# Notes for Curves and Surfaces
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**Abstract**

These are my live-texed notes for the Spring 2017 offering of MATH GR8293 Algebraic Curves & Surfaces . Let me know when you find errors or typos. I'm sure there are plenty.

# Chapter 1

# Curves on a surface

A surface for us will be smooth, projective, and connected (over $\mathbb{C}$). Equivalently, it is a connected compact complex 2-fold.

## 1.1 Topological invariants

**Definition 1.1.1.** There are two main topological invariants:

1. the **Betti numbers** $b_i(X) := \operatorname{rank} H_i(X)$;
2. the **intersection pairing** $H_2(X;\mathbb{Z}) \otimes H_2(X;\mathbb{Z}) \to \mathbb{Z} \cong H_4(X;\mathbb{Z})$.

*Remark.* By Poincaré duality, $b_i(X) = b_{4-i}(X)$ and $H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$, and the intersection pairing is just the cup product under this identification.

**Definition 1.1.2.** Let $\bar{H}^2(X;\mathbb{Z}) := H^2(X;\mathbb{Z})/\text{tors}$. This is sensible because the torsion dies in the intersection pairing, which induces a pairing $\bar{H}^2 \otimes \bar{H}^2 \to \mathbb{Z}$.

**Definition 1.1.3.** A **lattice** is a free finite rank $\mathbb{Z}$-module $\Lambda$ together with a symmetric bilinear map $(\cdot,\cdot)\colon \Lambda \otimes_{\mathbb{Z}} \Lambda \to \mathbb{Z}$. It is:

1. **non-degenerate** if $\Lambda \to \Lambda^\vee$ is injective;
2. **uni-modular** if $\Lambda \to \Lambda^\vee$ is an isomorphism.

If we pick a $\mathbb{Z}$-basis $\{\alpha_1, \ldots, \alpha_n\}$ of $\Lambda$ to get the **intersection matrix** $A_{ij} := (\alpha_i, \alpha_j)$, then

1. $\Lambda$ is non-degenerate iff $\det A \neq 0$, and
2. $\Lambda$ is uni-modular iff $\det A = \pm 1$.

Write $\alpha^2 := (\alpha, \alpha)$. We say $\Lambda$ is

1. **type I** or **odd** if there exists $\alpha \in \Lambda$ such that $\alpha^2 \equiv 1 \bmod 2$, and
2. **type II** or **even** otherwise, i.e. $\alpha^2 \equiv 0 \bmod 2$ for all $\alpha \in \Lambda$.

Accordingly, one speaks of the **type** or **parity** of the lattice. When $\Lambda$ is non-degenerate, it has **signature** $(r, s)$ if when $A$ is diagonalized over $\mathbb{R}$, there are $r$ positive eigenvalues and $s$ negative eigenvalues.

**Theorem 1.1.4.** *An indefinite unimodular lattice is characterized up to isometry by its type and signature.*

**Example 1.1.5.** Clearly $\bar{H}^2$ is a free finite rank $\mathbb{Z}$-module, and, equipped with the intersection pairing, is a lattice. By Poincaré duality it is unimodular. To determine its type we use the **Wu formula**

$$\alpha^2 \equiv \alpha \cdot c_1(X) \bmod 2$$

where $c_1(X) \coloneqq -c_1(K_X)$, the top Chern class of the canonical bundle. Hence

$$\bar{H}^2 \text{ is of type II} \iff c_1(X) \text{ is divisible by 2.}$$

Let $b_2^{\pm}(X)$ denote the number of positive and negative eigenvalues when the intersection pairing is diagonalized over $\mathbb{R}$, so that $(b_2^+(X), b_2^-(X))$ is the signature. Then

$$\bar{H}^2 \text{ is indefinite} \iff b_2^-(X) = 0 \iff b_2(X) = b_2^+(X).$$

This almost never happens.

**Example 1.1.6.** For smooth projective surfaces, $b_2^- = 0$ iff $b_2(X) = 1$. In this case, $b_1(X) = 0$, so $X$ has the same Betti numbers as $\mathbb{P}^2$. Now, it does not have to be the case that $X = \mathbb{P}^2$, but there are only a finite number of such surfaces, and their universal covers are the unit ball in $\mathbb{C}^2$.

## 1.2 Holomorphic invariants

**Definition 1.2.1.** The most basic holomorphic invariant is the **irregularity** $q(X) \coloneqq h^1(X, \mathcal{O}_X) = h^{0,1}(X)$. By Hodge theory,

$$b_1(X) = h^{0,1}(X) + h^{1,0}(X) = 2h^{0,1}(X) = 2q.$$

We say $X$ is **regular** if $q(X) = 0$; equivalently, $H^1(X; \mathbb{R}) = 0$.

**Definition 1.2.2.** Let $H$ be a very ample divisor on $X$. Then there is the **sheaf restriction exact sequence**

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_H(H) \to 0.$$

Here $\mathcal{O}_X(H)$ is the associated sheaf of sections of $H$, i.e. meromorphic functions with poles allowed along $H$, and $\mathcal{O}_H(H)$ is its restriction $\mathcal{O}_X(H) \otimes \mathcal{O}_H$ of $\mathcal{O}_X(H)$ to the hypersurface $H$.

*Remark.* We can view $\mathcal{O}_H(H)$ as the normal sheaf of $H$ in $X$, because for divisors, $\mathcal{O}_X(-H)$ is the conormal sheaf. In fact, this whole sequence arises simply by tensoring $\mathcal{O}_X(H)$ onto the ideal sheaf sequence $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_Y = \mathcal{O}_Y \to 0$.

*Remark.* If $X$ is regular, then $H^1(X, \mathcal{O}_X(H)) \to H^1(H, \mathcal{O}_H(H))$ is surjective, because in the exact sequence, $H^1(X, \mathcal{O}_X) = 0$.

**Example 1.2.3.** If $X$ is a smooth surface in $\mathbb{P}^3$, then $X$ is automatically regular.

**Definition 1.2.4.** Another holomorphic invariant is the **geometric genus** $p_g(X) \coloneqq \dim_{\mathbb{C}} H^0(X, \Omega^2 X) = h^2(X, \mathcal{O}_X)$ (where the equality is by Serre duality). By Hodge theory,

$$b_2 = h^{2,0} + h^{1,1} + h^{0,2} = 2p_g + h^{1,1}.$$

**Definition 1.2.5.** The **topological Euler characteristic** is

$$\begin{aligned}\chi(X) &\coloneqq 1 - b_1(X) + b_2(X) - b_3(X) + 1 \\ &= 2 - 2b_1(X) + b_2(X) = 2 - 4q(X) + 2p_g(X) + h^{1,1}.\end{aligned}$$

The **holomorphic Euler characteristic** is

$$\chi(\mathcal{O}_X) \coloneqq h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 1 - q(X) + p_g(X).$$

**Theorem 1.2.6** (Hodge index theorem). *Take the Hodge decomposition* $H^2(X;\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$*:*

1. *The space* $H^{2,0}(X) \oplus H^{0,2}(X)$ *is self-conjugate and therefore is the complexification of the real vector space* $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$*. On this space the intersection pairing is automatically positive-definite;*

2. *The space* $H^{1,1}(X)$ *is self-conjugate and therefore is the complexification of the real vector space* $H^{1,1}_{\mathbb{R}}$*. There exists an element* $x \in H^{1,1}_{\mathbb{R}}$ *such that* $x^2 > 0$*, and the intersection form on* $\operatorname{span}\{x\}^{\perp}$ *in* $H^{1,1}_{\mathbb{R}}$ *is negative definite.*

**Corollary 1.2.7.** $b_2^+ = 2p_g(X) + 1$*, and* $b_2^- = h^{1,1} - 1$*.*

*Remark.* This element $x$ is the divisor class of a hyperplane section $H$ in a given projective embedding. In particular, $H \cdot H = d$, the degree of $X$ in the embedding.

**Theorem 1.2.8.** *Two identities involving invariants:*

1. *(Noether's formula)* $c_1^2(X) + c_2(X) = 12\chi(\mathcal{O}_X)$*, which is a consequence is Hirzebruch–Riemann–Roch;*

2. *(Hirzebruch signature formula)* $b_2^+ - b_2^- = (1/3)(c_1^2 - 2c_2)$*.*

*Remark.* Any two of the Noether formula, Hirzebruch signature formula, and the Hodge index theorem imply the other.

**Definition 1.2.9.** The **plurigenera** $P_n(X) \coloneqq \dim H^0(X, K_X^{\otimes n})$ are defined for $n \geq 1$ and are "higher" holomorphic invariants of $X$.

1. They are *not* homotopy or homeomorphism invariants.

2. (Seiberg & Witten) They are diffeomorphism invariants.

## 1.3 Divisors

**Definition 1.3.1.** Given a divisor $D$ on $X$, there is an associated sheaf $\mathcal{O}_X(D)$ given by

$$\mathcal{O}_X(D)(U) \coloneqq \{g \text{ meromorphic on } U \text{ s.t. } (g) + D|_U \geq 0\},$$

i.e. meromorphic functions with poles only along $D$.

*Remark.* Note that $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ iff $D_1 \equiv D_2$.

*Remark.* If $D|_U = V(f)$ and $f$ is meromorphic on $U$, then

$$\mathcal{O}_X(D)(U) = \{h/f : h \in \mathcal{O}_X(U)\}$$

so that $\mathcal{O}_X(D)$ is a line bundle. Conversely, every line bundle on $X$ is isomorphic to $\mathcal{O}_X(D)$ for some $D$.

**Lemma 1.3.2.** *There is an exact sequence*

$$\{1\} \to \mathbb{C}^{\times} \to k(X)^{\times} \to \operatorname{Div} X \to \operatorname{Pic} X \to 0.$$

*Remark.* If the line bundle is holomorphic, then $D$ is effective, and $1 \in k(X)^{\times}$ is a global section of $\mathcal{O}_X(D)$ which vanishes along $D$. Conversely, if $L$ is a line bundle and $s$ a non-zero global section, then $L \cong \mathcal{O}_X(D)$ where $D = (s)$.

**Definition 1.3.3.** Given a divisor $D$, the **complete linear system** associated to $D$ is

$$|D| \coloneqq \Gamma(\mathcal{O}_X(D) - \{0\})/\mathbb{C}^\times = \{E \text{ effective} : E \equiv D\}$$

where $\equiv$ is linear equivalence. The **base locus** of $|D|$ is

$$\mathrm{Bs}(|D|) \coloneqq \{x \in X : s(x) = 0\ \forall s \in \Gamma(\mathcal{O}_X(D))\} = \{x \in X : x \in E\ \forall E \in |D|\}.$$

We get a function

$$X \setminus \mathrm{Bs}(|D|) \to (\mathbb{P}^N)^\vee, \quad x \mapsto \text{hyperplane } \{s \in \Gamma(\mathcal{O}_X(D)) : s(x) = 0\}.$$

An effective divisor $C \in |D|$ is a **fixed curve** of $|D|$ if $E - C \geq 0$ for every $E \in |D|$.

*Remark.* The analytic picture is $\operatorname{Pic} X = H^1(X, \mathcal{O}_X)$. It is involved in the **exponential sheaf sequence**

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

which induces an exact sequence

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}) \xrightarrow{c_1} H^2(X, \mathcal{O}_X) \to \cdots$$

where $c_1$ is the **(first) Chern class map**. Then for $L \in \mathcal{O}_X(C)$ with $C$ a smooth curve on $X$, the class $c_1(L)$ is the fundamental class of $C$.

