}]} \Lambda_{g,k}^{W,G}(\alpha_1, \dots, \alpha_k) \prod_{i=1}^k \psi_i^{\ell_i}$$

where $\Lambda_{g,k}^{W,G}$ is the factor arising from pushing down from $\overline{W}_{g,k}$ to the moduli of curves $\overline{M}_{g,k}$.

BHK mirror symmetry constructs a "dual" (W^T, G^T) to a pair (W, G), in the situation where W is invertible, meaning that there is the same number of variables as monomials.

- W^T comes from taking the exponent matrix A_W of W, taking its transpose, and getting the resulting polynomial. For example,

$$W = x_1^4 + x_2^2 x_3 + x_3^4 + x_4^8$$

is quasi-homogeneous with respect to $(2, 3, 2, 1; 8)$. Then

$$A_W = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}.$$

The dual polynomial is therefore

$$W^T = x_1^4 + x_2^2 + x_2 x_3^4 + x_4^8,$$

which is quasi-homogeneous with respect to $(2, 4, 1, 1; 8)$.

- G^T comes from

$$G^T \coloneqq \{h \in \mathrm{Aut}(W^T) : h A_W g^T \in \mathbb{Z}\ \forall g \in G\}.$$

Equivalently, this is $\mathrm{Hom}(\mathrm{Aut}(W)/G, \mathbb{C}^\times)$.

Given a pair (W, G), we can get an associated stack $[X_W/\widetilde{G}]$, where

$$X_W \coloneqq \{W = 0\} \subset \mathbb{P}(w_1, \dots, w_N), \quad \widetilde{G} \coloneqq G/\langle j_W\rangle.$$

Then we can look at the GW theory with target $[X_W/\widetilde{G}]$. The output is a state space $H^*_{CR}([X_W/\widetilde{G}], \mathbb{C})$, a moduli space of stable curves, and invariants.

Theorem 2.1 (Chiodo–Ruan, LG/CY correspondence)**.**

$$\mathcal{H}_{W,G} \cong H^*_{CR}([X_W/\widetilde{G}])$$

as bigraded vector spaces.

Conjecture 2.2.

$$FJRW(W, G) \cong GW([X_W/\widetilde{G}]).$$

This has been proved in a handful of cases and different levels. For example, if W is a sum of Fermat polynomials then this correspondence is proved in genus 0. We can compare it to other forms of mirror symmetry.

1. Quintic/mirror quintic:

$$W = \sum_{i=1}^{5} x_i^5, \quad G = \langle j_W\rangle \quad \leftrightarrow \quad Q = \{W = 0\} \subset \mathbb{P}^4$$

$$W^T = \sum_{i=1}^{5} x_i^5, \quad G^T = \langle \begin{matrix} (1/5, 4/5, 0, 0, 0) \\ (1/5, 0, 4/5, 0, 0) \\ (1/5, 0, 0, 4/5, 0) \\ (1/5, 0, 0, 0, 4/5) \end{matrix} \rangle \quad \leftrightarrow \quad M = \{W = 0\}/\widetilde{G},\ \widetilde{G} \cong \mathbb{Z}_5^3$$

where

$$\widetilde{G} = \langle (\xi_5, \xi_5^4, 1, 1, 1), (\xi_5, 1, \xi_5^4, 1, 1), (\xi_5, 1, 1, \xi_5^4, 1)\rangle.$$

In this setting, there is a diagram

$$\begin{array}{ccc} \text{FJRW}(W, G) & \longrightarrow & \text{GW}(Q) \\ \text{mirror theorem}\big\downarrow & & \big\downarrow\text{mirror theorem} \\ \text{B-model}(W^T, G^T) & \xrightarrow[\substack{\text{analytic continuation} \\ U\colon \widetilde{I}^{\text{GW}} = I^{\text{FJRW}} \\ \text{symplectic transformation}}]{} & \text{B-model}(M) \end{array}$$

On the rhs, the invariants involved are

$$J^{\text{FJRW}} = \sum \langle \tau_{\ell_1}(\alpha_1), \dots, \tau_{\ell_N}(\alpha_N)\rangle_{0,k}^{W,G} \frac{t_{\ell_1}^{(\alpha_1)} \cdots t_{\ell_N}^{(\alpha_N)}}{k!}$$

and one can show

$$(J^{\text{GW}} = \frac{I^{\text{GW}}}{I_0^{\text{GW}}}) \text{ after change of vars.}$$

A similar thing holds on the lhs.

2. Mirror symmetry for K3 surfaces. Take a K3 surface X. Then $H^2(X, \mathbb{Z}) = U^3 \oplus E_8^2 =: L_{K3}$ where U is the hyperbolic lattice. Take a *polarization*, i.e. a lattice M with a primitive embedding $M \hookrightarrow \text{Pic}(X)$; there is a moduli of M-polarized K3 surfaces. If there is a decomposition

$$M^\perp_{L_{K3}} = U \oplus M^\vee,$$

we say $X^\vee$ is *mirror* to X if there exists $M^\vee \hookrightarrow \text{Pic}(X^\vee)$.

If we let

$$W := x_0^n + f(x_1, x_2, x_3)$$

and we choose W right, then

$$X_{W,G} = \widetilde{X_W/\widetilde{G}}$$

is a K3 surface. This means for such K3 surfaces we have two candidates for how to do mirror symmetry. It turns out the dual X_{W^T,G^T} is also a K3 surface.

Theorem 2.3. *$X_{W,G}$ and X_{W^T,G^T} are K3 mirror.*

Why? If we take the automorphism

$$\sigma_n \colon (x_0, x_1, x_2, x_3) \mapsto (\xi_n x_0, x_1, x_2, x_3)$$

then we can look at the invariant lattice

$$S(\sigma_n) = \{x \in H^2(X, \mathbb{Z}) : \sigma_n^* x = x\}$$

and one can show $S(\sigma)^\vee = S(\sigma_n^T)$.

Finally we can look at non-abelian LG models. Take W as before, but now

$$G = H \cdot K, \quad H \subset \mathrm{Aut}(W) \subset \mathrm{GL}_n(\mathbb{C}), \quad K \in A_n(\text{permuting vars}).$$

Then there is a mirror $(W^\vee, G^\vee)$, where

$$W^\vee := W^T$$

as before, but

$$G^\vee := H^T \cdot K.$$

This *should* be mirror symmetry. The first indication this should work is an isomorphism of state spaces.

Theorem 2.4.

$$(\mathcal{H}_{W,G})_{untwisted} \cong (B_{W^\vee,G^\vee})_{narrow\ diagonal}$$
$$(\mathcal{H}_{W,G})_{narrow\ diagonal} \cong (B_{W^\vee,G^\vee})_{untwisted}.$$

This allows us to think about e.g. GW theory of *quotients* of the quintic, or symmetric products of elliptic curves.

3 Fenglong (Oct 01): Structures of relative Gromov–Witten theory

Let X be a smooth projective variety. In GW theory we are interested in counting curves in X. To define invariants, we consider the moduli space of stable maps

$$\overline{\mathcal{M}}_{g,n,d}(X) := \{f \colon (C, p_i) \to X : \text{genus } g \text{ degree } d \text{ and } n \text{ markings}\}.$$

Let $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(X) \to X$ be the i-th evaluation map $p_i \mapsto f(p_i)$. To define invariants we use ev_i to pull back some cohomology classes $\gamma_i \in H^*(X)$ and cap with the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X)$. There are also classes $\psi := c_1(\mathcal{L}_i)$ where $\mathcal{L}_i$ is the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fibers are the cotangent spaces of the i-th marking. We define GW invariants

$$\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i) \rangle_{g,n,d}^X := \int_{[\overline{\mathcal{M}}]^{\mathrm{vir}}} \prod \psi_i^{a_i}\, \mathrm{ev}_i^*(\gamma_i).$$

Such invariants have many structural properties. For example, we can define quantum cohomology, WDVV relations, topological recursion relations, Givental's formalism, Virasoro constraints, and cohomological field theory.

To define relative GW invariants we need a smooth divisor $D \subset X$, and we count curves with tangency conditions along the divisor. The relevant moduli space is

$$\overline{\mathcal{M}}_{g,\mathbf{k},n,d}(X,D) \coloneqq \{f\colon (C,p_i) \to X : \text{genus } g \text{ degree } d \text{ and } n \text{ interior markings}\}$$

with relative condition $\mathbf{k} = (k_1, \ldots, k_m)$ where $k_i \in \mathbb{Z}_{\geq 0}$ and $\sum k_i = \int_d [D]$. There are now additional evaluation maps $\mathrm{ev}_j \colon \overline{\mathcal{M}} \to D$ for $1 \leq j \leq m$, using which we can pull back $\delta_i \in H^*(D)$. Then invariants are

$$\langle \prod \tau_{a_i}(\delta_i) | \prod \tau_{a_i}(\gamma_i) \rangle^{(X,D)}_{g,\mathbf{k},n,d} = \int_{[\overline{\mathcal{M}}]^{\mathrm{vir}}} \prod \psi_i^{a_i} \,\mathrm{ev}_i^*(\delta_i) \prod \psi_i^{a_i} \,\mathrm{ev}_i^*(\gamma_i).$$

The virtual dimension constraint of this relative moduli space is

$$\begin{aligned} \mathrm{vdim} &= (1-g)(\dim_{\mathbb{C}} X - 3) + \int_d c_1(T_X) - \int_d [D] + m + n \\ &= \frac{1}{2} \sum \deg(\gamma_i) + \frac{1}{2} \sum \deg(\delta_i) + \sum a_i. \end{aligned}$$

Question: how to obtain structural properties of relative GW theory? (E.g. relative quantum cohomology, WDVV, topological recursion, Givental's formalism, mirror theorem.) Answer: using stacks to impose tangency conditions. This was originally part of C. Cadman's dissertation.

The specific stack we will use is the r-th **root stack** $X_{D,r}$ of X along D, where $r \in \mathbb{Z}_{>0}$. Geometrically, $X_{D,r}$ is smooth away from D and has μ_r stabilizer along D. Then consider the evaluation map

$$\overline{\mathcal{M}}_{g,\mathbf{k},n,d}(X_{D,r}) \xrightarrow{\mathrm{ev}_i} IX_{D,r}.$$

Here $IX_{D,r}$ is the inertia stack. In general inertia stacks are complicated, but in this case

$$IX_{D,r} = X \cup \underbrace{\coprod_{i=0}^{r-1} D}_{\text{twisted sectors}}.$$

The twisted sectors are labeled by **ages** $1/r, 2/r, \ldots, (r-1)/r$. The ages of the divisor evaluations $\mathrm{ev}_j \colon \overline{\mathcal{M}} \to D$ now have ages k_i/r. We can define GW invariants for $X_{D,r}$ as well. The virtual dimension constraint is

$$(1-g)(\dim_{\mathbb{C}} X - 3) + \int_d c_1(T_X) - \int_d [D] + \frac{\int_d [D]}{r} + m + n - \sum \text{ages} = \frac{1}{2} \sum \deg(\gamma_i) + \frac{1}{2} \sum \deg(\delta_i) + \sum a_i.$$

Remark. Not all orbifold GW invariants of $X_{D,r}$ are defined using relative data; there are extra orbifold invariants which will be very important later.