**Definition 1.3.4.** Hodge theory says $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ is a complex torus, i.e. the image of $H^1(X, \mathbb{Z})$ inside $H^1(X, \mathcal{O}_X)$ is discrete. Define

$$\operatorname{Pic}^0(X) \coloneqq H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}).$$

*Remark.* Note that $\operatorname{Pic}(X)/\operatorname{Pic}^0(X) = A$ is a finitely generated abelian group, with

$$A_{\mathrm{tors}} = H^2(X; \mathbb{Z})_{\mathrm{tors}}, \quad A/A_{\mathrm{tors}} = \bar{H}^2(X; \mathbb{Z}) \oplus H^{1,1},$$

the group of Hodge classes.

## 1.4 Algebraic intersection theory

We can define the intersection

$$D_1 \cdot D_2 = \int [D_1] \cup [D_2] \quad D_1, D_2 \in \operatorname{Div} X,$$

but it is important to have an algebraic definition.

**Definition 1.4.1** (Local intersection theory)**.** Start with $C_1, C_2$ reduced, irreducible and distinct with no component in common. Given $x \in C_1 \cap C_2$, the curve $C_i$ near $x$ looks like $V(f_i)$ for $f_i \in \mathcal{O}_{X,x}$. Since $f_1$ and $f_2$ must be relatively prime, $\mathcal{O}_{X,x}/(f_1, f_2)$ is zero-dimensional and therefore has finite length. Define the **local intersection multiplicity** to be

$$C_1 \cdot_x C_2 \coloneqq \dim_{\mathbb{C}}(\mathcal{O}_{X,x}/(f_1, f_2)).$$

*Remark.* Note that $C_1 \cdot_x C_2 = 0$ iff $x \notin C_1 \cap C_2$, and $C_1 \cdot_x C_2 = 1$ iff $(f_1, f_2) = \mathfrak{m}_x$, which is the definition of $C_1$ and $C_2$ intersecting **transversally**.

**Definition 1.4.2.** More generally, if $C_1, C_2$ are effective and have no component in common, we can still define the local intersection number by taking the sum

$$C_1 \cdot C_2 \coloneqq \sum_{x \in C_1 \cap C_2} C_1 \cdot_x C_2.$$

**Lemma 1.4.3.** *If $C_1$ is a smooth irreducible curve and $C_1$ has no component in common with $C_2$, then*

$$C_1 \cdot C_2 = \deg \mathcal{O}_X(C_2)|_{C_1}.$$

*Proof.* We have $0 \to \mathcal{O}_X(-C_2) \to \mathcal{O}_X \to \mathcal{O}_{C_2} \to 0$. Tensoring with $C_1$, we get

$$\mathcal{O}_X(-C_2) \otimes \mathcal{O}_{C_1} \to \bigoplus_{x \in C_1 \cap C_2} \mathcal{O}_{X,x}/(f_1, f_2) \to 0.$$

Since $f_2$ is not a zero-divisor in $\mathcal{O}_{C_1}$, this sequence is also exact on the left. Tensoring this with $\mathcal{O}_X(C_2)|_{C_1}$, we get

$$0 \to \mathcal{O}_{C_1} \to \mathcal{O}_{C_1}(C_2) \to \bigoplus_{x \in C_1 \cap C_2} \mathcal{O}_{X,x}/(f_1, f_2) \to 0.$$

Hence there exists a section $s$ of $\mathcal{O}_{C_1} \to \mathcal{O}_{C_1}(C_2)$ such that

$$\deg \mathcal{O}_{C_1}(C_2) = \deg(s) = \sum_x C_1 \cdot_x C_2 = C_1 \cdot C_2. \qquad \square.$$

**Theorem 1.4.4.** *There is a unique symmetric bilinear pairing* $\operatorname{Div} X \to \mathbb{Z}$*, denoted $D_1 \cdot D_2$, which factors through linear equivalence and is such that if $C_1, C_2$ are two smooth curves meeting transversally, then*

$$C_1 \cdot C_2 = \#(C_1 \cap C_2).$$

**Lemma 1.4.5.** *Every divisor $D \in \operatorname{Div} X$ is linearly equivalent to a difference $H' - H''$ of two very ample divisors.*

*Proof of theorem.* We begin with uniqueness. Given $D_i$, assume $D_i = H_i' - H_i''$ where the $H_i', H_i''$ are smooth, and possible intersections are transverse (by Bertini). Then

$$\begin{aligned} D_1 \cdot D_2 &= (H_1' - H_1'') \cdot (H_2' - H_2'') \\ &= \#(H_1' \cap H_2') - \#(H_1'' \cap H_2') - \#(H_1' \cap H_2'') + \#(H_1'' \cap H_2''). \end{aligned}$$

Now we show existence. Given $D_i$, pick $H_i', H_i''$ smooth with transverse intersections. Define $D_1 \cdot D_2$ by the equation above. By the lemma,

$$D_1 \cdot D_2 = \deg(\mathcal{O}(H_1' - H_1''))|_{H_2'} - \deg(\mathcal{O}(H_1' - H_1''))|_{H_2''}.$$

As defined, $D_1 \cdot D_2$ are independent of the choice of $H_i', H_i''$, and only depends on $D_1$ and its linear equivalence class. By symmetry, the same is true for $D_2$. Finally, if $D_i = C_i$ where the $C_i$ are smooth and meet transversally, take $H_i' = C_i$ and $H_i'' = \emptyset$. $\square$

*Remark.* If $D_1$ is smooth, then $D_1 \cdot D_2 = \deg(\mathcal{O}_X(D_2))|_{D_1}$. If $D_1$ is reduced irreducible but not necessarily smooth, the same formula holds. Recall that the degree of a line bundle $L$ over a reduced irreducible curve $C$ is defined in any of the following ways:

1. given the normalization $\gamma\colon \tilde{C} \to C$, define $\deg L \coloneqq \deg \gamma^* L$;
2. writing $L = \mathcal{O}_C(\sum_i n_i p_i)$, where $p_i \in C$ are in the smooth part of $C$, define $\deg L \coloneqq \sum_i n_i$;
3. from the exponential sheaf sequence $0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$, we get the map $H^1(\mathcal{O}_C^*) \xrightarrow{\deg} H^2(C, \mathbb{Z}) \cong \mathbb{Z}$, where the isomorphism holds for $C$ irreducible.

*Remark.* The uniqueness part of the theorem shows $D_1 \cdot D_2 = [D_1] \cdot [D_2] = \int_X [D_1] \cup [D_2]$, the intersection pairing.

*Remark.* Some useful facts: if $D \geq 0$ and $H$ is ample, then $D \cdot H \geq 0$ with equality iff $D = 0$. Also, if $C_1, C_2$ are distinct irreducible curves, then $C_1 \cdot C_2 \geq 0$ with equality iff $C_1 \cap C_2 = \emptyset$.

## 1.5 Arithmetic genus

**Definition 1.5.1.** For $C$ a smooth curve (reduced, irreducible), $C^2 = C \cdot C = \deg \mathcal{O}_X(C)|_C$. Note that $\mathcal{O}_X(C)|_C$ is the **normal bundle** $(I_C/I_C^2)^\vee$. There are short exact sequences

$$0 \to T_C \to T_X|_C \to \mathcal{O}_C(C) \to 0$$
$$0 \to I_C/I_C^2 \to \Omega_X^1|_C \to \Omega_C^1 \to 0.$$

Taking determinants, we get the **adjunction formula** $\mathcal{O}_C(-C) \otimes K_C = K_X|_C$, or equivalently, $K_C = (K_X + C)|_C$.

*Remark.* Numerically, this means that $2g(C) - 2 = \deg K_C = (K_X + C) \cdot C$.

**Definition 1.5.2.** Let $D \geq 0$ be an effective, not necessarily reduced, and smooth. Define the **dualizing sheaf** $\omega_D := (K_X \otimes \mathcal{O}_X(D))|_D$. This is an intrinsically defined line bundle, and makes Serre duality work, i.e. there is a trace map $H^1(D; \omega_D) \xrightarrow{\text{tr}} \mathbb{C}$ such that for $L$ a line bundle,

$$H^0(D; L) \otimes H^1(D; L^{-1} \otimes \omega_D) \to \mathbb{C}$$

is a perfect pairing.

**Definition 1.5.3.** For $C$ smooth, note that $g(C) = 1 + (1/2)(K_X \cdot C + C^2)$. For $D \geq 0$, define the **arithmetic genus** of $D$ by

$$p_a(D) := 1 + \frac{1}{2}(K_X \cdot D + D^2).$$

*Remark.* We will see later using Riemann–Roch that $p_a(D) \in \mathbb{Z}$. In fact, $p_a(D) = 1 - \chi(D; \mathcal{O}_D)$. If $h^0(\mathcal{O}_D) = 1$ (e.g. if $D$ is reduced irreducible), then $p_a(D) = h^1(\mathcal{O}_D)$. Let's see some recipes for calculating $p_a(D)$.

**Definition 1.5.4.** Let $C$ be reduced irreducible with normalization map $\nu \colon \tilde{C} \to C$ (so $\tilde{C}$ is smooth and connected). There is a **normalization exact sequence**

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}} \to \bigoplus_{x \in C_{\text{sing}}} \nu_* \mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x} \to 0.$$

For $x \in C$, the **local genus drop** at $x$ is

$$\delta_x := \dim_{\mathbb{C}} [\nu_* \mathcal{O}_{\tilde{C}}/\mathcal{O}_C]_x.$$

**Lemma 1.5.5.** *$p_a(C) = g(\tilde{C}) + \delta$ where $\delta := \sum_{x \in C} \delta_x$.*

*Proof.* Look at the exact sequence

$$0 \to H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_{\tilde{C}}) \to (\dim d \text{ vector space}) \to H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_{\tilde{C}}) \to 0.$$

Since $C$ is connected, $H^0(\mathcal{O}_C) = H^0(\mathcal{O}_{\tilde{C}})$. □

**Example 1.5.6.** Note that $\delta_x = 0$ iff $x$ is a regular point of $C$. Similarly, $\delta_x = 1$ iff $x$ is analytically a node or a cusp. We will relate $\delta_x$ to blowups.

**Corollary 1.5.7.** *Suppose $C \subset X$ is reduced irreducible. Then $p_a(C) \geq 0$, and in fact $p_a(C) = 0$ iff $C \cong \mathbb{P}^1$.*

*Proof.* If $p_a(C) = g(\tilde{C}) + \delta = 0$, then $\delta = 0$ and $C = \tilde{C}$. Then $g(\tilde{C}) = g(C) = 0$, and hence $C \cong \mathbb{P}^1$. □

*Remark.* For non-reduced divisors $D = D_1 + D_2$ with $D_i \geq 0$ but possibly having a component in common, we can use the exact sequence

$$0 \to \mathcal{O}_{D_1}(-D_2) \to \mathcal{O}_D \to \mathcal{O}_{D_2} \to 0.$$

Problem: it's hard to compute $H^0(\mathcal{O}_{D_2}) \to H^1(\mathcal{O}_{D_1}(-D_2))$. For example, for $D = nC$, we have

$$0 \to \mathcal{O}_C(-(n-1)C) \to \mathcal{O}_{nC} \to \mathcal{O}_{(n-1)C}.$$

## 1.6 Riemann–Roch formula

**Theorem 1.6.1.** $\chi(\mathcal{O}_X(D)) = (1/2)(D^2 - D \cdot K_X) + \chi(\mathcal{O}_X)$.

*Proof.* This is trivially true if $D = 0$. Suppose $D = C$ where $C$ is a smooth (connected) curve. Taking the Euler characteristic of the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$$

gives $\chi(\mathcal{O}_X(C)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_C(C))$. But we know

$$g(C) = (C^2 + K_X \cdot C)/2 + 1, \quad \deg \mathcal{O}_X(C)|_C = C^2,$$

so that Riemann–Roch for curves gives

$$\chi(\mathcal{O}_X(C)) = C^2 - \left(\frac{1}{2}(C^2 + K_X \cdot C) + 1\right) + 1 = \frac{1}{2}(C^2 - K_X \cdot C).$$

This proves the special case. In the general case, we can assume $D \equiv C_1 - C_2$ where $C_1, C_2$ are both smooth. Then use the exact sequence

$$0 \to \mathcal{O}_X(C_1 - C_2) \to \mathcal{O}_X(C_1) \to \mathcal{O}_X(C_1)|_{C_2} \to 0.$$

Thus $\chi(\mathcal{O}_X(C_1 - C_2)) = \chi(\mathcal{O}_X(C_1)) - \chi(\mathcal{O}_X(C_1)|_{C_2})$, but we can compute these two terms using Riemann–Roch and adjunction. □

**Corollary 1.6.2** (Wu formula)**.** *For $D \in \operatorname{Div} X$, we get $D^2 \equiv D \cdot K_X \bmod 2$. It follows that $p_a(D) \in \mathbb{Z}$.*

*Remark.* The Riemann–Roch formula is usually applied along with either:

1. (Serre duality) $H^i(X, \mathcal{O}_X(D))^\vee = H^{2-i}(X, K_X \otimes \mathcal{O}_X(-D))$;
2. (Kodaira vanishing) for $D$ ample, $H^i(X, \mathcal{O}_X(-D)) = 0$ for $i = 0, 1$, or dually, $H^i(X, K_X \otimes \mathcal{O}_X(D)) = 0$ for $i = 1, 2$.

This is because we need some criterion for the vanishing of $H^i(X, \mathcal{O}_X(D))$.

## 1.7 Hodge index theorem

**Lemma 1.7.1.** *Suppose $D^2 > 0$ and there exists $H$ ample such that $D \cdot H > 0$. Then $nD$ is effective for $n \gg 0$.*

*Proof.* By Riemann–Roch, $\chi(\mathcal{O}_X(nD)) = O(n^2)$, of the form $D^2/2n^2 + \cdots$. Hence $\chi(\mathcal{O}_X(nD)) > 0$ for sufficiently large $n$. Therefore $h^0(\mathcal{O}_X(nD)) + h^2(\mathcal{O}_X(nD)) > 0$, so

$$h^0(\mathcal{O}_X(nD)) > 0 \text{ or } h^2(\mathcal{O}_X(nD)) = h^0(\mathcal{O}_X(-nD) \otimes K_X) > 0.$$

But taking intersections,

$$H \cdot (-nD + K_X) = -n(D \cdot H) + H \cdot K_X < 0$$

so $-nD + K_X$ is not effective. Hence $h^2(\mathcal{O}_X(nD)) = 0$, and therefore $h^0(\mathcal{O}_X(nD)) > 0$. It follows that $nD$ is effective. □

**Theorem 1.7.2** (Hodge index theorem, algebraic version)**.** *Let $H$ be an ample divisor on $X$, and let $D \in \operatorname{Div} X$. If $D \cdot H = 0$, then $D^2 \leq 0$ and equality holds iff for all divisors $E$, $D \cdot E = 0$.*

*Proof.* Say $H \cdot D = 0$ and $D^2 > 0$. By Serre's theorem, for all $m \gg 0$, the divisor $mH + D$ is ample. Also, $(mH + D) \cdot D = D^2 > 0$. Apply the lemma with $H$ replaced by $mH + D$ to get that $nD$ is effective for $n \gg 0$. Then $H \cdot D > 0$, contradicting the assumption $H \cdot D = 0$.

Now assume $D^2 = 0$ and there exists a divisor $E$ such that $D \cdot E \neq 0$. WLOG assume $D \cdot E > 0$. Let $E' \coloneqq H^2 E - (H \cdot E)H$. This is set up so that $E' \cdot H = 0$ and

$$E' \cdot D = H^2(E \cdot D) - (H \cdot E)(H \cdot D) > 0.$$

Let $D' \coloneqq nD + E'$ for $n \gg 0$. By construction, $D' \cdot H = 0$. But

$$(D')^2 = n^2 D^2 + 2nD \cdot E' + (E')^2 = 2n(D \cdot E') + (E')^2 > 0.$$

This contradicts the first part of the theorem. □

**Definition 1.7.3.** A divisor $D$ is **numerically trivial**, written $D \sim 0$, iff $D \cdot E = 0$ for all $E \in \operatorname{Div} X$. More generally, $D_1 \sim D_2$ iff $D_1 - D_2 \sim 0$. Define

$$\operatorname{Num} X \coloneqq \operatorname{Div} X / \sim,$$

the **group of divisors mod numerical equivalence**.

*Remark.* The intersection pairing descends to $\operatorname{Num} X$. We will see shortly that $\operatorname{Num} X$ is a lattice under the intersection pairing. (The only content of this statement is that $\operatorname{Num} X$ is a finite rank $\mathbb{Z}$-module.) It is non-degenerate, but very rarely unimodular. Restatement of Hodge index theorem: given $H$ the class of an ample divisor, $H^\perp$ is a negative definite sublattice.

**Proposition 1.7.4.** *$D$ is numerically equivalent to 0 iff $[D] = 0$ in $\bar{H}^2(X, \mathbb{Z})$ (or $H^2(X, \mathbb{R})$). In other words, numerical equivalence is equivalent to homological equivalence mod torsion.*

*Proof.* If $[D] = 0$ in $\bar{H}^2(X, \mathbb{Z})$, then for all $E \in \operatorname{Div} X$,

$$E \cdot D = \int_X [E] \cup [D] = 0,$$

so that $D$ is numerically trivial. Conversely, suppose $[D] \neq 0$ in $\bar{H}^2(X, \mathbb{Z}) \subset H^{1,1}_{\mathbb{R}}(X)$. Pick $[H] \in \bar{H}^2(X, \mathbb{Z})$ where $H$ is ample. There are two cases.

1. If $\int_X [D] \cup [H] \neq 0$, then $D \cdot H \neq 0$ and therefore $D$ is not numerically trivial.
2. If $\int_X [D] \cup [H] = 0$, then $[D] \in [H]^\perp$ in $H^{1,1}_{\mathbb{R}}$. By assumption, $[D]$ is non-zero, so by the (topological) Hodge index theorem, $D^2 = [D]^2 < 0$. Hence $D \cdot D \neq 0$, and $D$ is again not numerically trivial. □

**Corollary 1.7.5.** *This gives a natural inclusion $\operatorname{Num} X \subset \bar{H}^2(X, \mathbb{Z}) \cap H^{1,1}_{\mathbb{R}}$. In fact, this is an isomorphism, so $\operatorname{Num} X$ is free abelian with rank at most $h^{1,1}$.*

## 1.8 Ample and nef divisors

**Theorem 1.8.1** (Nakai–Moishezon criterion)**.** *Let $D \in \operatorname{Div} X$. Then $D$ is ample iff $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves $C \subset X$.*

*Remark.* One of the main points here is that ampleness is a strictly numerical property, i.e. it descends to $\operatorname{Num} X$. This is not true for many other properties, e.g. for very ampleness.

**Example 1.8.2** (Mumford)**.** There exists a surface $X$ and a divisor $D \subset X$ such that $D \cdot C > 0$ for every $C$ irreducible (which implies $D^2 \geq 0$), but $D^2 = 0$.

*Proof sketch of Nakai–Moishezon criterion.* The main point is that there exist some $N \gg 0$ such that the linear system $|ND|$ is base-point free (bpf). This gives a morphism $\varphi\colon X \to \mathbb{P}^A$ with the property that $\varphi^*\mathcal{O}_{\mathbb{P}^A}(1) = \mathcal{O}_X(ND)$. Claim: $\varphi$ is finite. If not, there must exist an irreducible curve $C \subset X$ such that $\varphi(C) = \text{pt}$. Then

$$\deg \varphi^*\mathcal{O}_{\mathbb{P}^A}(1)|_C = N(D \cdot C) = 0,$$

contradicting the assumption $D \cdot C > 0$. Hence $\varphi$ is finite. Now use the general fact that for a finite morphism, the pullback $\varphi^*$ of an ample divisor is still ample. In particular, $\mathcal{O}_X(ND)$ is ample, and therefore $\mathcal{O}_X(D)$ is ample.

We saw that there exists $n \gg 0$ such that $nD$ is effective. For simplicity, replace $nD$ by $D$, so we can assume $D$ is effective. In fact, we can assume $D$ is some fixed 1-dimensional scheme (possibly singular and non-reduced). Let $D_1, \dots, D_s$ be irreducible components of $D_{\text{red}}$. Fact: a line bundle $L$ on $D$ is ample iff $\deg(L|_{D_i}) > 0$ for all $i$. In particular, $\mathcal{O}_X(D)|_D$ is ample, so there exists $n_0$ such that $\mathcal{O}_X(nD)$ is generated by its global sections for all $n \geq n_0$. Look at

$$0 \to \mathcal{O}_X((n-1)D) \to \mathcal{O}_X(nD) \to \mathcal{O}_X(nD)|_D \to 0.$$

We want $H^0(\mathcal{O}_X(nD)) \to H^0(\mathcal{O}_X(nD)|_D)$ to be surjective, which would imply $nD$ is bpf. This is because of the following.

1. If $x \notin D$, then there exists a section $s$ of $\mathcal{O}_X(D)$ defining $D$, so $s(x) \neq 0$ and $x \notin D$. Hence $s^n \in H^0(\mathcal{O}_X(nD))$.
2. If $x \in D$, then there exists a section $\bar{t}$ of $\mathcal{O}_X(nD)|_D$ such that $\bar{t}(x) \neq 0$. Lift it to a section $t$ of $\mathcal{O}_X(nD)$, where still $t(x) \neq 0$.

We make sure it is surjective by showing for $n \gg 0$ that $H^1(\mathcal{O}_X((n-1)D)) \to H^1(\mathcal{O}_X(nD))$ is an isomorphism. Use that $\mathcal{O}_X(D)|_D$ is ample, so that there exists some $n_0$ such that , $H^1(\mathcal{O}_X(nD)|_D) = 0$ for $n \geq n_0$. Hence for $n \geq n_0$,

$$H^1(\mathcal{O}_X((n-1)D)) \to H^1(\mathcal{O}_X(nD)) \to 0$$

is surjective. Consider the sequence of surjections

$$H^1(\mathcal{O}_X(n_0D)) \to \cdots \to H^1(\mathcal{O}_X((n-1)D)) \to H^1(\mathcal{O}_X(nD)).$$

Since $\dim H^1(\mathcal{O}_X(nD))$ is non-increasing, the dimensions stabilize for all $n \geq M_0$. So for $n \geq M_0 + 1$, the map $H^1(\mathcal{O}_X((n-1)D)) \to H^1(\mathcal{O}_X(nD))$ is surjective and the two terms have the same dimension, so it must be an isomorphism. ☐

**Definition 1.8.3.** We will work in $\operatorname{Num}_{\mathbb{R}} X := \operatorname{Num} X \otimes_{\mathbb{Z}} \mathbb{R} \subset H^2(X, \mathbb{R})$. Let $A(X)$ be the **ample cone**, which is the convex hull of the classes of ample divisors.