Q: what is the relation between the orbifold invariant $\langle \cdots \rangle^{X_{D,r}}$ and the relative invariant $\langle \cdots \rangle^{(X,D)}$? Before we talk about this, we state some facts.

1. (Cadman) $\overline{\mathcal{M}}_{g,\mathbf{k},n,d}(X_{D,r})$ provides an alternative compactification of the space of relative stable maps.
2. (Maulik–Pandharipande) Relative invariants can be determined from absolute invariants of X and D. (Tseng–You) The orbifold invariants can also be determined from this data, with the extra data of r.
3. $-K_{X_{D,r}} = -K_X - D + D/r$. As $r \to \infty$ note that this becomes $-K_X - D$.

What is the precise relation? In genus-0, they are equal (Abramovich–Cadman–Wise 2017, arXiv 2010):

$$\langle \cdots \rangle^{(X,D)} = \langle \cdots \rangle^{X_{D,r}}, \quad r \gg 1.$$

This ACW result includes all relative GW invariants of (X, D) but *not* all orbifold GW invariants of $X_{D,r}$. Orbifold GW invariants may involve large ages $1 - k_i/r$, which are not included in this relation. It would be nice if the ACW relation holds in higher genus as well, but in $g = 1$ there is a counterexample given by Maulik: take

$$X = E \times \mathbb{P}^1, \quad D = E_0 \cup E_\infty$$

and compute for $[f] \in H_2(E)$ the invariants

$$\langle \rangle^{(X,D)}_{1,0,[f]} = 0, \quad \langle \rangle^{X_{(D_0,r_0),(D_\infty,r_\infty)}}_{1,0,[f]} = r_0 + r_\infty.$$

So what is the relation in higher genus? The invariant $\langle \rangle^{X_{D,r}}$ is a function of r. As $r \to \infty$ this is constant in $g = 0$. For general g, it is a *polynomial* in r.

Theorem 3.1 (Tseng–You)**.** *The relation is that*

$$[\langle \rangle^{X_{D,r}}]_{r^0} = \langle \rangle^{(X,D)},$$

i.e. the constant term of the polynomial.

Proof. The ACW proof doesn't generalize to higher genus. Instead, the idea is as follows.

1. The comparison is local over D, so we can degenerate to the normal cone $Y := \mathbb{P}(N_D \oplus \mathcal{O}_D)$, by

$$X \rightsquigarrow X \cup_D Y, \quad X_{D,r} \rightsquigarrow X \cup_D Y_{(D_\infty,r)}.$$

   Here $Y_{D_\infty,r}$ is the r-th root stack of Y along ∞. By the degeneration formula,

$$\begin{aligned} \langle \rangle^{X_{D,r}} &= \sum \langle \rangle^{(X,D)} \langle \rangle^{(Y_{D_\infty,r},D_0)} \\ \langle \rangle^{(X,D)} &= \sum \langle \rangle^{(X,D)} \langle \rangle^{(Y,D_0 \cup D_\infty)}. \end{aligned}$$

   The sums are exactly the same, so it remains to compare the orbifold and relative invariants for Y.

2. Localize with respect to the $\mathbb{C}^\times$ scaling on the $\mathbb{P}^1$ fibers of Y. There are two fixed loci: one orbifold, one relative. The key idea is from double ramification cycles (Janda–Pandharipande–Pixton–Zvonkine, 2018), and is a polynomiality in the pushforward of the resulting Hurwitz–Hodge classes to $\overline{\mathcal{M}}_{g,n}$. What remains is a rubber integral.

In genus 0, we can actually get this without needing polynomiality. For higher genus we need to take the constant term, which removes some extra contributions from the orbifold side. □

Question: do orbifold invariants with large ages stabilize? What is the relation with relative invariants? The answer to the first question is *no*! Actually $\langle \rangle^{X_{D,r}}$ depends on r and tends to 0 when $r \to \infty$. We should instead set

$$m_- := \#(\text{large ages}),$$

and look at $r^{m_-} \langle \rangle^{X_{D,r}}$ (Fan–Wu–You). In genus 0,

$$r^{m_-} \langle \rangle^{X_{D,r}} = \langle \rangle^{(X,D)} \text{ with negative contact orders.}$$

For positive contact orders, curves in the rubber satisfy kissing conditions with the original curve ramifying at the divisor D. If instead the curve in the rubber ramifies at D with no corresponding ramification in the original curve, it is a negative contact order. The m_- is the number of such negative contact orders. (This is expected to be related to the *punctured* GW invariants of the Gross–Seibert program, but the precise relation is unknown.)

Using this, we get topological recursion relations, WDVV, relative quantum cohomology, Givental's formalism, and Virasoro constraints directly from the orbifold GW theory. The underlying state space $\mathcal{H} = \bigoplus_{a\in\mathbb{Z}} \mathcal{H}_a$ has $\mathcal{H}_0 = H^*(X)$, for the untwisted sector, and $\mathcal{H}_a = H^*(D)$ when $a \neq 0$, for the twisted sectors. Write $[\gamma]_i \in \mathcal{H}_i$. There is a pairing

$$([\alpha]_i, [\beta]_j) = \begin{cases} 0 & i+j \neq 0 \\ \alpha \cup \beta & i+j=0,\, i,j \neq 0 \\ \int_X \alpha \cup \beta & i=j=0. \end{cases}$$

In higher genus, we do have

$$[r^{m_-} \langle\rangle^{X_{D,r}}]_{r^0} = \langle\rangle^{(X,D)} \text{ negative contact orders.}$$

For this, we need to prove that $r^{m_-}\langle\rangle^{X_{D,r}}$ is a polynomial, which is not clear even from preceding results (Fan–Wu–You). This implies a *partial* CohFT structure, which is the usual CohFT structure without the loop axiom.

4 Kostya (Oct 08): Special geometry for invertible singularities and localization in GLSM

The first paper in this direction was by Candelas, de la Ossa, Green, Parker ('91) where they computed periods for the quintic threefold. Another later paper was by Cecotti and Vafa. We'll generalizes to the cases of arbitrary number of deformations, for invertible singularities.

Mathematically, the genus zero B-model corresponds to variation of polarized Hodge structure. Special geometry is a case of this. Let $\mathcal{X} \to \mathcal{M}$ be a family of quasi-smooth CY 3-folds. The example worked out first was the family

$$\mathcal{X} = \{\sum_{i=1}^5 x_i^5 - \phi \prod x_i = 0\}.$$

This is a family over $\mathbb{C}$, parameterized by ϕ. The special fiber at $\phi = 0$ is the Fermat quintic. There is a Kähler metric on $\mathcal{M}$; if ϕ^a are coordinates on $\mathcal{M}$, then the Kähler potential is

$$\exp\left(-K(\phi_a, \overline{\phi}_b)\right) = \int_{X_\phi} \Omega_\phi \wedge \overline{\Omega}_\phi$$

where $\Omega \in \Gamma(\mathcal{M}, H^{3,0}(X_\phi))$ is a family of 3-forms. The Kähler metric itself is

$$G_{ab} = -\partial_{\phi^a}\overline{\partial}_{\phi^b} \log \int_{X_\phi} \Omega_\phi \wedge \overline{\Omega}_\phi = \frac{\int_{X_\phi} \chi_a \wedge \overline{\chi}_b}{\int_{X_\phi} \Omega_\phi \wedge \overline{\Omega}_\phi}.$$

These χ are Beltrami $(2,1)$-forms. In the case of the quintic three-fold,

$$\Omega = \frac{x_5 dx_1 dx_2 dx_3}{\partial W(x,\phi)} = \operatorname{Res}_{x_5=0} \operatorname{Res}_{w=0} \frac{d^5 x}{W(x,\phi)}.$$

Pick a flat basis $\{p^a\}$ of $H^3(X_\phi, \mathbb{Z})$; this is do-able because there is a canonical Gauss–Manin connection. Let $\Omega_\phi = \omega_a(\phi) p^a$ by taking Poincaré duals. Then

$$\int_{X_\phi} \Omega_\phi \wedge \overline{\Omega}_\phi = \omega_a(\phi) C^{ab} \overline{\omega_b(\phi)}$$

where C^{ab} is the intersection matrix $\int p^a \wedge p^b$. Usually we pick $\{p^a\}$ such that C^{ab} is *symplectic*. Then we introduce

$$X^a := \int_{q_a} \Omega_\phi$$

These are projective coordinates on the moduli space. They have dual coordinates

$$\mathcal{F}_a = \int_{q_{a-1-h^{2,1}}} \Omega_\phi.$$

Then we can form the superpotential

$$\mathcal{F}(X) = \frac{1}{2}\sum_a X^a \mathcal{F}_a$$

and say $\mathcal{F}_a = \partial_{X^a}\mathcal{F}(X)$.

Finally, we define Yukawa constants. These are mirror to 3-point GW invariants. In the B-model they are computed as

$$C_{abc} = \int_{X_\phi} \Omega \wedge \partial_{\phi_a}\partial_{\phi_b}\partial_{\phi_c}\Omega = \frac{1}{X^0}\partial_{X^a}\partial_{X^b}\partial_{X^c}\mathcal{F}(X)$$

where ∂_ϕ is the Gauss–Manin connection. Define

$$\widetilde{\mathcal{F}}(X, \overline{X}) = \big(\text{poly in } \overline{X}, X_0\big) + X_0 \cdot \mathcal{F}(X).$$

This is a superpotential defining a Frobenius manifold isomorphic to the quantum cohomology of the mirror.

What is an invertible singularity? They are natural generalizations of the quintic which do not lose many nice properties. They are defined by equations of the type

$$\{W(X,\phi) = 0\} \subset \mathbb{P}^5_{(v_1,\ldots,v_5)}$$

inside *weighted* projective space. Invertible singularities are distinguished by the fact that we can write

$$W(X,\phi) = W_0(X) + \sum e^5\phi_5$$

where W_0 is a sum $\sum\prod_j x_j^{M_{ij}}$ of five monomials such that (M_{ij}) is an invertible matrix. Actually most formulas are written in terms of the *inverse* of this matrix. For the quintic,

$$\exp(-K(\phi,\overline{\phi})) = \sum_{i=0}^{3}(-1)^i \frac{\Gamma((i+1)/5)^5}{\Gamma((4-i)/5)^5}\sum_{n\geq 0}\left(\frac{n+1}{5}\right)_n^5 \frac{\phi^{5n+i}}{(5n+i)!}.$$

We'll see the general formula is similar to this.

To describe $H^3(X_\phi)$, there is a certain subspace $H^3_{\text{poly}})$ which stands for *polynomial* deformations. We have

$$H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

and polynomial deformations live in $H^{2,1}$. In the Fermat case all deformations are polynomial. But in general weighted projective spaces have orbifold singularities. Then the hypersurface may intersect such singularities, and the blow-up may contribute non-trivially to H^3.

The space $H^3_{\text{poly}}(X_\phi)$ is isomorphic to the Jacobi ring $\text{Jac}(W)^{\mathbb{Z}/d}$. The Jacobi ring is the space of infinitesimal deformations of a polynomial:

$$\text{Jac}(W) = \frac{\mathbb{C}[x_1,\ldots,x_5]}{\langle\partial_1 W,\ldots,\partial_5 W\rangle}.$$

Invertible singularities have a huge symmetry group:

$$W_0(\lambda_i^{\sum_j M_{ij}^{-1}} x_i) = \sum \lambda_i \prod_j x_j^{M_{ij}}.$$

If we take the λ_i as exponents of $e^{2\pi i}$, then they generate symmetries. The $\mathbb{Z}/d$ is the *diagonal* subgroup where $\lambda_1 = \cdots = \lambda_5$.