*Remark.* If $x \in A(X)$, then $x^2 > 0$ and $x \cdot C > 0$ for all irreducible curves $C$. Then $A(X) \cap \operatorname{Num} X$ is precisely the set of ample divisors.

**Definition 1.8.4.** Let $\operatorname{NE}(X) \subset \operatorname{Num} X$ be the convex hull of classes of effective (equivalent irreducible) curves. Nakai–Moishezon says $D$ ample iff $D \cdot (\operatorname{NE}(X)) > 0$ and $D^2 > 0$.

**Theorem 1.8.5** (Kleiman's criterion)**.** *$D$ is ample iff* $D \cdot (\overline{\operatorname{NE}(X)} \setminus \{0\}) > 0$.

**Definition 1.8.6.** A divisor $D$ is **nef** if for all $C$ irreducible, $D \cdot C \geq 0$. A divisor $D$ is **big** if $D^2 > 0$.

**Proposition 1.8.7.** *If $D$ is nef, then* $D^2 \geq 0$.

**Example 1.8.8.** Mumford's example shows nef divisors are not necessarily big. There are other ways this can happen too, e.g. if $D$ is effective and a fiber of a morphism $X \to C$ (where $C$ is a curve), then $D$ is nef but $D^2 = 0$.

*Proof.* Suppose $D^2 < 0$. Fix an ample $H$ and consider $D + tH$. For $t \gg 0$, this is ample. Consider

$$f(t) := (D + tH)^2 = D^2 + 2tD \cdot H + t^2 H^2.$$

This is a parabola with $f(0) < 0$. So there is a $t_0$ such that $f(t_0) = 0$ and $f(t) > 0$ for all $t > t_0$. Clearly nef is a convex property, so $D + tH$ is nef for all $t > 0$, and in fact $(D + tH) \cdot C > 0$ for all $C$ irreducible. Hence if $t \in \mathbb{Q}$ and $t > t_0$, then $D + tH$ is ample. Therefore some large multiple of $D + tH$ is effective, and $D \cdot (D + tH) \geq 0$. Now let $t \to t_0$ in $\mathbb{Q}$, so that $D \cdot (D + t_0 H) \geq 0$. By construction,

$$0 = (D + t_0 H)^2 = D \cdot (D + t_0 H) + t_0 D \cdot H + t_0^2 H^2.$$

We also know $D \cdot H \geq 0$ because $D$ is nef, and $H^2 > 0$ because $H$ is ample. By contradiction, $D^2 \geq 0$. □

*Remark.* We have $\operatorname{Num} X \subset \operatorname{Num}_{\mathbb{Q}} X \subset \operatorname{Num}_{\mathbb{R}} X$ and similarly for Div. We can define nef for both $\mathbb{Q}$- and $\mathbb{R}$-divisors, and ample makes sense for $\mathbb{Q}$-divisors. Nakai–Moishezon applies to $\mathbb{Q}$-divisors word-for-word.

**Theorem 1.8.9** (Ramanujan vanishing theorem)**.** *In characteristic* 0*, if $D$ is nef and big then*

$$H^i(X, \mathcal{O}_X(-D)) = 0 \quad i = 0, 1.$$

*In other words, $D$ behaves cohomologically like an ample divisor. Equivalently, $H^i(X, \mathcal{O}_X(D) \otimes K_X) = 0$ for $i = 1, 2$.*

**Definition 1.8.10.** A divisor system $|D|$ is **eventually bpf** if for all $N \gg 0$, the system $|ND|$ is bpf. Note that if there exists $n_0$ such that $|n_0 D|$ is bpf, then $|Nn_0 D|$ is bpf.

*Remark.* Clearly if $|D|$ is bpf, then $D$ is nef. More strongly, if $|D|$ is eventually bpf, then $D$ is nef. Also, if $|n_0 D|$ is bpf, then $D$ is big iff $\varphi_{n_0 D}(X) \subset \mathbb{P}^N$ is a surface. (Here $\varphi_{n_0 D}$ is the morphism to projective space associated to $n_0 D$.)

**Proposition 1.8.11.** *Suppose $D$ is big, and $|D|$ is eventually bpf. Then for all $N \gg 0$, the image $\varphi_{ND}(X) = \bar{X} \subset \mathbb{P}^A$ is a normal surface. There exists finitely many points $x_i$ such that:*

1. *$\varphi^{-1}(x_i) = \bigcup_j C_{ij}$ is an connected curve with irreducible components $C_{ij}$;*
2. *the $C_{ij}$ are linearly independent in $\operatorname{Num}_{\mathbb{Q}} X$;*
3. *the intersection matrix spans a negative definite sublattice, and*
4. *the $C_{ij}$ are exactly the curves $C$ such that $D \cdot C = 0$.*

*Proof.* Fix $n_0$ such that $|n_0 D|$ is bpf, giving a morphism $\varphi_0 \colon X \to X_0 \subset \mathbb{P}^B$. This surface $X_0$ is not necessarily normal, but by general theory there exists a Stein factorization $X \xrightarrow{\pi} \bar{X} \xrightarrow{\nu} X_0$, where $\bar{X}$ is normal and $\bar{X} \to X_0$ is finite and $\pi$ has connected fibers.

Let $\bar{L} = \nu^* \mathcal{O}_{\mathbb{P}^B}(1)$ so that $L := n \cdot \bar{L} = \mathcal{O}_X(n_0 D)$ so that $\bar{L}$ is ample. Then $(\bar{L})^{\otimes m}$ is very ample and embeds $\bar{X}$ in $\mathbb{P}^A$. The normality of $\bar{X}$ says $\pi_* \mathcal{O}_X = \mathcal{O}_{\bar{X}}$, so

$$\pi_* L = \pi_* \pi^* \bar{L} = \bar{L} \otimes \pi_* \mathcal{O}_X = \bar{L}.$$

Similarly, $\pi_* L^{\otimes m} = \bar{L}^{\otimes m}$. This says that

$$H^0(X, L^{\otimes m}) = H^0(X, \mathcal{O}_X(mn_0 D)) = H^0(\bar{X}, \bar{L}^{\otimes m}).$$

So the image of $X$ under the morphism determined by $|mn_0 D|$ is equal to $\bar{X}$. In fact, the possibilities for fibers of $\pi$ are:

1. if $\dim \pi^{-1}(x) = 0$, then $\pi^{-1}(x) = \{\text{pt}\}$;

2. if $\dim \pi^{-1}(x) = 1$, then $\pi^{-1}(x)$ is a connected curve.

By counting, there can only be finitely many points $x_1, \ldots, x_n$ such that $\dim \pi^{-1}(x) = 1$. For each of these, we can write $\pi^{-1}(x_i) = \bigcup_j C_{ij}$, and clearly $D \cdot C_{ij} = 0$. Conversely if $D \cdot C = 0$, then $\varphi_{mn_0 D}(C) = \{\mathrm{pt}\}$. The classes of the $C_{ij}$ in $\operatorname{Num} X$ must lie in $D^\perp$. By the Hodge index theorem, they span a negative definite sublattice of $\operatorname{Num}_{\mathbb{Q}} X$.

It remains to show the linear independence of the classes of the $C_{ij}$. This follows from the following general lemma. □

**Lemma 1.8.12.** *If $X$ is a surface, $C_1, \ldots, C_k$ are irreducible curves such that their classes span a negative definite sublattice of $\operatorname{Num}_{\mathbb{Q}} X$, then $[C_1], \ldots, [C_k]$ are linearly independent over $\mathbb{Q}$.*

*Proof.* If not, there exists a linear relation $\sum_{i=1}^n r_i[C_i] = 0$ with $r_i \in \mathbb{Q}$ all non-zero. We can assume, by re-indexing, that $r_i > 0$ for $i \le \ell$, and $r_i = -s_i$ with $s_i > 0$ for $i > \ell$. So we have

$$\sum_{i=1}^{\ell} r_i C_i = \sum_{i=\ell+1}^{k} s_i C_i.$$

Claim: we can't have $\ell = k$ or $\ell = 0$. Otherwise $\sum_{i=1}^k r_i C_i$ is an effective non-zero curve, so that $(\sum_{i=1}^k r_i C_i) \cdot H > 0$, which can't happen. Now by negative definiteness, $(\sum_{i=\ell+1}^k s_i C_i)^2 < 0$. But

$$\left(\sum_{i=\ell+1}^{k} s_i C_i\right)^2 = \left(\sum_{i=1}^{\ell} r_i C_i\right)\left(\sum_{j=k+1}^{\ell} s_j C_j\right) = \sum_{i,j} r_i s_j (C_i \cdot C_j) \ge 0,$$

a contradiction. □

*Remark.* For ample, nef, and nef + big, there are strictly numerical criteria. But for eventually bpf and big, there is no numerical criterion. Suppose $D$ is nef and big, $D \cdot C = 0$. Then for $n \gg 0$ (maybe divisble), $\mathcal{O}_X(nD)|_C \cong \mathcal{O}_C$ if $|nD|$ is bpf. What could happen is that $\deg(\mathcal{O}_X(D)|_C) = 0$ but $\mathcal{O}_X(D)$ has infinite order in $\operatorname{Pic} C$.

**Example 1.8.13.** Let $X = \mathbb{P}^2$ and $C$ have degree $d \ge 3$ (so that $g(C) \ge 1$). If we blow up more than $d^2 + 1$ points on $C$ and let $C'$ be the proper transform, then $(C')^2 < 0$. For generic choices, $\operatorname{Pic} X \to \operatorname{Pic} C$ is injective. We can find $D$ nef and big such that $D \cdot C' = 0$.

## 1.9 Ample cone and its closure

**Definition 1.9.1.** Recall $\operatorname{Num}_{\mathbb{R}} X \subset H^{1,1}_{\mathbb{R}}(X)$ has an intersection form of type $(1, \rho - 1)$, where $\rho = \operatorname{rank} \operatorname{Num} X$. So there exists a $\mathbb{R}$-basis $\{e_i\}$ of $\operatorname{Num}_{\mathbb{R}} X$ such that $e_1^2 = 1$ and $e_i^2 = -1$ for $i > 1$. So

$$x = x_1 e_1 + \sum_{i>1} x_i e_i \implies x^2 = x_1^2 - \sum_{i>1} x_i^2.$$

Let $\mathcal{C}$ be the positive "cone" $\mathcal{C} \coloneqq \{x \in \operatorname{Num}_{\mathbb{R}} X : x^2 \ge 0\}$ (it is not really a cone). Divide this into two pieces $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$, where

$$\mathcal{C}_+ \coloneqq \{x \in \mathcal{C} : x_i > 0\}, \quad \mathcal{C}_- \coloneqq \{x \in \mathcal{C} : x_i < 0\}.$$

**Lemma 1.9.2** (Light cone lemma)**.** *1. $\mathcal{C}_\pm$ are open convex (hence connected) subsets of $\operatorname{Num}_{\mathbb{R}} X$.*

*2. If $x \in \mathcal{C}_+$ and $y \in \bar{\mathcal{C}}$ (the closure of $\mathcal{C}$), then $x \cdot y = 0$ iff $y = 0$, and otherwise*

*(a) $x \cdot y > 0$ if $y \in \mathcal{C}_+$, and*

(b) $x \cdot y < 0$ *if* $y \in \mathcal{C}_-$.