Lemma 4.1 (Key lemma)**.** *For any* $\gamma \in H_3^{poly}(X_\phi)$,

$$\int_\gamma \Omega_\phi = \int_\Gamma e^{-W(x,\phi)}\, d^5x$$

where $\Gamma \in H_5(\mathbb{C}^5, \Re W \ll 0, \mathbb{Q})$ *is some Lefschetz thimble.*

This lemma lets us reduce computations about the CY geometry to oscillating integrals. It is a version of a well-known fact in singularity theory, that period integrals are the same as singularity integrals. Hence we just need to compute

$$\int_{\Gamma_a^+} e^{W(x,\phi)}\, d^5x\, C^{ab} \int_{\Gamma_b^-} e^{-W(x,\phi)}\, d^5x.$$

What are these cycles Γ? The space of *integrands* of such integrals is just $\Omega^5(\mathbb{C}^5)/(d+dW)\Omega^5(\mathbb{C}^5)$. This differential comes from Stokes' formula

$$\int d(e^{W(x,\phi)}\alpha) = \int e^{W(x,\phi)}(d\alpha + dW \wedge \alpha).$$

This space is isomorphic to $\mathrm{Jac}(W)$. A basis $\{[e^a]\}$ of $\mathrm{Jac}(W)$ gives a basis $\{[e^a d^5x]\}$ of this space.

For example, the *whole* quintic has a 101-dimensional deformation

$$\sum x_i^5 + \sum_{a=1}^{101} e^a \phi_a.$$

The corresponding oscillating integral is relatively easy to compute:

$$\int_\Gamma e^{W(x,\phi)} d^5x = \sum_{a=1}^{101} \frac{\phi_1^{m_1} \cdots \phi_{101}^{m_{101}}}{m_1! \cdots m_{101}!} \int e^{W_0(x)} \prod_{i=1}^{5} x_i^{\sum_s m_s s_i}\, d^5x.$$

But then we can write

$$x_1^{b_1} \cdots x_5^{b_5}\, d^5x = \prod_{i=1}^{5} \left(\frac{b_i \bmod 5 + 1}{5} \right) x_1^{b_1 \bmod 5} \cdots x_5^{b_5 \bmod 5}\, d^5x.$$

(Missing notes.)

5 Gus (Oct 22): Coulomb branches and cluster algebras

Fix a reductive group G over $\mathbb{C}$, and a complex representation N. Loosely speaking, the BFN construction associates to (G, N) some sort of "generalized affine Springer theory". The building block of this theory is the affine Grassmannian Gr_G, which is the moduli of pairs $(\mathcal{P}, \varphi)$ where

- $\mathcal{P}$ is a principal G-bundle on $\mathbb{D} \coloneqq \operatorname{Spec} \mathcal{O} \coloneqq \operatorname{Spec} \mathbb{C}[[z]]$, and
- φ is a trivialization of $\mathcal{P}|_{\mathbb{D}^\times}$.

A fruitful way of thinking about Gr_G is as the homogeneous space $G(\mathcal{K})/G(\mathcal{O})$, where $\mathcal{K} \coloneqq \mathbb{C}((z))$. It has a stratification

$$\mathrm{Gr}_G = \bigsqcup_\lambda \mathrm{Gr}^\lambda, \quad \mathrm{Gr}^\lambda \coloneqq G(\mathcal{O})z^\lambda$$

indexed by dominant coweights λ, with

$$\overline{\mathrm{Gr}^\lambda} = \bigsqcup_{\mu \le \lambda} \mathrm{Gr}^\mu .$$

From (G, N), BFN produce a "variety of triples" $\mathcal{R}_{G,N}$ parameterizing *triples* $(\mathcal{P}, \varphi, s)$ where

- $(\mathcal{P}, \varphi) \in \mathrm{Gr}_G$, and
- s is a section of $\mathcal{P} \times_G N$ such that $\varphi \circ s$ is regular at $z = 0$.

More concretely,

$$\mathcal{R}_{G,N} = \{[g], s : [g] \in \mathrm{Gr}_G,\ s \in N[[z]] \cap gN[[z]]\}.$$

It is roughly an affine analogue of the usual Steinberg variety of triples $\mathrm{St} = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$. This presentation of St gives it a convolution structure, and $\mathbb{C}^\times \times G$ act on St. A theorem by Ginzburg shows

$$K_{G\times\mathbb{C}^\times}(\mathrm{St}) \cong \text{affine Hecke}(\mathsf{Weyl}(G)).$$

Similarly, $\mathcal{R}_{G,N}$ carries an action by $G(\mathcal{O}) \ltimes \mathbb{C}_q^\times$, where the $\mathbb{C}_q^\times$ is by loop rotation, and $\mathcal{R}_{G,N}$ has a very similar convolution structure. Hence BFN considered

$$\mathcal{A}_{G,N} \coloneqq K_{G(\mathcal{O})\ltimes\mathbb{C}_q^\times}(\mathcal{R}_{G,N})$$

and showed it is an associative algebra which becomes commutative at $q = 1$. It is called the **Coulomb branch algebra** for (G, N).

To understand $\mathcal{A}_{G,N}$ better, we have a few tools. First, there is a forgetful map $\pi \colon \mathcal{R}_{G,N} \to \mathrm{Gr}_G$ and we can look at

$$\mathcal{R}^\lambda_{G,N} = \pi^{-1}(\overline{\mathrm{Gr}^\lambda_G}).$$

Second, we can use equivariant localization with respect to $T \subset G$ to get an algebra embedding

$$\mathcal{A}_{G,N} \hookrightarrow \mathcal{A}^{\mathrm{loc}}_{T,0}.$$

This Coulomb branch algebra $\mathcal{A}_{T,0}$ is very explicit algebra of rational q-difference operators on $K_T(\mathrm{pt})$. When λ is minuscule, we can actually compute the image of $[\mathcal{O}_{\mathcal{R}^\lambda_{G,N}}]$ under this map.

Theorem 5.1 (Weekes, '19)**.** *Such classes, ranging over all minuscule coweights λ, generate $\mathcal{A}_{G,N}$ as an algebra over $K_G(pt)$, for (G, N) coming from quivers.*

We can add flavor symmetry G_F. If N carries an action by a larger group $\widetilde{G} = G \times G_F$, then we can consider

$$K_{\widetilde{G}(\mathcal{O})\ltimes\mathbb{C}_q^\times}(\mathcal{R}_{G,N}).$$

One place where this shows up is when we add *framing vertices* to the quiver.

Example 5.2. Consider the Jordan quiver, where $G = \mathrm{GL}(d)$ and N is the adjoint rep. Let $\mathbb{C}_t^\times$ scale N. Then

$$\mathcal{A}_{T,0} = \mathbb{C}\langle \lambda_i^\pm, D_i^\pm \rangle_{i=1}^d / D_i\lambda_j = q^{\delta_{ij}}\lambda_j D_i.$$

Then the images of the classes $[\mathcal{O}_{\mathcal{R}^{\omega_i}}]$ are

$$\sum_{\substack{J\subset\{1,\dots,d\}\\ |J|=i}} \prod_{\substack{r\in J\\ s\notin J}} \frac{t\lambda_r - \lambda_s}{\lambda_r - \lambda_s} \prod_{r\in J} D_r,$$

which are Macdonald operators. In this case, $\mathcal{A}_{G,N}$ is exactly spherical DAHA of GL_d.

What is the physical meaning of $\mathcal{A}_{G,N}$? At $q = 1$, the space $\operatorname{Spec} \mathcal{A}_{G,N}^{q=1}$ is a component $\mathcal{M}_C$, called the **Coulomb branch**, in the moduli of vacua of a 4d $N = 2$ G_c-gauge theory on $\mathbb{R}^3 \times S^1$. There are physical expectations about these moduli of vacua. For one, expectations of line operators should give interesting functions on $\mathcal{M}_C$. Gaiotto–Moore–Neitzke explain that these line operators should be described by the *cluster algebra* for the BPS quiver of the theory. The 1-particle Hilbert space decomposes over the charge lattice

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda,$$

where Λ comes with a skew-symmetric $\mathbb{Z}$-bilinear form.

Let's discuss cluster algebras. Suppose we are in exactly this situation, where $\Lambda \cong \mathbb{Z}^d$ has some $\mathbb{Z}$-valued skew form $(\cdot, \cdot)$. Let $\{e_i\}_{i=1}^d$ be a basis in $\mathbb{Z}^d$, and set $\epsilon_{ij} := (e_i, e_j)$. We can let ϵ_{ij} denote multiplicities of arrows in some quiver Q. Associated to the lattice is a quantum torus

$$\mathcal{T}_\Lambda^q := \mathbb{Z}[q^\pm]\Lambda,$$

with multiplication

$$Y_\lambda Y_\mu := q^{(\mu,\lambda)} Y_{\lambda+\mu}.$$

(Importantly, note that this is *basis-dependent*: choosing a different basis gives a different quiver.) For each node $k \in Q$, define a new basis

$$\{e_i'\} = \mu_k(\{e_i\}),$$

which we think of as a **mutation** of the old basis "in the direction k", by

$$e_i' := \begin{cases} -e_k & i = k \\ e_i + [\epsilon_{ik}]_+ e_k & i \neq k \end{cases}$$

where $[a]_+ := \max(0, a)$. To each such mutation we want to associate a birational automorphism of the quantum torus $\mathcal{T}_\Lambda^q$, given by $\operatorname{Ad}_{\Psi^q(Y_{e_k})}$ where

$$\Psi^q(z) := \prod_{n \geq 0} (1 + q^{2n+1} z)^{-1}.$$

Example 5.3. Consider the quiver with one arrow and two edges. Then the quantum torus has generators Y_1, Y_2 with relation

$$Y_1 Y_2 = q^{-1} Y_{e_1+e_2}, \quad Y_2 Y_1 = q Y_{e_1+e_2},$$

and hence $Y_1 Y_2 = q^{-2} Y_2 Y_1$. Then

$$\mu_1(Y_2) = \Psi^q(Y_1) Y_2 \Psi^q(Y_1)^{-1} = Y_2 \frac{\Psi^q(q^2 Y_1)}{\Psi^q(Y_1)} = Y_2(1 + qY_1).$$

The **cluster algebra** $\mathcal{C}_\Lambda = \mathcal{C}_Q \subset \mathcal{T}_\Lambda^q$ is

$$\mathcal{C}_\Lambda := \{a \in \mathcal{T}_\Lambda^q : \mu(a) \in \mathcal{T}_\Lambda^q \text{ for all finite sequences of mutations } \mu\}.$$

This seems like a very restrictive condition and it is not clear that $\mathcal{C}_\Lambda$ contains more than scalars. But Fomin–Zelevinsky proved a "Laurent phenomenon" which implies for each node $i \in Q$ there exists a monomial $A_i \in \mathcal{C}_\Lambda$. In the previous example, $A_1 = Y_2$ and $A_2 = 1/Y_1$. These A_i in general are called **cluster A-variables**.

The relation with the BFN construction is clearest in the case $G = \mathrm{GL}_n$ and $N = 0$. Bezrukavnikov–Finkelberg–Mirkovic (BFM) studied $\mathcal{A}_{\mathrm{GL}_n,0}$ and showed it is isomorphic to an algebra $\mathsf{Toda}_q^{(n)}$, which is the algebra of observables in the quantum difference open Toda system. Then BFZ identified $\mathsf{Toda}_q^{(n)}$ with a cluster algebra $\mathcal{C}_{Q_n}$. There is a functional realization of this cluster algebra acting on Laurent polynomials $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$, by

$$p_j \cdot f \coloneqq f(q^2 x_j).$$

Via this action,

$$H_1^{\mathsf{Toda}} = Y_0 + Y_0 Y_1 + Y_0 Y_1 Y_2 + \cdots .$$

There is a spectral transform (Schrader–Shapiro '18) for the q-Toda system

$$\mathcal{W} \colon \mathsf{Fun}(x_1^{\pm}, \dots, x_n^{\pm}) \xrightarrow{\sim} \mathsf{Fun}^{\mathrm{sym}}(\lambda_1^{\pm}, \dots, \lambda_n^{\pm})$$

such that

$$\mathcal{W} H_k = e_k(\lambda) \mathcal{W}$$

where e_k is the k-th elementary symmetric polynomial. Applied to the frozen vertices in Q,

$$\mathcal{W} Y_{\text{frozen}} = \left(\sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{1 - w_j/w_i} D_i \right) \mathcal{W}$$

which is the $t = 0$ limit of the first Macdonald operator.

Now suppose (G, N) come from a quiver Γ. Construct another (cluster algebra) quiver $Q(\Gamma)$ in the following way:

- take a copy of GL_{d_i}-Toda quiver with no frozen vertex at the bottom, for each node $i \in \Gamma_0$ with $\dim V_i = d_i$;
- for each edge $e \in \Gamma_1$, add a new vertex to $Q(\Gamma)$ which we use to glue the corresponding Toda subquivers.

This recipe manifestly depends on the orientation of Γ. So now we have recipes

$$\begin{array}{ccccc} \Gamma & \longrightarrow & \mathcal{A}_\Gamma & & \\ \big\downarrow & & Q(\Gamma) & \longrightarrow & \mathcal{C}_Q \end{array}$$

producing a Coulomb algebra and a cluster algebra associated to Γ. We know there is an embedding

$$\mathcal{C}_Q \hookrightarrow D^{\mathrm{pol}}(\{X_j^{(i)}\}_{i \le j \le d_i}^{i \in \Gamma_0}).$$

Via the spectral transform, we can look at

$$\begin{array}{ccc} \mathcal{A}_\Gamma & \longrightarrow & D^{\mathrm{rat}}(\{\lambda_j^{(i)}\}) \\ & & \big\downarrow {\scriptstyle \bigotimes_{i \in \Gamma_0} \mathcal{W}^{d_i}} \\ \mathcal{C}_Q & \longrightarrow & D^{\mathrm{pol}}(\{X_j^{(i)}\}_{i \le j \le d_i}^{i \in \Gamma_0}) \end{array}$$

and we want to fill in the lhs arrow. This is not clear because the spectral transform produces denominators. In other words, we need to show $\mathcal{A}_\Gamma$ stays Laurent under any finite sequence of mutations. It is enough to use Weekes' result and show it is true for minuscule monopole operators M_i associated to vertex $i \in \Gamma$. There is a special case we can check by pure calculation: if there is a node $i \in \Gamma$ which is a sink, then the monopole operator M_i goes to a cluster A-variable which is in $\mathcal{C}_Q$ by the Laurent phenomenon. In general, observe that the BFN construction is *independent* of the orientation of the quiver Γ. If we reverse the orientation of an edge in Γ to get Γ', on the cluster algebra side there exists a sequence of mutations

$$\mu_e \colon Q(\Gamma) \to Q(\Gamma').$$

Hence by reversing all arrows incident to a sink i, we can mutate M_i to a cluster A-variable, and therefore $M_i \in \mathcal{C}_Q$.

Theorem 5.4 (Schrader–Shapiro)**.** *For Γ without self loops, $\mathcal{A}_\Gamma \subset \mathcal{C}_{Q(\Gamma)}$.*

6 Catherine (Nov 12): HMS for the complex genus two curve

Here is the statement of HMS in a special case.

- On the symplectic side, take T^2 with $\int_{T^2} \omega = a$. Associated to it is the *Fukaya category* $\mathsf{Fuk}(T^2)$, whose objects are Lagrangians and morphisms are their intersection points.
- On the complex side, take $E = \mathbb{C}/(\mathbb{Z} + ia\mathbb{Z})$. Associated to it is $D^b\mathsf{Coh}(E)$, whose objects are equivalence classes of bounded chain complexes of coherent sheaves, inverting chain maps which induce isomorphisms on homology.

Then HMS says that there is an equivalence

$$\mathsf{Fuk}(T^2) \simeq D^b\mathsf{Coh}(E).$$

The way to do this is as follows. Anything one-dimensional in T^2 is Lagrangian. Take them to be indexed by *slope* s and denote them by ℓ_s. For example, $|\ell_0 \cap \ell_1| = 1$ and $|\ell_0 \cap \ell_2| = 2$. A linear Lagrangian of slope i is mirror to a fixed degree-1 ample line bundle to the i-th tensor power $\mathcal{L}^{\otimes i}$. The homology of the complex

$$HF \coloneqq \frac{CF = \bigoplus_{\cap \text{pts } p} \mathbb{C}p}{\partial = 0}$$

should match with Ext groups. For example,

$$HF(\ell_0, \ell_2) = CF(\ell_0, \ell_2) = \bigoplus_{p \in \ell_0 \cap \ell_2} \mathbb{C}p \xrightarrow{\sim} H^0(\mathcal{L}^2).$$

Counting triangles corresponds to multiplying theta functions, so actually composition agrees as well. This was worked out by Zaslow and Polishchuk.

How does one get a mirror geometrically? For SYZ mirror symmetry, start with a symplectic manifold equipped with Lagrangian torus fibration. Then SYZ produces a mirror as follows. Start with

$$\begin{array}{ccc} L = T^2 & \longrightarrow & M \\ & & \downarrow \\ & & B. \end{array}$$

The mirror has the same base, but parameterizes (L, ∇) where ∇ is a flat connection on $L \times \mathbb{C}$.

What if M has no Lagrangian torus fiber? By work of AAK, the idea is to embed it into something that does, as a critical locus. Specifically, embed M as a hypersurface in a toric or abelian variety V, in which case we denote $H \coloneqq M$. Then consider

$$X \coloneqq \mathrm{Bl}_{H \times 0} V \times \mathbb{C}_y.$$

We can do SYZ on this. The projection onto $\mathbb{C}_y$ has the right critical locus, namely $\mathrm{Crit}(y) = H$. The hypersurface $H \subset V$ is embedded as a genus-2 curve of general type. The abelian variety is

$$V \coloneqq (\mathbb{C}^\times)^2 / \Gamma_B,$$

where Γ_B scales norms of (x_1, x_2). Namely

$$\Gamma_B \coloneqq \mathbb{Z}\langle \gamma_1 \coloneqq \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \gamma_2 \coloneqq \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle$$

and

$$\gamma = (\gamma_1, \gamma_2) \cdot (x_1, x_2) = (\tau^{-\gamma_1} x_1, \tau^{-\gamma_2} x_2).$$

Now take

$$\mathcal{L} := (\mathbb{C}^\times)^2 \times \mathbb{C}/\Gamma_B, \quad \gamma \cdot (x_1, x_2, v) := (\gamma \cdot (x_1, x_2), x^{\lambda(\gamma)}\tau^{\kappa(\gamma)} v)$$

where $\kappa(\gamma)$ is the associated quadratic form on Γ_B, and embed $V \xrightarrow{s} \mathcal{L}$ via the section

$$s := \sum_{\gamma \in \Gamma_B} x^{-\lambda(\gamma)}\tau^{-\kappa(\gamma)} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} x^{-n}\tau^{\cdots}\theta(\cdots).$$

Now look at $X \to \mathbb{C}$ where we view

$$X = \{(x, y, (u, v)) : s(x)v = yu\}.$$

Looking at the structure of the exceptional fiber vs H, actually it turns out there is a semi-orthogonal decomposition

$$D^b\mathsf{Coh}(X) = \langle D^b\mathsf{Coh}(V \times \mathbb{C}), D^b\mathsf{Coh}(H)\rangle.$$

What is the SYZ mirror to X? It is defined by a toric polytope of infinite type

$$\Delta_Y := \{(\xi_1, \xi_2, \eta) \in \mathbb{R}^3 : \eta \geq \mathrm{Trop}(s)(\gamma)\}/\Gamma_B$$

This tropicalization is essentially

$$\max_{\gamma\in\Gamma}\langle\lambda(\gamma), \xi\rangle + \kappa(\gamma).$$

This lattice Δ_Y looks like some honeycomb lattice mod Γ_B. If we take $v_0 := xyz$, it corresponds to η. Then the fibers over v_0 are generically $T^4 = V^\vee$, and special fibers correspond to a hexagon in the honeycomb with edges glued, i.e. $\mathrm{Bl}_{3\text{ pts}}\,\mathbb{CP}^2/\Gamma_B$.

Theorem 6.1 (C)**.** *The right vertical arrow is a fully faithful embedding*

$$\begin{array}{ccc} D^b\mathsf{Coh}(X) & \xrightarrow{\iota^*} & D^b\mathsf{Coh}(\Sigma_2) \\ \subset\downarrow & & \subset\downarrow \\ H^*\mathsf{Fuk}(V^\vee) & \xrightarrow{u_{u\text{-}shape}} & H^*\mathsf{FS}(Y, v_0), \end{array}$$

and the diagram commutes.

Parameterize $V^\vee \ni (\xi_1, \xi_2, \theta_1, \theta_2)$. Just as in Zaslow–Polishchuk, define ℓ_i to be the Lagrangian with a certain slope:

$$\ell_i := \{(\xi_1, \xi_2, i\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}(\xi)\}_{\xi\in T_B}.$$

These are topologically T^2's. Let $\ell_\infty =: t_z$, which corresponds to $(a_1, a_2, \theta_1, \theta_2)$ where $\theta_i \in [0, 2\pi)$ are allowed to vary. It corresponds to a skyscraper sheaf $\mathcal{O}_{z(a)}$. Points in the intersection $\ell_i \cap \ell_j$ are parameterized by

$$(i-j)\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \in \mathbb{Z}^2,$$

i.e. $|\ell_i - \ell_j| = (i-j)^2$. The algebra and triangle counting matches with the structure of $\mathrm{Ext}(\mathcal{L}^i, \mathcal{L}^j) = H^0(V, \mathcal{L}^{j-i})$ (of rank $(i-j)^2$). In other words,

$$D^b\mathsf{Coh}(V) \to \mathsf{Fuk}(V^\vee)$$

is a functor.