*Proof.* Clearly $\mathbb{R}^+\mathcal{C} = \mathcal{C}$, but also $\mathbb{R}^+\mathcal{C}_\pm = \mathcal{C}_\pm$. To prove $\mathcal{C}_\pm$ are convex, it is enough to prove if $x, y \in \mathcal{C}_+$ then $x + y \in \mathcal{C}_+$ (and similarly for $\mathcal{C}_-$). Write $x = x_1 e_1 + \sum_{i>1} x_i e_i$ and similarly for $y$, with $x_1, y_1 > 0$. Then

$$x^2 = x_1^2 - \|x'\|^2 > 0, \quad y^2 = y_1^2 - \|y'\|^2 > 0,$$

so that

$$(x+y)^2 > 2(x \cdot y) = 2(x_1 y_1 - \sum_{i>1} x_i y_i).$$

But $x \cdot y > \|x'\|\|y'\| - \langle x', y' \rangle \geq 0$. by Cauchy–Schwarz.

Assume $x \in \mathcal{C}_+$ and $y \in \bar{\mathcal{C}}$, so that $y^2 \geq 0$. If $x \cdot y = 0$ and $y \neq 0$, then $y^2 < 0$ by Hodge index theorem, a contradiction. Finally, suppose $y \in \bar{\mathcal{C}}$ is non-zero. It is easy to see that $\bar{\mathcal{C}} \setminus \{0\}$ has two connected components $\bar{\mathcal{C}}_+ \setminus \{0\}$ and $\bar{\mathcal{C}}_- \setminus \{0\}$. Look at the function

$$(y \in \bar{\mathcal{C}} \setminus \{0\}) \mapsto \operatorname{sign}(x \cdot y) = (x \cdot y)/|x \cdot y|.$$

This is a continuous function with values in $\{\pm 1\}$, so it must be constant on each connected component. □

*Remark.* There exists a component of $\mathcal{C}_+ \cup \mathcal{C}_-$ which contains the class of an ample divisor $H$. Choose notation such that $\mathcal{C}_+$ contains the class of $H$, so that $D$ nef and $[D] \neq 0$ implies $[D] \in \bar{\mathcal{C}}_+$. Equivalently,

$$\mathcal{C}_+ = \{x \in \operatorname{Num}_{\mathbb{R}} X : x^2 > 0,\ x \cdot H > 0,\ H \text{ ample}\}.$$

We see then that $A(X) \subset \mathcal{C}_+$.

**Proposition 1.9.3.** $A(X)$ *is an open convex subset of* $\mathcal{C}_+$.

*Proof.* We know $A(X)$ is convex, so we prove it is open. Fix a $\mathbb{Z}$-basis $e_1, \ldots, e_\rho$ of $\operatorname{Num} X$ corresponding to classes of divisors $[D_i]$. If $H$ is ample, there exists $N \gg 0$ such that $NH \pm D_i$ is also ample. Then $[H] \pm (1/N)e_i \in A(X)$. Their convex hull is contained in $A(X)$, and its interior is an open neighborhood of $[H]$.

Now let $x \in A(X)$. Then $x = \sum_{i=1}^k \lambda[H_i]$ is some convex combination of ample divisors. If $U_i$ is the open set around $[H_i]$, then $A(X) \supset \sum_{i=1}^k \lambda_i U_i$, which is also open. □

*Remark.* Within $H^{1,1}_{\mathbb{R}}$, we can also talk about the **Kähler cone** $\mathcal{K}(X)$, which is the set of classes of Kähler forms of Kähler metrics on $X$. This is also an open convex cone.

*Remark.* The normalized ample cone $\{x \in \mathcal{C}_+ : x^2 = 1\}$ is a model of a hyperbolic space $\mathbb{H}$. We want to understand what $A(X)$ looks like inside this $\mathbb{H}$.

**Theorem 1.9.4.** $A(X) = \{x \in \operatorname{Num}_{\mathbb{R}} X : x^2 > 0,\ x \cdot C > 0\, \forall C$ *irreducible*$\}$.

*Proof.* We obviously have $\subset$. The main point is $\supset$. □

*Remark.* This description of $A(X)$ writes it as the countable intersection of open sets (because of the "for all irreducible $C$").

**Lemma 1.9.5.** *If* $x$ *is an* $\mathbb{R}$*-divisor, then* $x \sim 0$ *iff* $x = \sum_{i=1}^k r_i D_i$ *for* $r_i \in \mathbb{R}$ *and* $D_i \in \operatorname{Div} X$ *and* $D_i \sim 0$.

**Definition 1.9.6.** $x \in \operatorname{Div}_{\mathbb{R}} X$ is **ample** iff there exist ample divisors $H_1, \ldots, H_r$ in $\operatorname{Div} X$ and $t_i \in \mathbb{R}^+$ such that $x = \sum_i t_i H_i$ is a strictly convex combination.

**Lemma 1.9.7.** *If* $x$ *is ample and* $x \sim y$, *then* $y$ *is ample.*

*Proof.* Assume $x = \sum_i t_i H_i$ as in the definition. Assume $y = \sum_i r_i E_i$ where $r_i \in \mathbb{R}$ and $E_i \in \operatorname{Div} X$ are actual divisors and $E_i \sim 0$. It suffices to prove the lemma in the case $x = tH$ and $y = tH + E = t(H + sE)$ where $H + sE$ is ample. If $H + sE$ is a $\mathbb{Q}$-divisor, we are done by Nakai–Moishezon. In general, find $s_1, s_2 \in \mathbb{Q}$ with $s_1 < s < s_2$, so that $s = ts_1 + (1-t)s_2$ where $t \in (0,1)$. Then

$$H + sE = t(H + s_1 E) + (1-t)(H + s_2 E)$$

and by hypothesis $H + s_i E$ are $\mathbb{Q}$-divisors and therefore ample. □

**Corollary 1.9.8.** *For $x \in \operatorname{Div}_{\mathbb{R}} X$, the class of $x$ in $\operatorname{Num}_{\mathbb{R}} X$ is in $A(X)$ iff $x$ is ample.*

**Lemma 1.9.9.** *If $x$ is a nef $\mathbb{R}$-divisor, then $x^2 \geq 0$.*

*Proof.* We are done if $x$ is a $\mathbb{Q}$-divisor. In general, there exists a basis $\{h_1, \ldots, h_\rho\}$ of $\operatorname{Num}_{\mathbb{R}} X$ such that the $h_i$ are ample. So for every $\epsilon_i > 0$, there exists $0 < \epsilon'_i < \epsilon_i$ such that $x + \sum_i \epsilon'_i h_i$ is a $\mathbb{Q}$-divisor (which defines an open set in $\operatorname{Num}_{\mathbb{R}} X$). Then $x + \sum_i \epsilon'_i h_i$ is nef, so because it is also a $\mathbb{Q}$-divisor, $(x + \sum_i \epsilon'_i h_i)^2 \geq 0$. By continuity, $x^2 \geq 0$. □

**Lemma 1.9.10.** *If $x$ is a nef $\mathbb{R}$-divisor and $y$ is an ample $\mathbb{R}$-divisor, then $x + y$ is an ample $\mathbb{R}$-divisor.*

*Proof.* If $x + y$ is a $\mathbb{Q}$-divisor, then $(x+y)^2 = x^2 + 2(x \cdot y) + y^2 > 0$ since $x \cdot y \geq 0$ by the light-cone lemma, and $(x + y) \cdot C > 0$ so we are done by Nakai–Moishezon.

In general, find a basis $h_1, \ldots, h_\rho$ of $\operatorname{Num}_{\mathbb{R}} X$ where $h_i$ is the class of an ample $H_i \in \operatorname{Div} X$. Then there exist $t_i$ such that $0 < t_i \ll 1$ and $y - \sum_i t_i H_i$ is ample (since the ample cone is open), and $x + y - \sum_i t_i h_i$ is a $\mathbb{Q}$-divisor. So we are in the same situation as the beginning of the proof. It follows that $x + y - \sum_i t_i h_i$ is ample, and therefore so is $x + y = (x + y - \sum_i t_i h_i) + \sum_i t_i h_i$ as the sum of two ample divisors. □

**Theorem 1.9.11** (Campana–Peirenell)**.**

$$\{x \text{ is ample in } \operatorname{Num}_{\mathbb{R}} X\} \iff \{x^2 > 0,\ x \cdot C > 0\, \forall \text{irred } C.\}$$

*Proof.* The direction $\Longrightarrow$ is obvious. We prove the converse. Choose a basis $h_1, \ldots, h_\rho$ as before, with $h_i = [H_i]$. Look at $h = \sum_i t_i h_i$ with $0 < t_i \ll 1$. We can assume $(x - h)^2 > 0$, and $(x - h) \cdot H > 0$ where $H$ is some fixed ample divisor, and $x - h$ is a $\mathbb{Q}$-divisor. So by a previous lemma, for all $N \gg 0$ we know $N(x - h)$ is effective. So write $N(x - h) = \sum_i n_j C_j$ where $n_j \in \mathbb{Z}$ and the $C_j$ are irreducible curves. By assumption, $x \cdot C_j > 0$ for all $j$. Choose $0 < \epsilon \ll 1$ such that $(x - \epsilon h) \cdot C_j > 0$. If $C \neq C_j$, then $N(C \cdot (x - h)) = \sum_j n_j (C \cdot C_j) \geq 0$. Therefore

$$C \cdot (x - \epsilon h) = C \cdot (x - h + (1 - \epsilon) h) > 0.$$

If $C = C_j$, then $C_j \cdot (x - \epsilon h) > 0$. Hence $x + \epsilon h$ is nef, and $x = (x - \epsilon h) + \epsilon h$ is a sum of nef and ample, and is therefore ample. □

*Remark.* With minor variations, this is true in all dimensions.

## 1.10 Closure of the ample cone

**Definition 1.10.1.** Let $\overline{\overline{A}}(X)$ be the closure of $A(X)$ in $\operatorname{Num}_{\mathbb{R}} X$, and $\overline{A}(X)$ be the closure of $A(X)$ in $\mathbb{C}_+$.

**Proposition 1.10.2.** *We have*

$$\overline{\overline{A}}(X) = \{x \in \operatorname{Num}_{\mathbb{R}} X : x^2 \geq 0, x \cdot C \geq 0\, \forall \text{irred } C\}$$
$$\overline{A}(X) = \{x \in \operatorname{Num}_{\mathbb{R}} X : x^2 > 0, x \cdot C \geq 0\, \forall \text{irred } C\}.$$

*Proof.* The inclusion $\subset$ is clear. For $\supset$, take $x$ satisfying $x^2 \geq 0$ and $x \cdot C \geq 0$ for all irreducible $C$. Pick $h$ ample, so $x + \epsilon h$ is nef plus ample, which is ample. So $x + \epsilon h \in A(X)$. When $\epsilon \to 0$, then $x \in \overline{A}(X)$. $\square$

*Remark.* If $x \in \partial\overline{\overline{A}}(X)$, then either $x^2 = 0$ or there exists a $C$ irreducible such that $x \cdot C = 0$. Similarly, if $x \in \partial\overline{A}(X)$, then there exists a $C$ irreducible such that $x \cdot C = 0$.

**Definition 1.10.3.** Define the **wall defined by** $C$

$$W^C := \{x \in \operatorname{Num}_{\mathbb{R}} X : x \cdot [C] = 0\}.$$

*Remark.* Note that $W^C = W^{C'}$ iff $C' = rC$ for some $r \in \mathbb{R}^+$, iff $C' = C$. This is because $C^2 < 0$ implies $C' \cdot C = rC^2 < 0$, but if $C \neq C'$ then $C \cdot C' \geq 0$, a contradiction. Conversely, if $C$ is irreducible and $C^2 < 0$, then $\overline{A}(X) \cap W^C$ is a non-empty open subset of $W^C$.