To get the picture on Σ_2, we need to restrict to $\Sigma_2 \subset V$. The *Fukaya–Seidel category* $\mathsf{FS}(Y, v_0)$ has objects which are parallel transports along u-shapes of ℓ_i, starting over -1, such that v_0 outside of a compact set is just a union of rays in $\Re v_0 > 0$. Morphisms are defined by *categorical localization*, where one:

1. defines
$$\operatorname{Hom}(K,L) \coloneqq \begin{cases} CF(K,L) & K > L \\ \mathbb{C}\cdot e_L & \\ 0 & \end{cases};$$
2. inverts e_L-quasiisomorphisms $K < L$.

So a morphism $K \to L$ is a roof. To identify $\operatorname{Ext}(\mathcal{L}^i, \mathcal{L}^j)$ with $HF(L_i, L_j)$, consider

$$\begin{array}{ccccccc} \operatorname{Hom}(\mathcal{L}^{i+1}, \mathcal{L}^j) & \xrightarrow{\otimes s} & \operatorname{Hom}(\mathcal{L}^i, \mathcal{L}^j) & \longrightarrow & \operatorname{Hom}(\mathcal{L}^i, \mathcal{L}^j \otimes \mathcal{O}_U) & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & & & & \\ CF_{V^\vee}(\ell_{i+1}, \ell_j) & \xrightarrow{\partial} & CF_{V^\vee}(\ell_i, \ell_j) & \longrightarrow & HF(L_i, L_j) & \longrightarrow & 0. \end{array}$$

The first two vertical arrows are isomorphisms from what we know in the abelian variety case. If one has $\partial \propto s$, then the missing last vertical arrow is also an isomorphism. (Missing notes for the argument why this is true.)

7 Song (Nov 19): Open crepant transformation conjecture for toric CY3 orbifolds

What is the crepant transformation conjecture? It was first proposed by Ruan in '06. Take X to be a Gorenstein (K_X is a line bundle) orbifold and let $Y \to X$ be a crepant resolution. In this setting, the conjecture is that
$$\mathrm{QH}^*(Y;\mathbb{C}) \cong \mathrm{QH}^*_{CR}(X;\mathbb{C}).$$

Example 7.1. If M is a smooth surface, then
$$\pi\colon \operatorname{Hilb}^n M \to \operatorname{Sym}^n M$$
is a crepant resolution.

Example 7.2 ($\mathcal{A}_n$ singularities)**.** When $n = 1$, the resolution is
$$Y \to X \coloneqq \mathbb{C}^2/\mathbb{Z}_2$$
where Y has an extra exceptional curve.

These examples satisfy the *hard Lefschetz condition*, which is some condition on orbifold cohomology involving an automorphism swapping components but preserving ages.

More generally, let X and Y be K-equivalent manifolds/orbifolds. Here *K-equivalence* means there is a common resolution
$$X \xleftarrow{\phi} Z \xrightarrow{\psi} Y$$
such that $\phi^* K_X = \psi^* K_Y$. In this case, Ruan also conjectured that there is an isomorphism
$$\mathrm{QH}^*_{CR}(X;\mathbb{C}) \cong \mathrm{QH}^*_{CR}(Y;\mathbb{C}).$$

Since the quantum product is formulated in terms of GW theory, there is a reformulation of the crepant transformation conjecture in terms of GW theory (Bryan–Graber '09). If $\pi\colon Y \to X$ is a crepant resolution, the statement is that there is an identification of genus-0 GW potentials
$$F_0^X(Q_X, \gamma_X) \coloneqq \sum_{\beta \in \mathsf{Eff}(X)} \sum_{n \geq 0} \langle \gamma_X, \dots, \gamma_X \rangle^X_{0,n,\beta} \frac{Q_X^\beta}{n!}$$
and $F_0^Y(Q_Y, \gamma_Y)$. The identification is not equality, but rather:

1. a degree-preserving linear isomorphism

$$H^*(Y;\mathbb{C}) \cong H^*_{\mathrm{CR}}(X;\mathbb{C});$$

2. a relation between Novikov parameters Q_X and Q_Y;

3. some kind of analytic continuation.

Let's look at this from the perspective of mirror symmetry. On the B-model side there is a Kähler moduli space. We view X and Y as points p_X, p_Y in this space, where p_X is a large radius limit, where the local coordinates q_X is sent to 0. Then the two GW potentials become local solutions to some global system on the Kähler moduli space, called the Picard–Fuchs system or the quantum D-module. It is a flat connection on some bundle on this space. The analytic continuation corresponds to moving from p_X to p_Y.

Known cases: $\mathcal{A}_n$ singularities (Perroni, Bryan et al, Maulik). The $\mathcal{A}_n$ case was fully proved in Coates–Corti–Iritani–Tseng '09. In higher genus, there is a paper by Zhou.

Another formalism is due to Givental. Consider two infinite-dimensional symplectic vector spaces. One is $\mathcal{H}_X$, which is $(H^*_{CR}(X;\mathbb{C})[z,z^{-1}],\omega_X)$, and similarly for X. Inside them are *Lagrangian cones* $\mathcal{L}_X$ and $\mathcal{L}_Y$. Then in Givental's formalism, the crepant resolution conjecture is that there is an isomorphism

$$U\colon \mathcal{H}_X \xrightarrow{\sim} \mathcal{H}_Y$$

which preserves these Lagrangian cones. This can be related to K-theory by Iritani's integral class, to form a square

$$\begin{array}{ccc} \mathcal{H}_X & \xrightarrow{U} & \mathcal{H}_Y \\ \uparrow & & \uparrow \\ K_X & \xrightarrow{FM} & K_Y \end{array}$$

where FM is some Fourier–Mukai transform. In this formalism, Coates–Iritani–Jiang '17 proves the CTC for general toric crepant transformations.

We can now consider the CTC in open GW theory. In usual CTC, we look at maps from *closed* Riemann surfaces. For the open version of CTC, we look at maps from *bordered* Riemann surfaces $(C,\partial C)$ into (X,L) where X is a toric CY3-orbifold and L is a Lagrangian. The geometry of X is specified by an *extended stacky fan*, which consists of:

1. a 3d lattice $N \cong \mathbb{Z}^3$;

2. a simplicial fan $\Sigma \subset N_{\mathbb{R}}$;

3. a surjection $\beta\colon \mathbb{Z}^R \twoheadrightarrow N$, thought of as a selection of vectors $e_i \mapsto b_i$.

If we consider

$$0 \to L \to \mathbb{Z}^R \xrightarrow{\beta} N \to 0,$$

the CY condition is that there exists an affine hyperplane

$$N' = \{v \in N : \langle v,u\rangle = 1\}$$

containing all the b_i.

Example 7.3. Consider $\mathbb{C}^3$, with presentation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Consequently the hyperplane is $N' = \{z = 1\}$. Then on N' the fan becomes a a triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$.

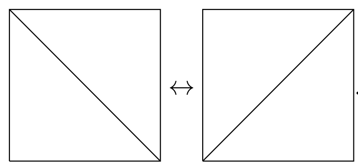
Example 7.4. Consider the $\mathcal{A}_1$ singularity, with $\beta\colon \mathbb{Z}^4 \to N$ given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then on the hyperplane $N' = \{z = 1\}$, we now get a triangle with vertices $(0,0)$, $(1,0)$ and $(0,2)$. Hence one of the T-invariant divisors has a $\mathbb{Z}/2$ stabilizer. The extended stacky fan contains the vector $(0,1)$, which resolves the singularity.

Example 7.5. For the $\mathcal{A}_2$ singularity, picking different β corresponds to picking different partial resolutions. Now there are two different partial resolutions.

Example 7.6. The classical flop is given by



The Lagrangian $\mathcal{L}$ is an *Aganagic–Vafa brane*, defined as follows. The 3-dimensional torus T acting on X has a CY subtorus T'. Consider the maximal compact $T'_{\mathbb{R}} \subset T'$.

1. We require that $T'_{\mathbb{R}}$ preserves $\mathcal{L}$. In other words, $\mathcal{L}$ will be a union of orbits of $T'_{\mathbb{R}}$.
2. We require $\mathcal{L}$ to intersect a unique T-invariant curve $V(\tau)$, where τ is some 2-dimensional cone in X.

If $\mathcal{L}$ has generic stabilizer, then we call it *ineffective*; otherwise it is effective. If $\mathcal{L}$ intersects a compact T-invariant curve, we call it *inner*.

We can define open GW invariants using localization with respect to $T'_{\mathbb{R}}$. For genus zero with one boundary component we get disk invariants $F_{0,1}^{X,\mathcal{L}}$. Then the open CTC says that for a pair $(X_+, \mathcal{L}_+)$ and $(X_-, \mathcal{L}_-)$ of toric CY3 orbifolds related by some crepant transformation, there should be an identification

$$F_{0,1}^{X_+,\mathcal{L}_+} \approx F_{0,1}^{X_-,\mathcal{L}_-},$$

again up to some analytic continuation. This is imprecise, because it could be that a Lagrangian splits into multiple upon some crepant transformation. Cavalieri–Ross '13 computes both sides for the $\mathcal{A}_2$ case and shows they are equal. Brini–Cavalieri–Ross '17 prove it for $\mathcal{A}_n$ using the Givental formalism, where the disk potentials D_X and D_Y are viewed as automorphisms of $\mathcal{H}_X$ and $\mathcal{H}_Y$. In the mirror symmetry formalism, the orbifold equivariant mirror map should send

$$F_{0,1}^{X,\mathcal{L}} \mapsto W^{X,\mathcal{L}}(q,x)$$

where x is the open parameter. From physics, this should correspond to period integrals $\int \check{\Omega}_q$ over the Hori–Vafa mirror $(\check{X}_q, \check{\Omega}_q)$, which via dimensional reduction correspond to some integrals $\int \log y \frac{dx}{x}$ over the mirror curve $C_q \coloneqq \{(x,y) \in (\mathbb{C}^\times)^2 : H_q(x,y) = 0$.

Theorem 7.7 (F.L, F.L.T)**.** *Expand the object* $W^{X,\mathcal{L}} \in H^*_{CR}(V(\tau); \mathbb{C})$ *as*

$$W = W_m 1 + \sum_{j=1}^{m-1} W_j 1_j.$$

Then

$$\left(x\frac{\partial}{\partial x}\right)^2 \begin{pmatrix} W_1 \\ \vdots \\ W_m \end{pmatrix} = Ux\frac{\partial}{\partial x} \begin{pmatrix} \log k_1 \\ \vdots \\ \log k_m \end{pmatrix}$$

where k_i *are solutions to* H_q.

Song's approach is to look at X_+, X_- related by a single wall-crossing. In this setting there are two scenarios: they differ by a flop, or by a (partial) resolution. On the two sides we can relate the disk potentials W^+ and W^-. Via mirror symmetry, the potentials can be related to local coordinates on the mirror curves C_q^+ and C_q^-. The observation is that these two mirror curves fit into a global family of mirror curves C over the Kähler moduli space. Then matching the disk potentials is just analytic continuation of coordinates on C.

8 Renata (Nov 26): Stable maps with p-fields on smooth projective varieties

Let X be a smooth projective variety of dimension n over $\mathbb{C}$, and let $E \to X$ be a rank-r vector bundle. Pick a section $s \in \Gamma(X, E)$ cutting out some $Z \subset X$, such that s intersects the zero section transversely. Then Z is smooth projective of dimension $n - r$. The goal is to relate the Gromov–Witten theory on X to that of Z.