More generally, suppose $C_1, \ldots, C_r$ are irreducible curves on $X$. When is $\overline{A}(X) \cap W^{C_1} \cap \cdots \cap W^{C_n} \neq \emptyset$?

**Proposition 1.10.4.** *The intersection $\overline{A}(X) \cap W^{C_1} \cap \cdots \cap W^{C_n} \neq \emptyset$ iff $C_1, \ldots, C_r$ span a negative definite sublattice of $\operatorname{Num}_{\mathbb{R}} X$. In this case, the intersection is an open subset of $W^{C_1} \cap \cdots \cap W^{C_n}$.*

**Corollary 1.10.5.** *If $C_1, \ldots, C_r$ is above, then $r \leq \rho - 1$, and hence the set of walls $\{W^C : C^2 < 0\}$ is locally finite at every point of $\overline{A}(X)$.*

*Proof.* $C_1, \ldots, C_r$ are linearly independent in $\operatorname{Num}_{\mathbb{R}} X$. So $r \leq \rho$, but because there is a positive eigenvalue in $\operatorname{Num}_{\mathbb{R}} X$, they cannot in fact span the whole space. So $r \leq \rho - 1$. $\square$

*Proof of proposition.* If $x \in \overline{A}(X) \cap W^{C_1} \cap \cdots \cap W^{C_r}$, then $x^2 > 0$ but $[C_1], \ldots, [C_r] \in (x)^\perp$ in $\operatorname{Num}_{\mathbb{R}} X$, which is negative definite. The openness follows from the following proposition. $\square$

**Proposition 1.10.6.** *If $C_1, \ldots, C_r$ are distinct and irreducible and span a negative definite sublattice, then there exists a divisor $H \in \operatorname{Div} X$ such that $H$ is nef and big and $H \cdot C = 0$ iff $C = C_i$ for some $i$.*

**Lemma 1.10.7.** *Let $C_1, \ldots, C_r$ be irreducible curves such that the $C_i$ span a negative definite sublattice in $\operatorname{Num}_{\mathbb{R}} X$. Suppose $F$ is an effective divisor on $X$ such that no $C_i$ is contained in $F$.*

1. *If there exist $s_i \in \mathbb{R}$ such that $(F + \sum_i s_i C_i) \cdot C_j = 0$ for all $j$, then $s_i \geq 0$. Moreover if $I \subset \{1, \ldots, r\}$ such that $\bigcup_{i \in I} C_i$ is connected, and for some $j \in I$ we have $F \cdot C_j > 0$, then $s_i > 0$ for all $i \in I$.*
2. *If there exist $s_i \in \mathbb{R}$ such that $[F] + \sum_i s_i [C_i] = 0$ in $\operatorname{Num}_{\mathbb{R}} X$, then $F = 0$ and $s_i = 0$ for all $i$.*
3. *If there exist $s_i, t_i \in \mathbb{R}$ such that $[F] + \sum_i s_i [C_i] = \sum_i t_i [C_i]$ in $\operatorname{Num}_{\mathbb{R}} X$, then $F = 0$ and $s_i = t_i$ for all $i$.*

*Proof.* Write $F + \sum_i s_i C_i = F + \sum_{i \in A} s_i C_i + \sum_{j \in B} (-t_j) C_j$ where the $s_i \geq 0$ and the $t_i > 0$. By assumption, $A \cup B = \{1, \ldots, r\}$. Compute that

$$(F + \sum_i s_i C_i) \cdot (\sum_{j \in B} f_j C_j) = (F + \sum_{i \in A} s_i C_i)(\sum_{j \in B} t_j C_j) - (\sum_{j \in B} t_j C_j)^2 \geq 0.$$

By negative definiteness, $(\sum_j t_j C_j)^2 \geq 0$ and equality holds iff $t_j = 0$ for all $j$. Hence both terms in the above expression must be 0. In particular, $(\sum_{j \in B} t_j C_j)^2 = 0$.

Suppose $\bigcup_{i \in I} C_i$ is connected and $F \cdot C_k \neq 0$ for some $k \in I$. Since $F \cdot C_k > 0$ and $C_i \cdot C_k \geq 0$,

$$0 = (F + \sum_i s_i C_i) \cdot C_k \implies s_k \neq 0 \implies s_k > 0.$$

Say $C_k$ meets $C_j$. Then by the same argument, $s_j > 0$, and then apply connectedness inductively.

Now assume $[F] + \sum_i s_i[C_i] = 0$. This implies in particular that

$$(F + \sum_i s_i C_i) \cdot C_j = 0.$$

Hence $s_i \geq 0$ for all $i$. But $F + \sum_i s_i C_i$ is effective, and when dotted with $H$ ample, we get 0. Hence $F + \sum_i s_i C_i$ is actually 0, and hence $s_i = 0$. □

*Proof of proposition.* Take some ample divisor $H_0$. Then $H_0$ defines a $\mathbb{Z}$-linear function $\mathbb{Z}^r = \bigoplus \mathbb{Z}[C_i] \to \mathbb{Z}$. Since the $C_i$ are negative definite, there exist $r_i \in \mathbb{Q}$ such that $H \cdot C_j = (-\sum_i r_i C_i) \cdot C_j$ for all $j$, by looking at the intersection pairing $\mathbb{Q}^r \to (\mathbb{Q}^r)^\vee$. This implies

$$(H_0 + \sum_i r_i C_i) \cdot C_j = 0 \quad \forall j.$$

Since $H_0$ meets every $C_i$, the lemma implies $r_i > 0$. Hence there exists a $\mathbb{Q}$-divisor $H \coloneqq H_0 + \sum_i r_i C_i$ such that $H \cdot C_i = 0$ for all $i$, and $H_0 \cdot C > 0$ and $C_i \cdot C \geq 0$ for $C \neq C_i$, i.e. $H \cdot C > 0$. These facts together imply $H$ is nef. In addition, $H$ is big because

$$H^2 = H \cdot (H_0 + \sum_i r_i C_i) = H \cdot H_0.$$

Since $H$ is nef (i.e. $H^2 \geq 0$) and $H_0$ is ample (i.e. $H_0^2 \geq 0$), by the light-cone lemma $H \cdot H_0 > 0$. Now take $N \gg 0$ divisible, so $NH \in \operatorname{Div} X$ with the desired properties. □

*Remark.* By previous, $NH$ is effective, but we don't know that $NH$ is eventually bpf.

*Remark.* $\mathbb{Z}^2 = \bigoplus \mathbb{Z}[C_i]$ is negative definite, and the same is true for the corresponding $\mathbb{R}^r$. So $\operatorname{Num}_{\mathbb{R}} X$ is an orthogonal direct sum

$$\operatorname{Num}_{\mathbb{R}} X = \bigoplus_i \mathbb{R}[C_i] \oplus^\perp \{C_1, \ldots, C_r\}^\perp.$$

Therefore there is an orthogonal projection $p\colon \operatorname{Num}_{\mathbb{R}} X \to \{C_1, \ldots, C_r\}^\perp$. By definition, this perpendicular space is the same thing as the intersection $W^{C_1} \cap \cdots \cap W^{C_r}$ of the walls. The projection $p$ is always an open map, so $p(A(X))$ is an open subset of $W^{C_1} \cap \cdots \cap W^{C_r}$. This image consists of big and nef $\mathbb{R}$-divisors $x$ such that $x \cdot C > 0$ for all irreducible $C \neq C_i$. Also, $\overline{A}(X) \cap W^{C_1} \cap \cdots \cap W^{C_r}$ contains this open set $p(A(X))$.

## 1.11 Div and Num as functors

**Definition 1.11.1.** Let $X, Y$ be smooth projective surfaces, and $f\colon X \to Y$ be a generically finite morphism, i.e. surjective in this case where $\dim X = \dim Y$. The generic fiber has $d$ points, where $d = \deg f = [k(X) : k(Y)]$. There are **pullbacks**

$$f^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X, \quad L \mapsto f^* L$$
$$f^*\colon \operatorname{Div} Y \to \operatorname{Div} X, \quad g \mapsto g \circ f$$

where $C$ is an irreducible curve locally defined by $\{g = 0\}$. Note that $f^*(g)$ is effective but is no longer reduced or irreducible. Also, $f^*$ takes principal divisors to principal divisors, so it induces a map $\operatorname{Pic} Y = \operatorname{Div} Y/ \equiv \to \operatorname{Div} X/ \equiv = \operatorname{Pic} X$.

There is also a pushforward $f_*\colon \operatorname{Div} X \to \operatorname{Div} Y$, defined by extending the following by linearity:

1. if $f(C) = \{\mathrm{pt}\}$, then define $f_*(C) \coloneqq 0$;
2. if $f|_C\colon C \to f(C)$ has degree $n > 0$, then define $f_*(C) \coloneqq n f(C)$.

**Lemma 1.11.2.** *For all* $D \in \operatorname{Div} Y$ *and* $E \in \operatorname{Div} X$,

$$(f^*D) \cdot E = D \cdot (f_*E).$$

*Hence if* $D \sim 0$, *then* $f^*D \sim 0$. *Likewise, if* $E \sim 0$, *then* $f_*E \sim 0$.

**Lemma 1.11.3.** *For* $D, D' \in \operatorname{Div} Y$,

$$(f^*D) \cdot (f^*D') = d(D \cdot D'), \quad f_*f^*D = dD.$$

*Hence* $f^*\colon \operatorname{Num} Y \to \operatorname{Num} X$ *is injective.*

# Chapter 2

# Birational geometry

## 2.1 Blowing up and down

**Definition 2.1.1.** Given a point $x \in X$, there is a morphism $\pi\colon \operatorname{Bl}_x X = \tilde{X} \to X$ such that if $y \neq x \in X$, then $\pi^{-1}(y) = y$, and $\pi^{-1}(x) = E \cong \mathbb{P}^1$ and $E^2 = -1$. In fact, $\mathcal{O}_{\tilde{X}}(E)|_E = \mathcal{O}_{\mathbb{P}^1}(-1)$. Since $-2 = 2g(\mathbb{P}^1) - 2 = K_{\tilde{X}} \cdot E + E^2$, this implies $K_{\tilde{X}} \cdot E = -1$.

*Remark.* We can define $\tilde{X}$ as the relative proj $\operatorname{Proj}(\bigoplus_{n\geq 0} \mathfrak{m}_x^n)$ where $\mathfrak{m}_x$ is the maximal ideal of $x$.

*Remark.* We can pick local coordinates. Choose $U$ some small analytic neighborhood of $x$, and say $x_1, x_2$ are local analytic coordinates in $U$. Then $\tilde{U} = \pi^{-1}(U) = \tilde{U}_1 \cup \tilde{U}_2$ where

1. $\tilde{U}_1$ has analytic coordinates $x_1', x_2'$ where $x_1 = x_1'$ and $x_2 = x_1'x_2'$, and
2. $\tilde{U}_2$ has analytic coordinates $x_1'', x_2''$ where $x_1 = x_1''x_2''$ and $x_2 = x_2''$.

The **exceptional divisor** $E$ is defined in $\tilde{U}_1$ by $x_1' = 0$ and in $\tilde{U}_2$ by $x_2' = 0$.

*Remark.* Note that $\pi^{-1}(\mathfrak{m}_x)\mathcal{O}_{\tilde{X}} = I_E = \mathcal{O}_{\tilde{X}}(-E)$. The universal property is as follows. For all morphisms $\varphi\colon Y \to X$ from any scheme to $X$ such that $\varphi^{-1}(\mathfrak{m}_x)\mathcal{O}_Y$ is the ideal sheaf of a Cartier divisor, there is a unique factorization $\tilde{\varphi}\colon Y \to \tilde{X}$ via $\pi\colon \tilde{X} \to X$.