One example of this is when $X = \mathbb{P}^4$ and $E = \mathcal{O}_X(5)$. Then $Z = Z(\sum x_i^5)$ is a quintic threefold. Being able to relate $\mathsf{GW}(Z)$ with $\mathsf{GW}(X)$ means we can use techniques on $\mathbb{P}^4$ on Z.

Some notation: fix a genus g, number of marked points n, and a curve class $\beta \in H_2(X, \mathbb{Z})$. The moduli of stable maps is denoted $\mathcal{X} \coloneqq \overline{\mathcal{M}}_{g,n}(X, \beta)$, with universal curve $\mathcal{C}_\mathcal{X} \to \mathcal{C}$. Let $\mathrm{ev}_\mathcal{X} \colon \mathcal{C}_\mathcal{X} \to X$ be the universal evaluation. Similarly, let

$$\mathcal{Z} \coloneqq \bigcup_{i_*\beta' = \beta} \overline{\mathcal{M}}_{g,n}(Z, \beta').$$

Then $i \colon Z \hookrightarrow X$ induces $i \colon \mathcal{Z} \to \mathcal{X}$. Let $\mathcal{M} \coloneqq \mathcal{M}_{g,n}^{\text{prestable}}$, so that both moduli spaces have forgetful maps to $\mathcal{M}$ compatible with i. Since GW invariants come from integrating against a virtual cycle, we want to compare virtual cycles on $\mathcal{Z}$ and $\mathcal{X}$.

We briefly recall how to construct $[\mathcal{X}]^{\text{vir}}$. Recall that the tangent-obstruction theory is

$$\check{\mathbb{E}}_{\mathcal{X}/\mathcal{M}} \coloneqq R^\bullet \pi_{\mathcal{X},*} \operatorname{ev}_\mathcal{X}^* T_X.$$

The virtual class then arises as

$$[\mathcal{X}]^{\text{vir}} = 0^!_{\underline{E}_{\mathcal{X}/\mathcal{M}}}([\mathcal{M}]),$$

where $\underline{E}_{\mathcal{X}/\mathcal{M}} = h^1/h^0(\check{\mathbb{E}}_{\mathcal{X}/\mathcal{M}})$ is the vector bundle stack containing the normal cone $C_\mathcal{X}\mathcal{M}$. It turns out that the morphism $\mathcal{Z} \to \mathcal{X}$ has tangent-obstruction theory

$$\check{\mathbb{E}}_{\mathcal{Z}/\mathcal{X}} = [0 \to i^* R^0 \pi_{\mathcal{X},*} \operatorname{ev}_\mathcal{X}^* E \to i^* R^1 \pi_{\mathcal{X},*} \operatorname{ev}_\mathcal{X}^* E].$$

From now on, write $\mathcal{E}_\mathcal{X} \coloneqq \operatorname{ev}_\mathcal{X}^* E$. If

$$R^1 \pi_{\mathcal{X},*} \operatorname{ev}_\mathcal{X}^* E = 0,$$

then this is perfect in $[0, 1]$ and $\underline{E}_{\mathcal{Z}/\mathcal{X}} = h^1/h^0(\check{\mathbb{E}}_{\mathcal{Z}/\mathcal{X}}) =: E_d$ is an actual vector bundle. This condition is called *convexity* and happens in genus-0 under good assumptions. Then

$$i_*[\mathcal{Z}]^{\text{vir}} = c_{\text{top}}(\pi_{\mathcal{X},*}\mathcal{E}_\mathcal{X}) \cap [\mathcal{X}]^{\text{vir}},$$

first shown by Kim, Kresch, and Pantev.

In general, $R^1\pi_{\mathcal{X},*}\mathcal{E}_\mathcal{X}$ does not vanish, so we need to take it into account to define the "E-twisted" theory on $\mathcal{X}$. One way to do so is to look at

$$\overline{\mathcal{M}}_{g,n}(X, \beta)^p =: \mathcal{X}^E \coloneqq \operatorname{Spec}_\mathcal{X}(\operatorname{Sym} R^1\pi_*\mathcal{E}_\mathcal{X}).$$

Note that $R^1\pi_*$ commutes with base change, since π is flat of dimension 1. So objects in $\mathcal{X}^E$ are $((C, x), f, p)$ where:

- (C, x) is a marked prestable curve;

- $f\colon C \to X$ is a stable map;
- $p \in H^0(f^*E^\vee \otimes \omega_C)$ is a p**-field**.

Then $\mathcal{X} \subset \mathcal{X}^E$ as the zero locus $\{p = 0\}$. The locus $\mathcal{Z} \subset \mathcal{X}^E$ is no longer a zero section of a vector bundle as before, but rather is a degeneracy locus of a cosection.

There is a construction of cosection-localized virtual classes, due to Kiem and Li. If M is a DM stack with some perfect obstruction theory $\mathbb{E}_M$, a cosection is a map

$$\mathsf{Ob}_M \coloneqq h^1(\check{\mathbb{E}}_M) \to \mathcal{O}_M.$$

We can define cosection-localized intrinsic normal cones and cosection-localized Gysin pullback, and use them to define

$$[M]^{\mathrm{vir}}_\sigma \in A_*(D(\sigma))$$

where $D(\sigma)$ is the degeneracy locus

$$D(\sigma) \coloneqq \{x \in M : \sigma|_x \text{ is not surjective}\}.$$

This cosection-localized virtual class has virtual dimension equal to the virtual dimension of M, and:

- has good functorial properties;
- has a notion of torus localization;
- has a relative construction.

These properties were proved later, in a paper by Chang, Kiem and Li. So we can find $\mathbb{E}_{\mathcal{X}^E}$ and σ such that

$$D(\sigma) = \mathcal{Z} \subset \mathcal{X}^E.$$

This gives a cosection-localized virtual class

$$[\mathcal{X}^E]^{\mathrm{vir}}_\sigma \in A_*(\mathcal{Z}).$$

Theorem 8.1.

$$[\mathcal{X}^E]^{vir}_\sigma = (-1)^{\int_\beta c_1(E) + r(1-g)}[\mathcal{Z}]^{vir}.$$

This has been proved for:

- $(X, E) = (\mathbb{P}^4, \mathcal{O}(5))$ by H.L. Chang and J. Li;
- $(X, E) = (\mathbb{P}^n, \bigoplus_{i=1}^r \mathcal{O}(d_i))$, i.e. complete intersections, by H.L. Chang and M.L. Li;
- for quasimaps in these two settings by Kim and Oh;
- X a smooth projective DM stack (and also quasimaps) by Chen, Janda and Webb.

Proof. We'll give an alternate approach. The idea is to work relative to

$$\mathcal{B} = \mathsf{Bun}^{\mathrm{GL}_r}_{g,n,d} \coloneqq \{((C,x), F) : F \in \mathsf{Bun}(\mathrm{GL}_r) \text{ of degree } d\}$$

where $d = \int_\beta c_1(E)$. On top of $\mathcal{B}$ lies

$$\mathcal{E}_\mathcal{B} \to \mathcal{C}_\mathcal{B} \to \mathcal{B},$$

arising from the pullback from $\mathcal{M}$ of $\mathcal{C}_\mathcal{M}$ and $\mathcal{E}_\mathcal{M}$. There is a map

$$\mathcal{X} \to \mathcal{B}, \quad ((C,x), f) \mapsto ((C,x), f^*E)$$

which only remembers the pullback of E. The diagram

$$\begin{array}{ccc} \mathcal{E}_{\mathcal{X}} & \longrightarrow & \mathcal{E}_{\mathcal{B}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}_{\mathcal{X}} & \longrightarrow & \mathcal{C}_{\mathcal{B}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{B} \end{array}$$

is Cartesian. The perfect obstruction theory comes from studying the automorphisms/deformations of the pair $E \to X$. It turns out these are represented by the Atiyah extension A_E, defined as

$$0 \to \operatorname{End}(E) \to A_E \to T_X \to 0.$$

The map $A_E \to T_X$ sends the Atiyah class to the usual Kodaira–Spencer class. Then

$$\check{\mathbb{E}}_{\mathcal{X}/\mathcal{B}} = R^\bullet \pi_{\mathcal{X},*} \operatorname{ev}_{\mathcal{X}}^* A_E.$$

This is compatible with the usual tangent-obstruction theory $\check{\mathbb{E}}_{\mathcal{X}/\mathcal{M}}$. Hence we can define

$$[\mathcal{X}]^{\mathrm{vir}} \coloneqq 0^!_{\underline{E}_{\mathcal{X}/\mathcal{B}}}([\mathcal{B}])$$

and this will agree with the usual virtual fundamental class on $\mathcal{X}$. Consequently, we can split

$$\check{\mathbb{E}}_{\mathcal{X}^E/\mathcal{B}} = \pi^*_{\mathcal{X}^E/\mathcal{X}} R^\bullet \pi_{\mathcal{X},*}(\operatorname{ev}_{\mathcal{X}}^* A_E \oplus \omega_\pi).$$

Note that $\mathcal{X}^E$ is not proper, so it does not have a virtual class, but we can get a cosection-localized class. The section $s \in H^0(X, E)$ gives a map $\delta_s \colon A_E \to E$. It induces

$$\operatorname{ev}_{\mathcal{X}}^* A_E \to \operatorname{ev}_{\mathcal{X}}^* E = \mathcal{E}_{\mathcal{X}}.$$

Hence there is an induced

$$\operatorname{ev}_{\mathcal{X}}^* A_E \oplus \mathcal{E}_{\mathcal{X}}^\vee \otimes \omega_\pi \to \mathcal{E}_{\mathcal{X}} \oplus \mathcal{E}_{\mathcal{X}}^\vee \otimes \omega_\pi \to \omega_\pi.$$

To get the cosection, we take $R^1\pi_{\mathcal{X},*}$ of this. One can check that the cosection vanishes for $f^*s = 0$ and $p = 0$, which is exactly $\mathcal{Z}$.

Now to compare the two virtual classes, we can use deformation to the normal cone to turn $Z \hookrightarrow X$ into $Z \hookrightarrow C_Z X = E|_Z$. Under this deformation, the moduli $\mathcal{X}$ deforms to $\mathcal{N}$, which is the moduli of stable maps to $E|_Z$. Equivalently, it is the moduli of stable maps to Z with the additional choice of a section $q \in H^0(C, f^*E|_Z)$. Similarly, $\mathcal{X}^E$ deforms to $\mathcal{N}^E$. The perfect obstruction theory of $\mathcal{X}^E$ deforms to a nice

$$\check{\mathbb{E}}_{\mathcal{N}^E/\mathcal{B}} = \pi^* \left(\check{\mathbb{E}}_{\mathcal{Z}/\mathcal{B}} \oplus R^0\pi_{\mathcal{Z},*}(\mathcal{E}_{\mathcal{Z}}^\vee \otimes \omega_{\pi_{\mathcal{Z}}} \oplus \mathcal{E}_{\mathcal{Z}}) \right)$$

where $\mathcal{E}_{\mathcal{Z}} \coloneqq \operatorname{ev}_{\mathcal{Z}}^* E|_Z$. We also need to deform the cosection; it becomes, on points,

$$\begin{aligned} \sigma|_{\mathcal{Z}} \colon H^1(C, f^*E|_Z^\vee \otimes \omega_C) \oplus H^1(C, f^*E|_Z) &\to H^1(\omega_C) = \mathbb{C} \\ (\dot{p}, \dot{q}) &\mapsto \langle p, \dot{q} \rangle + \langle \dot{p}, q \rangle. \end{aligned}$$

It is easy to see then that $D(\sigma) = \mathcal{Z} \subset \mathcal{N}^E$. By deformation-invariance of cosection-localized virtual classes,

$$[\mathcal{X}^E]^{\mathrm{vir}}_\sigma = [\mathcal{N}^E]^{\mathrm{vir}}_\sigma \in A_*(\mathcal{Z}).$$

But there is a natural torus action on $\mathcal{N}^E$ which we can use to study $[\mathcal{N}^E]^{\mathrm{vir}}_\sigma$. Using it, we can compare $[\mathcal{N}^E]^{\mathrm{vir}}_\sigma$ and $[\mathcal{Z}]^{\mathrm{vir}}$.