**Lemma 2.1.2.** *$\pi_*\mathcal{O}_{\tilde{X}}(nE)$ is $\mathcal{O}_X$ if $a \geq 0$, and $\mathfrak{m}_x^n$ if $a = -n < 0$.*

*Proof.* Clearly $\pi'_*(\mathcal{O}_{\tilde{X}}(nE)|_{\tilde{X}-E}) \cong \mathcal{O}_X|_{X-\{x\}}$, where $\pi'\colon \tilde{X} - E \to X - \{x\}$ is the isomorphism. Then if $j\colon X - \{x\} \to X$ is the inclusion, $j_*(\mathcal{O}_{X-\{x\}} = \mathcal{O}_X$. (This is a Hartogs phenomenon.) So

$$\Gamma(U, \mathcal{O}_X|_U) \to \Gamma(U - \{x\}, \mathcal{O}_X|_{U-\{x\}})$$

is an isomorphism. In all cases, this implies $\pi_*\mathcal{O}_{\tilde{X}}(nE) \subset \mathcal{O}_X$. If $a \geq 0$, then $\mathcal{O}_{|tildeX} \subset \mathcal{O}_{\tilde{X}}(nE)$, so

$$\mathcal{O}_X = \pi_*\mathcal{O}_{\tilde{X}} \subset \pi_*\mathcal{O}_{\tilde{X}}(nE) \subset \mathcal{O}_X,$$

which gives the first part of the lemma. Now suppose $a = -n < 0$. Look at $f \in \mathcal{O}_{X,x}$. Then $f = \sum_{\nu=m}^{\infty} g_\nu(x_1, x_2)$, where $g_\nu$ is homogeneous of degree $\nu$ and moreover, $g_m \neq 0$. (We say the **multiplicity** is $\operatorname{mult}_x f = m$.) This is equivalent to saying $f \in \mathfrak{m}_x^n - \mathfrak{m}_x^{n+1}$. In this case, $\pi^* f$ in $\pi^{-1}(U) = \tilde{U}_1$ is of the form

$$\sum_{\nu=m}^{\infty} g_\nu(x_1', x_1'x_2') = \sum_{\nu=m}^{\infty} (x_1')^\nu g_\nu(1, x_2') = (x_1')^m (g_m(1, x_2') + x_1'G).$$

The term $x_1'G$ vanishes on $E$, but $g_m(1, x_2')$ does not. This says $\pi^* f$ in $\tilde{U}_1$ is a section of $\mathcal{O}_{\tilde{X}}(-mE) - \mathcal{O}_{\tilde{X}}(-(m+1)E)$. Hence $\pi^* f \in \mathcal{O}_{\tilde{X}}(-mE)$ iff $f \in \mathfrak{m}_x^m$. So

$$\mathfrak{m}_x^m = \pi_*\mathcal{O}_{\tilde{X}}(-mE)(U) = \mathcal{O}_{\tilde{X}}(-mE)(\pi^{-1}(U)) \subset \mathcal{O}_X(U).$$

(Note that if we were to do the calculation in $\tilde{U}_2$, the same conclusion holds.) $\square$

**Definition 2.1.3.** Let $C$ be an effective curve in $X$. Since it is effective, in $U$ a small neighborhood of $x \in X$, we have $C = V(f)$, where $f$ is well-defined up to (at least locally) an element of $\mathcal{O}_{X,x}^*$. The **multiplicity** $\operatorname{mult}_x C$ is defined to be $\operatorname{mult}_x f$. By the above, $\pi^* C = mE + C'$, where $C' \cdot E = \mathcal{O}_{\tilde{X}}(C')|_E$, defined by $g_m(1, x_2')$ (or $g_m(x_1'', 1)$), i.e. $\{g_m = 0\}$ on $\mathbb{P}^1$. We call $C'$ the **proper transform** of $C$ in $\tilde{X}$.

*Remark.* Note that $C'$ is the effective curve $\pi^* C - mE$. For simplicity, in the case where $C'$ is reduced irreducible, $C' \cap E$ is finite and

$$X \supset C - \{x\} \cong C - \{x\} \in \tilde{X} - E,$$

and we could define $C'$ as the closure of $C - \{x\}$ in $\tilde{X}$.

## 2.2 Numerical invariants of $\tilde{X}$

**Proposition 2.2.1.** *$\pi^* \colon \operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$ and $\pi^* \colon \operatorname{Num} X \to \operatorname{Num} \tilde{X}$ are injective. Moreover,*

$$\operatorname{Pic} \tilde{X} = \pi^* \operatorname{Pic} X \oplus \mathbb{Z}\mathcal{O}_{\tilde{X}}(E),$$

*i.e. every line bundle $\tilde{L}$ on $\tilde{X}$ is uniquely written as $\tilde{L} = \pi^* L \otimes \mathcal{O}_{\tilde{X}}(nE)$ for some integer $n \in \mathbb{Z}$.*

*Proof.* If $L \in \operatorname{Pic} X$, then $\pi^* L \in \operatorname{Pic} \tilde{X}$, but by the projection formula, $\pi_* \pi^* L = L \otimes \pi_* \mathcal{O}_{\tilde{X}} \cong L$. Hence $\pi^*$ is injective on $\operatorname{Pic} X$. Likewise, for $\operatorname{Num} X$, we have $(\pi^* D \cdot \pi^* D') = 1(D \cdot D')$. Say $D$ is an irreducible curve on $\tilde{X}$ and $D \neq E$, then $\pi_* D =: C$ is an irreducible curve on $X$. Moreover, $D$ is the proper transform of $C$. But then $D = \pi^* C - mE$, where $m = \operatorname{mult}_x C$ (which could be 0). In particular, every divisor $D \in \operatorname{Div} \tilde{X}$ is of the form $\pi^* D' + aE$ where $D' \in \operatorname{Div} X$. In particular,

$$\tilde{L} = \mathcal{O}_{\tilde{X}}(D) \cong \pi^* \mathcal{O}_X(D') \otimes \mathcal{O}_{\tilde{X}}(aE), \qquad a = -\deg(\tilde{L}|_E).$$

So there exists a surjective homomorphism

$$\pi^* \operatorname{Pic} X \otimes \mathbb{Z} \to \operatorname{Pic} \tilde{X}, \quad (L, a) \mapsto \pi^* L \otimes \mathcal{O}_{\tilde{X}}(aE).$$

If $(L, a) \mapsto 0 = \mathcal{O}_{\tilde{X}}$, then $a = -\deg(\mathcal{O}_{\tilde{X}}|_E) = 0$. Hence $\mathcal{O}_{\tilde{X}} = \pi^* L$, but that implies $L \cong \mathcal{O}_X$, i.e. $L$ is also trivial.

For Num, we get a surjection

$$\operatorname{Num} X \otimes \mathbb{Z} \to \operatorname{Num} \tilde{X}, \quad (D, a) \mapsto \pi^* D + aE.$$

Since this is a surjection, $a$ is determined by $-a = \tilde{D} \cdot E$, where $\pi^* D + aE = \tilde{D} \in \operatorname{Num} \tilde{X}$. □

**Corollary 2.2.2.** *$\rho(\tilde{X}) = \rho(X) + 1$ where $\rho = \operatorname{rank} \operatorname{Pic}$.*

**Example 2.2.3.** Let $C$ be an effective curve. We saw that $\pi^* C = C' + mE$ where $C'$ is the proper transform and $E$ is not a component of $C'$. Here $m$ is the multiplicity of $C$ at $x$. Hence we can view this as the statement that $C' = \pi^* C - mE$, and $C' \cdot E = m \geq 0$. In fact, $\mathcal{O}_{\tilde{X}}(C')|_E$ is the projective tangent cone to $C$ at $x$. More canonically, $E \cong \mathbb{P}T_{X,x}$. Write $C = V(g)$ with $g = \sum_m g_m$ its decomposition into homogeneous parts, so that $V(g_m)$ defines a subscheme of $\mathbb{P}^1 \cong E$. Furthermore, $(C')^2 = (\pi^* C)^2 - m^2 = C^2 - m^2$, i.e. $(C')^2 < C^2$ if $m \geq 1$.

**Proposition 2.2.4.** *As a line bundle, $K_{\tilde{X}} = \pi^* K_X \otimes \mathcal{O}_{\tilde{X}}(E)$, so that as divisor classes, $K_{\tilde{X}} = \pi^* K_X + E$.*

*Proof.* We know $K_{\tilde{X}} = \pi^* L \otimes \mathcal{O}_{\tilde{X}}(aE)$ for some $L$ and $a$, but $a = -\deg(K_{\tilde{X}}|_E) = 1$ and

$$\pi_* K_{\tilde{X}} = \pi_*(\pi^* L \otimes \mathcal{O}_{\tilde{X}}(E)) = L \otimes \pi_* \mathcal{O}_{\tilde{X}}(E) = L.$$

In particular, this means $L|_{X - \{x\}} \cong K_{\tilde{X}}|_{\tilde{X} - E} = K_X|_{X - \{x\}}$. But in general, for any smooth scheme of dimension $\geq 2$, we have $\operatorname{Pic}(X - \{x\}) \cong \operatorname{Pic}(X)$, so $L \cong K_X$. □

**Corollary 2.2.5.** *$c_1(\tilde{X})^2 = c_1(X)^2 - 1$, but $p_g(\tilde{X}) = p_g(X)$ and in fact all plurigenera $P_n$ are equal for all $n \geq 1$.*

*Proof.* Since $K_{\tilde{X}} \sim \pi^* K_X + E$, we know

$$c_1(\tilde{X})^2 = K_{\tilde{X}}^2 = (K_X)^2 - 1 = c_1(X) - 1.$$

For the plurigenus,

$$\begin{aligned} P_n(\tilde{X}) &= h^0(\tilde{X}, K_{\tilde{X}}^{\otimes n}) \\ &= h^0(\tilde{X}, (\pi^* K_X^{\otimes n}) \otimes \mathcal{O}_{\tilde{X}}(nE)) \\ &= h^0(X, K_X^{\otimes n} \otimes \pi_* \mathcal{O}_{\tilde{X}}(nE)) \\ &= h^0(X, K_X^{\otimes n} \otimes \mathcal{O}_X) = P_n(X) \end{aligned}$$

where the third equality is the trivial case of the Leray spectral sequence. □

*Remark.* Note that $H^0(\tilde{X}, K_{\tilde{X}}^{-1})$ is not the same as $H^0(X, K_X^{-1})$ and in general not for $P_n$ for $n \leq -1$.

**Proposition 2.2.6.** *$q(\tilde{X}) = q(X)$.*

*Remark.* The change in topology from blowing-up is well-understood. As 4-manifolds, $\tilde{X}$ is diffeomorphic to $X \# \overline{\mathbb{CP}}^2$ where $\overline{\mathbb{CP}}^2$ is $\mathbb{CP}^2$ with the opposite orientation. By van Kampen, $\pi_1(\tilde{X}, *) = \pi_1(X, *)$, and therefore $H_1(\tilde{X}; \mathbb{Z}) \cong H_1(X; \mathbb{Z})$. Hence $q(\tilde{X}) = q(X)$ since $b_1 = 2q$. Another observation is $b_2(\tilde{X}) = b_2(X) + 1$, since

$$\bar{H}^2(X; \mathbb{Z}) = \bar{H}^2(X, \mathbb{Z}) \oplus \mathbb{Z}[E].$$

Clearly $b_2^+(\tilde{X}) = b_2^+(X)$.