Let $T := \mathbb{C}^\times$ act on E by scaling fibers (diagonally, with weight 1). It lifts to $\mathcal{N}^E$, with $(\mathcal{N}^E)^T = \mathcal{Z}$. All constructions so far were equivariant. The cosection on the fixed locus is trivial, so torus localization says

$$[\mathcal{N}^E]^{\text{vir}}_\sigma = \frac{[\mathcal{Z}]^{\text{vir}}}{e_T(N^{\text{vir}}_{\mathcal{Z}/\mathcal{N}^E})}.$$

This Euler class can be explicitly computed using our previous description of $N^{\text{vir}}_{\mathcal{Z}/\mathcal{N}^E}$:

$$e_T(N^{\text{vir}}_{\mathcal{Z}/\mathcal{N}^E}) = \frac{e_T(H^0(E|_Z \oplus E|_Z^\vee \otimes \omega_C))}{e_T(H^1(E|_Z \oplus E|_Z^\vee \otimes \omega_C))}.$$

If we set $\mathcal{F} = H^0(E|_Z)$ and $\mathcal{G} = H^1(E|_Z)$, then this is just

$$\frac{e_T(\mathcal{F} \oplus \mathcal{G}^\vee)}{e_T(\mathcal{F}^\vee \oplus \mathcal{G})} = (-1)^{\chi(f^*E|_Z)}.$$

By Riemann–Roch, $\chi(f^*E|_Z) = \int_\beta c_1(E) + r(1-g)$. □

9 Qile Chen (Dec 03): Logarithmic gauged linear sigma model

(This is joint with Felix Janda and Yongbin Ruan.)

Definition 9.1. In log geometry we are interested in pairs $X := (\underline{X}, \mathcal{M}_X)$. Such a pair is a **log scheme** (or log stack) if:

1. $\underline{X}$ is a scheme (or a stack);
2. $\mathcal{M}_X$ is a sheaf of monoids such that
   - there is a structure morphism $\alpha\colon \mathcal{M}_X \to (\mathcal{O}_{\underline{X}}, \cdot)$,
   - $\alpha^{-1}\mathcal{O}^*_{\underline{X}} \xrightarrow{\alpha} \mathcal{O}^*_{\underline{X}}$ is an isomorphism.

A **morphism** $f = (\underline{f}, f^\flat)$ is a morphism $\underline{f}\colon \underline{X} \to \underline{Y}$ along with a morphism of log structures

$$f^*\mathcal{M}_Y \xrightarrow{f^\flat} \mathcal{M}_X$$

compatible with the structure morphisms $f^*\alpha_Y$ and α_X.

The definition is designed in such a way that the log structure keeps track of the boundary, where elements of $\mathcal{M}_X$ are *not* invertible.

Example 9.2. For any scheme, take $X = (\underline{X}, \mathcal{O}^*_{\underline{X}})$, called the **trivial** log structure.

Example 9.3. For any smooth scheme $\underline{X}$ and smooth divisor $D \subset \underline{X}$, the **divisorial** log structure is

$$\mathcal{M} := \{f \in \mathcal{O}_{\underline{X}} : f|_{\underline{X}\setminus D} \in \mathcal{O}^*\}.$$

This forms a sheaf of monoids under multiplication and has a natural injection into $\mathcal{O}_{\underline{X}}$.

The dream of log GLSM is to start with a non-compact (DM-type morphism) $\mathfrak{P}^\circ \to B\mathbb{C}^*_W$ and compactify it as follows. There is a universal line bundle $\mathcal{L}_W \to B\mathbb{C}^\times_W$, and let

$$W\colon \mathfrak{P}^\circ \to \mathcal{L}_W$$

be the superpotential. We would like a space

$$\mathfrak{P} \supset \mathfrak{P}^\circ$$

such that $\mathfrak{P} \to B\mathbb{C}^\times_W$ is proper DM, and also that $\partial\mathfrak{P} := \mathfrak{P} \setminus \mathfrak{P}^\circ$ is nice, namely it defines a log structure on $\mathfrak{P}$. The superpotential W should extend to $W\colon \mathfrak{P} \to \mathcal{L}_W$.

Definition 9.4. A **log R-map** is a commutative triangle

$$\begin{array}{ccc} & & \mathfrak{P} \\ & \overset{f}{\nearrow} & \downarrow \\ \mathcal{C} & \xrightarrow{\omega^{\log}_{\mathcal{C}/S}} & B\mathbb{C}^\times_W \\ \downarrow & & \\ S & & \end{array}$$

where $\mathcal{C} \to S$ is a family of prestable curves.

Without the $B\mathbb{C}^\times_W$, this is just the usual theory of stable maps.

The reality is as follows. Fix a projective DM stack $\mathcal{X}$, a vector bundle $\mathbb{E} := \bigoplus_{i>0} \mathbb{E}_i$ over $\mathcal{X}$, a line bundle $\mathbb{L}$ over $\mathcal{X}$, and $r \in \mathbb{Z}_{>0}$. The target is

$$\begin{array}{ccc} \mathfrak{P} & & \\ \downarrow & & \\ \mathfrak{X} & \longrightarrow & B\mathbb{G}_m = B\mathbb{C}^\times_R \\ \downarrow & & \downarrow{\scriptstyle r\text{-th root}} \\ B\mathbb{C}^\times_W \times \mathcal{X} & \xrightarrow{\mathcal{L}_W \boxtimes \mathbb{L}} & B\mathbb{G}_m \end{array}$$

where the log stack $\mathfrak{P}$ is

$$\mathfrak{P} := \mathbb{P}^w \left(\bigoplus_{i>0} \mathbb{E}^\vee_{i,\mathfrak{X}} \otimes \mathcal{L}^i_{\mathfrak{X}} \oplus \mathcal{O} \right)$$

with boundary $\infty_{\mathfrak{P}}$ defined by the vanishing of the last coordinate. The $\mathbb{P}^w$ means to weigh the projectivization by a weight w, chosen as follows. Let $d := \gcd(i : \mathbb{E}_i \neq 0)$ and pick $a \in (1/d)\mathbb{Z}_{>0}$. Then the i-th factor has weight $a + i$, and the last factor has weight 1. The freedom to choose this weighing affects certain multiplicities at $\infty_{\mathfrak{P}}$.

There are two classes of examples: p-fields, and FJRW theory.

Example 9.5 (r-spin)**.** Here $\mathcal{X} = \mathrm{pt}$ and $\mathbb{E} = \mathbb{E}_1 = \mathbb{L} = \mathbb{C}$. Then a log R-map is a morphism $\mathcal{C} \to B\mathbb{C}^\times_W$ inducing

$$\begin{array}{ccc} \mathbb{P}(\mathcal{L}_{\mathfrak{X}} \oplus \mathcal{O}) & \longleftarrow & \mathbb{P}(\mathcal{L}_{\mathcal{C}} \oplus \mathcal{O}) \\ \downarrow & & \downarrow \\ B\mathbb{C}^\times_R = \mathfrak{X} & & \\ \downarrow{\scriptstyle r\text{-th root}} & & \\ B\mathbb{C}^\times_W & \xleftarrow[\omega^{\log}]{} & \mathcal{C} \end{array}$$

In other words, it is such a morphism with a section $\mathcal{C} \to \mathbb{P}(\mathcal{L}_{\mathcal{C}} \oplus \mathcal{O})$.

Example 9.6 (p-fields)**.** Here $\mathcal{X} = \mathbb{P}^4$ and $\mathbb{E} = \mathcal{O}(5)$ and $r = 1$. Then a log R-map is a morphism $\mathcal{C} \to B\mathbb{C}^\times_W$ along with a section $\mathcal{C} \to \mathbb{P}(h^*\mathcal{O}(-5) \otimes \omega^{\log} \oplus \mathcal{O})$.

Definition 9.7. A log R-map is **stable** if:

1. f is representable;
2. $(\omega^{\log})^{1+\delta} \otimes f^*\mathcal{H}^{\otimes k} \otimes f^*\mathcal{O}(ar\infty_{\mathfrak{P}}) > 0$.

Example 9.8 (Line bundle)**.** If $\mathbb{E}$ is a line bundle, the condition (2) is equivalent to

$$\omega^{\log} \otimes f^*\mathcal{H}^{\otimes k} \otimes f^*\mathcal{O}(m0_{\mathfrak{P}}) > 0$$

for $k \gg 0$ and $m \gg 0$ since we just have a $\mathbb{P}^1$ bundle and it doesn't matter whether we use 0 or ∞. Note that without the last term, this is the usual stability condition for Gromov–Witten.

Let $R_{g,\mathbf{c}}(\mathfrak{P}, \beta)$ be the stack of stable log R-maps with fixed discrete data β.

Theorem 9.9. *$R \coloneqq R_{g,\mathbf{c}}(\mathfrak{P}, \beta)$ is a proper DM stack with a natural perfect obstruction theory.*

However $[R]^{\mathrm{vir}}$ is *not* the one we want. To simplify the story, let's assume there are no markings at all. The natural deformation theory comes from applying $R^1\pi_*$ to

$$f^*T_{\mathfrak{P}^\circ/B\mathbb{C}^\times_W} \xrightarrow{dW} f^*\mathcal{L}_W = \omega.$$

This yields a Kiem–Li cosection $\sigma\colon \mathsf{Ob} \to \mathcal{O}$. Kiem–Li tells us that $[R^0]^{\mathrm{vir}}$ is represented by a Chow cycle $[R^0]_\sigma$ supported on

$$\{\sigma = 0\} = R(\mathrm{Crit}(W)),$$

which is proper. The problem is that this mysterious cycle $[R^0]_\sigma$ is hard to compute. The solution is to study how the cosection extends to the compactification R.

Theorem 9.10. *There exists a proper surjective log étale morphism*

$$\phi\colon \mathcal{U} \to R$$

such that:

1. *it is an isomorphism over R°;*
2. *$\mathcal{U}$ has a natural perfect obstruction theory such that*
$$\phi_*[\mathcal{U}]^{vir} = [R]^{vir};$$
3. *$\mathcal{U}$ has a reduced perfect obstruction theory such that*
$$[\mathcal{U}]^{red} = [R^0]_\sigma;$$
4. *the boundary $\Delta_{\mathcal{U}} \coloneqq \mathcal{U} \setminus R^\circ$ has a reduced perfect obstruction theory such that*
$$[\mathcal{U}]^{vir} = [\mathcal{U}]^{red} + ar[\Delta_{\mathcal{U}}]^{red},$$
where ar is the order of the pole of W along $\infty_{\mathfrak{P}}$.

Example 9.11 (p-fields)**.** Recall $\mathcal{X} = \mathbb{P}^4$ and $\mathbb{E} = \mathcal{O}(5)$ and $W \in \Gamma(\mathbb{E})$. Equivalently, W is a function $\mathsf{Tot}(\mathbb{E}^\vee) \to \mathbb{C}$. Choose a generic section so that the zero locus is a smooth quintic. Then

$$\mathrm{Crit}(W) = Y \coloneqq \{W = 0\} \subset \mathbb{P}^4.$$

We know there is a moduli of stable maps $\mathcal{M}_g(Y, \beta)$ to the quintic, and people have proved that

$$[\mathcal{M}_g(Y, \beta)]^{\mathrm{vir}} = (\pm 1)[R^0]_\sigma.$$

This construction tells us the rhs can be written in two ways:

$$[R^0]_\sigma = [\mathcal{U}]^{\mathrm{red}} = [\mathcal{U}]^{\mathrm{vir}} - ar[\Delta_{\mathcal{U}}]^{\mathrm{red}}.$$

The usual virtual class on Y has little symmetry, but $[\mathcal{U}]^{\mathrm{red}}$ has non-trivial $\mathbb{C}^\times$-action.

10 Clara (Dec 10): Degenerations

Today we'll talk about the stack of expanded degenerations and the stack of expanded relative pairs. There are many reasons we care about such things:

1. degeneration/gluing formulas;
2. invariants for singular targets.

The theory was first described in Jun Li's "Stable morphisms to singular schemes and relative stable morphisms" (2001).

Motivation 1. We want to compute invariants (e.g. GW or DT) of smooth targets. To do so, we look at relative invariants of pairs (Y, D), where Y is an ambient space and $D \subset Y$ is a smooth divisor. The rough setup is a family $W \to C$ where C is a connected curve, such that W_t is smooth for $t \neq 0$ but $W_0 = Y_1 \sqcup_D Y_2$. From this we get two pairs (Y_1, D_1), (Y_2, D_2), for which we should have

$$\mathrm{GW}(W_t) = \mathrm{GW}(Y_1, D_1) \cdot \mathrm{GW}(Y_2, D_2).$$

Motivation 2. The family $W \to C$ gives rise to moduli spaces $\overline{\mathcal{M}}_{g,n}(W_t, \beta)$. When $t \neq 0$, these spaces have virtual classes, computable invariants, etc. We get a family

$$\bigcup_{t\neq 0} \overline{\mathcal{M}}_{g,n}(W_t, \beta) \to C \setminus 0.$$

The naive idea is to insert $\overline{\mathcal{M}}_{g,n}(W_0, \beta)$ over $0 \in C$. But this is missing a perfect obstruction theory. Jun Li's idea is to create an Artin stack $\mathfrak{M}$ of expanded degenerations, such that $\overline{\mathcal{M}}_{g,n}(\mathfrak{M}, \beta)$ has a perfect obstruction theory.

Setup. The remainder of the talk will describe degenerations in the special case of a family $\pi\colon W \to \mathbb{A}^1$, where:

1. W_t is smooth for $t \neq 0$;
2. W_0 is reducible with a normal crossing singularity;
3. in the normalization, we get two isomorphic copies of the singular locus D.

The guiding example will be the family

$$\pi\colon \mathbb{A}^2 \to \mathbb{A}^1, \quad (t_1, t_2) \mapsto t_1 t_2.$$

When $t \neq 0$, the fiber is $\mathbb{C}^\times$, and when $t = 0$ we get $\{t_1 t_2 = 0\}$. The idea of the degeneration space is:

1. away from the singular fiber, do nothing;
2. at the singular fiber, insert projective bundles over the divisor.

In general, the projective bundle is $\Delta = \mathbb{P}(1_D \oplus N_{D_2/Y_2})$. In the guiding example, $\Delta = \mathbb{P}^1$. There are distinguished divisors

$$D_- = \mathbb{P}(1_D \oplus 0), \quad D_+ = \mathbb{P}(0 \oplus N_{D_2/Y_2}).$$

The Y_1 and Y_2 are glued to Δ along $D_\pm$. Iterating this process yields a chain of $\mathbb{P}^1$'s attached to Y_1 and Y_2.

Q: how do we construct the whole degeneration? Generally, $W[n]$ is a desingularization of $W \times_{\mathbb{A}^n} \mathbb{A}^{n+1}$, where we take the map

$$\mathbb{A}^{n+1} \to \mathbb{A}^n, \quad (t_1, \ldots, t_{n+1}) \mapsto (t_1, \ldots, t_{n-1}, t_n t_{n+1}).$$

The result will be a family $W[n] \to \mathbb{A}^{n+1}$. So to understand $W[1]$ we need a map $W[1] \to \mathbb{A}^2$.

Step 1. Pull everything back to $\mathbb{A}^2$, via

$$\begin{array}{ccc} W \times_{\mathbb{A}^1} \mathbb{A}^2 & \longrightarrow & W \\ \downarrow & & \downarrow {\scriptstyle (x,y)\mapsto xy} \\ \mathbb{A}^2 & \xrightarrow{(t_1,t_2)\mapsto t_1 t_2} & \mathbb{A}^1 \end{array}$$

In our specific example,

$$W \times_{\mathbb{A}^1} \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y, t_1, t_2]/(xy - t_1 t_2),$$

and $D \times 0 = (0, 0, 0, 0)$.

Step 2. Blow up $D \times 0$ in $W \times_{\mathbb{A}^1} \mathbb{A}^2$. Of course,

$$\mathrm{Bl}_{D\times 0}(W \times_{\mathbb{A}^1} \mathbb{A}^2) \subset \mathrm{Bl}_0 \mathbb{A}^4 = \mathcal{O}_{\mathbb{P}^3}(-1).$$

We just need to restrict this to $W \times_{\mathbb{A}^1} \mathbb{A}^2$, to get

$$\mathcal{O}_{\mathbb{P}^3}(-1)|_{\{xy-t_1t_2\}} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1, -1),$$

via the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \quad ([a_0 : a_1], [b_0 : b_1]) \mapsto [a_0b_0 : a_1b_0 : a_0b_1 : a_1b_1].$$

We label these two $\mathbb{P}^1$'s as $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$. We can choose a $\mathbb{P}^1$ to contract; we'll choose $\mathbb{P}^1_1$. This yields $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, which can be checked either via the toric diagram, or by explicitly computing

$$\operatorname{Spec} \operatorname{Sym} \pi_{2*} \left(\pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)\right).$$

Explicitly, in coordinates, the maps to $\mathbb{A}^2$ are

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) & \xrightarrow{([a_0:a_1],[b_0:b_1],\xi)\mapsto([b_0:b_1],a_0\xi,a_1\xi)} & \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \\ \downarrow & & \downarrow {\scriptstyle ([b_0:b_1],\eta_1,\eta_2)\mapsto(b_0\eta_1,b_1\eta_2)} \\ \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2. \end{array}$$

Hence we get a family

$$W[1] = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathbb{A}^2, \quad ([b_0 : b_1], \eta_1, \eta_2) \mapsto (b_0\eta_1, b_1\eta_2).$$

Properties. Now we should convince ourselves that this $W[1]$ is actually what we wanted. Over $0 \in \mathbb{A}^2$,

$$W[1] \times_{\mathbb{A}^2} 0 = \{([b_0 : b_1], \eta_1, \eta_2) : b_0\eta_1 = b_1\eta_2 = 0\}.$$

This is exactly Y_1 and Y_2 pulled apart, with a $\mathbb{P}^1$ between them. Over the t_1-axis,

$$\begin{array}{ccc} W[1] \times_{\mathbb{A}^1} \mathbb{A}^2 & \longrightarrow & W[1] \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2, \end{array}$$

and when $t \neq 0$ we get

$$W[1] \times_{\mathbb{A}^2} \mathbb{A}^1 = \{([b_0 : b_1], \eta_1, \eta_2) : b_0\eta_1 = t, b_1\eta_2 = 0\}.$$

Hence this is a smoothing of the node connecting $\mathbb{P}^1$ and Y_1. Similarly, over the t_2 axis, we smooth the other node connecting $\mathbb{P}^1$ and Y_2. Away from the axes, we just have a $\mathbb{C}^\times$.

Constructing $W[2]$. The idea is to start with $W[1] \to \mathbb{A}^2$ and project

$$W[1] \to \mathbb{A}^2 \xrightarrow{\mathrm{pr}_2} \mathbb{A}^1$$

where pr_2 is projection to the second coordinate. Over 0,

$$\begin{array}{ccc} W[1] \times_{\mathbb{A}^1} 0 & \longrightarrow & W[1] \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{A}^1, \end{array}$$

we have

$$W[1] \times_{\mathbb{A}^1} 0 = \{([b_0 : b_1], \eta_1, \eta_2) : b_1\eta_2 = 0\}.$$

The singular locus in here is

$$D_1 = \{([1:0], \eta_1, 0)\} \subset W[1].$$

Away from 0, we just get $\mathbb{C}^2$. Then we can write a diagram

$$\begin{array}{ccc} W[1] \times_{\mathbb{A}^1} \mathbb{A}^2 & \longrightarrow & W[1] \\ \downarrow & & \downarrow{\scriptstyle \pi}\ V \\ \mathbb{A}^3 & \xrightarrow{(\mathrm{id}, m)} & \mathbb{A}^2 \\ {\scriptstyle \mathrm{pr}_{23}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \mathbb{A}^2 & \longrightarrow & \mathbb{A}^1 \end{array}$$

and repeat the construction along the bottom row. Globally, one does an analogous construction along the middle row.

Stack of relative stable pairs. This is "half" of a stack of expanded degenerations, because in the end we want to take two stacks of relative stable pairs and glue them back together. The setup is as before: $D \subset Z$ smooth. We expand along D as before, and insert projective bundles over D. For example, given $(Y, D) = (\mathbb{A}^1, 0)$, we should have a nodal curve as $Y[1] \times_{\mathbb{A}^1} 0$. In general, we want to get $(Y[n], D[n])$.

1. Start with a family $(Y \times \mathbb{A}^1, D \times \mathbb{A}^1)$.
2. Set $Y[1] \coloneqq \mathrm{Bl}_{D\times 0}(Y \times \mathbb{A}^1)$, and let $D[1]$ be the proper transform of $D \times \mathbb{A}^1$ in this blow-up.

For $(Y, D) = (\mathbb{A}^1, 0)$, we get

$$\mathrm{Bl}_{D\times 0}(Y \times \mathbb{A}^1) = \mathcal{O}_{\mathbb{P}^1}(-1),$$

and $D[1] = \{([0:1], \eta)\}$. Note that the map defining $W[1] \to \mathbb{A}^1$ is

$$\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{A}^1, \quad ([b_0 : b_1], \eta) \mapsto b_1\eta.$$

On the affine open $U_0 \subset \mathbb{P}^1$,

$$\mathcal{O}_{\mathbb{P}^1}(-1)|_{U_0} = \{([1:a], \eta) : a, \eta \in \mathbb{C}\}$$

and the map is

$$\mathcal{O}_{\mathbb{P}^1}(-1)|_{U_0} \to \mathbb{A}^1, \quad (a, \eta) \mapsto a\eta.$$