*Proof.* Claim: $R^1 \pi_* \mathcal{O}_{\tilde{X}} = 0$. Hence by Leray,

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^1(X, \pi_* \mathcal{O}_{\tilde{X}}) \oplus (0) = H^1(X, \mathcal{O}_X),$$

so taking dimensions we are done. The claim follows from the formal functions theorem:

$$R^1 \pi_* \mathcal{O}_{\tilde{X}} = \varprojlim_n H^1(\mathcal{O}_{nE})$$

so it suffices to show $H^1(\mathcal{O}_{nE}) = 0$. But we have the short exact sequence

$$0 \to \mathcal{O}_E(-(n-1)E) \to \mathcal{O}_{nE} \to \mathcal{O}_{(n-1)E} \to 0.$$

When $n = 2$, we have $H^1(\mathcal{O}_E) = 0$. Assume by induction that $H^1(\mathcal{O}_{(n-1)E}) = 0$. But then $H^1(\mathcal{O}_E(-(n-1)E)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-1)) = 0$ for $n \geq 0$. So $H^1(\mathcal{O}_{nE}) = 0$. □

*Remark.* As an exercise, compute $R^1 \pi_* \mathcal{O}_{\tilde{X}}(nE)$ for all $n \in \mathbb{Z}$.

## 2.3 Embedded resolutions for curves on a surface

Suppose $C \subset X$, and $x \in C$. Then $m \coloneqq \operatorname{mult}_x C \geq 1$, with equality iff $C$ is smooth at $x$. Assume $C$ is reduced irreducible, so $p_a(C) \geq 0$. Let $\tilde{X} \coloneqq \operatorname{Bl}_x X$ and $C'$ be the proper transform of $C$. Again $C'$ is reduced irreducible.

**Proposition 2.3.1.** *$p_a(C') = p_a(C) - m(m-1)/2$. Hence if $m \geq 2$, then $p_a(C') < p_a(C)$.*

*Proof.* Compute that

$$\begin{aligned}2p_a(C') - 2 &= (K_{\tilde{X}} \cdot C') + (C')^2 = (K_X \cdot C + m) + (C^2 - m^2) \\ &= K_X \cdot C + C^2 - (m^2 - m) = 2p_a(C) - (m^2 - m).\end{aligned}$$

□

*Remark.* This leads to an algorithm for finding a resolution of a (reduced irreducible) curve on a surface. Start with $C \subset X$. If $x \in C$ is smooth, stop. Otherwise $\operatorname{mult}_x C \geq 2$, so blow up at $x$ to get $C'$, which now has $p_a(C') < p_a(C)$. Since $p_a \geq 0$, this process must terminate.

Now say $\tilde{C}$ is the normalization of $C$. Then we have the genus drop $\delta = p_a(C) - g(\tilde{C})$. Hence

$$\delta = \sum_{y \to x \in C} \frac{m_y(m_y - 1)}{2}.$$

In principle, this is a formula for the genus drop of a curve.

**Example 2.3.2.** Take the local equation $\prod_{i=1}^m (x_1 - \lambda_i x_2)$ where the $\lambda_i$ are distinct, i.e. $m$ distinct branches passing through a point. After a single blow-up, $C'$ becomes smooth, and so the genus drop at $x$ is $m(m-1)/2$. In particular, for $m > 2$, the singularity is not obtained by taking $m$ smooth curves $C_i$ and identifying the points $y_i$. Specifically, if we look at $\{f \in \mathcal{O}_{\tilde{C}} : f(y_1) = \cdots = f(y_m)\}$, the resulting subscheme is not the same analytically (for $m \geq 3$). We can check that the genus drop is not the same.

**Example 2.3.3.** Take $x_1^2 - x_2^3$ and blow up the cusp at the origin. The proper transform in $\tilde{X}$ is given in $\tilde{U}_1$ by $(x_1')^2(1 - (x_1')(x_2')^3)$. Note that the first term meets $E$, but the second term does not meet $\tilde{U}_1 \cap E$. In $\tilde{U}_2$, it is given by $(x_2'')^2((x_1'')^2 - (x_2''))$. Note that the first term meets $E$, and the second term meets $E$ tangentially (with $C' \cdot E = 2$). So the genus drop is $\delta = 2(2-1)/2 = 1$.

**Theorem 2.3.4** (Castelnuovo's criterion)**.** *Let $Y$ be a (smooth, projective) surface, and suppose $E \subset Y$ is a (reduced irreducible) curve with $E \cong \mathbb{P}^1$ and $E^2 = -1$. Then there exists a smooth projective surface $X$ and $x \in X$ such that $Y = \operatorname{Bl}_x X$.*

**Definition 2.3.5.** Let $\rho\colon Y \to X$ be the projection. We say $E$ can be **(smoothly) contracted** if $\rho$ blows down $E$. We say $E$ is an **exceptional curve** (of the first kind).

*Remark.* If $E$ is an irreducible curve on $Y$, then $E$ is exceptional iff $E^2 = -1$ and $E \cdot K_Y = -1$. This is by adjunction: $2p_a(E) - 2 = K_Y \cdot E + E^2$. In fact, $E$ is exceptional iff $E^2 < 0$ and $E \cdot K_Y < 0$. One direction is clear, and the converse is true because $2p_a(E) - 2 \geq -2$ since $p_a(E) \geq 0$.

*Remark.* In higher dimensions, we can always blow down $\mathbb{P}^{n-1}$s analytically, but not necessarily projectively.

**Theorem 2.3.6** (Easy version of Zariski's main theorem)**.** *Let $f\colon Y \to X$ be a birational morphism between two smooth projective surfaces. Let $x \in X$. Then either $f$ is an isomorphism at $x$ or there exists a curve $C \subset Y$ such that $\pi_*(C) = \{x\}$.*

*In higher dimensions, we assume $Y$ is a smooth variety and $X$ is any variety, and $\mathcal{O}_{X,x}$ is a UFD. Then either $f$ is an isomorphism at $x$, or there exists a hypersurface $V \subset Y$ such that $x \in f(V)$ and* $\operatorname{codim}_X \overline{f(V)} \geq 2$.

*Proof sketch.* Take $y \in f^{-1}(X)$, and consider the pullback $f^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ which induces an isomorphism of function fields. We can assume $\mathcal{O}_{Y,y}$ is the localization of a finitely generated $\mathbb{C}$-algebra $\mathbb{C}[t_1, \ldots, t_n]I$. Write $t_i = f_i/g_i$ for $f_i, g_i \in \mathcal{O}_{X,x}$ (suppressing the $\pi^*$). Because $\mathcal{O}_{X,x}$ is a UFD, assume that $f_i, g_i$ are relatively prime in $\mathcal{O}_{X,x}$. If $g_i$ is a unit for all $i$, then $t_i \in \operatorname{im} f^*$, and hence $\mathcal{O}_{Y,y} = \mathcal{O}_{X,x}$, i.e. $f$ is a local isomorphism. Otherwise $V(g_i)$ is a hypersurface $V \subset Y$, and by assumption $y \in V(g_i)$. So $f(V) \subset V(g_i, t_i)$. But in $X$, we have $g_i, t_i$ relatively prime, and hence $V(g_i, t_i)$ is codimension $\geq 2$. □

*Proof of Castelnuovo's criterion.* There are two main steps in this proof:

1. construct a normal surface $X$ at the point $x \in X$ and a birational morphism $\rho\colon Y \to X$ such that $\rho(E) = \{x\}$ and such that $\rho|_{Y-E}\colon Y - E \to X - \{x\}$;

2. show that $X$ is in fact smooth at $x \in X$ and then that $Y \cong \mathrm{Bl}_x X$ factors $\rho$ via the projection $\pi\colon \mathrm{Bl}_x X \to X$.

Start with a very ample $H_0$ in $Y$. Let $a = H_0 \cdot E > 0$, and set $H = H_0 + aE$. Clearly $H \cdot E = 0$ by construction. Also, $H \cdot C > 0$ for all $C \neq E$. Assume $H^1(Y, \mathcal{O}_Y(H_0)) = 0$. Claim: $|H_0 + aE| = |H|$ is bpf, and defines a morphism $\varphi\colon Y \to \mathbb{P}^N$ which separates points $y_1, y_2 \notin E$, separates points $y \notin E$ from $E$, and separates tangent directions at $y \notin E$, and $\varphi(E) = \{\mathrm{pt}\}$. Once we have that, we take $X$ to be the normalization of $\varphi(Y)$.

To prove the claim, note that $|H_0| \subset |H_0 + aE|$ by taking sections in $|H_0|$ and adding $aE$ to them. The only possible base points are on $E$. Also, $\mathcal{O}_Y(H_0 + aE)|_E \cong \mathcal{O}_E$, so if there are no base points on $E$, the morphism $\varphi\colon Y \to \mathbb{P}^N$ satisfies $\varphi(E) = \{\mathrm{pt}\}$. Look at the exact sequence

$$0 \to \mathcal{O}_Y(H_0 + (a-1)E) \to \mathcal{O}_Y(H_0 + aE) \to \mathcal{O}_E \to 0.$$

We want $H^0(\mathcal{O}_Y(H_0 + aE)) \twoheadrightarrow H^0(\mathcal{O}_E)$, so that we can lift a constant section in $\mathcal{O}_E$ to get something that does not vanish along $E$. Hence we want $H^1(\mathcal{O}_Y(H_0 + (a-1)E)) = 0$. In general, look at

$$0 \to \mathcal{O}_Y(H_0 + (k-1)E) \to \mathcal{O}_Y(H_0 + kE) \to \mathcal{O}_E(a-k) \to 0.$$

For $k = 1$, we have $H^1(\mathcal{O}_Y(H_0)) \to H^1(\mathcal{O}_Y(H_0+E)) \to H^1(\mathcal{O}_E(a-1)) = 0$, because $a > 0$ implies $a-1 \geq 0$. Now we induct: for $k = j$, we use that $H^1(\mathcal{O}_E(a-j)) = 0$ since $a - j \geq 1$.

By the universal property of normalization, the map $Y \to \varphi(Y) \subset \mathbb{P}^N$ factors through the normalization to give a map $\pi\colon Y \to X$. Since $X$ is normal and $\varphi$ is birational, $\mathcal{O}_X = \pi_*\mathcal{O}_Y$. To show $X$ is smooth at $x \coloneqq \pi(E)$, it suffices to show $\widehat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[t_1, t_2]]$. By the formal functions theorem,

$$\widehat{\mathcal{O}}_{X,x} = \widehat{R^0\pi_*\mathcal{O}_Y}_x = \varprojlim_n H^0(Y, \mathcal{O}_Y/\mathfrak{m}_x^n\mathcal{O}_Y).$$

But $\mathcal{O}_Y/\mathfrak{m}_x^n\mathcal{O}_Y$ is supported on $E$, so it is annihilated by some power of $I_E \coloneqq \mathcal{O}_Y(-E)$. Hence there is a surjection $\mathcal{O}_Y/\mathcal{O}_Y(-nE) \to \mathcal{O}_Y/\mathfrak{m}_x^n\mathcal{O}_Y$ for some $n$, and it suffices to show $\varprojlim H^0(Y, \mathcal{O}_Y/\mathcal{O}_Y(-nE))$ is a formal power series ring (in two variables). Then $\varprojlim H^0(Y, \mathcal{O}_Y/\mathcal{O}_Y(-nE)) \to \varprojlim H^0(Y, \mathcal{O}_Y/\mathfrak{m}_x^n\mathcal{O}_Y)$ is a surjection between rings of Krull dimension 2, and therefore is an isomorphism. Take the exact sequence

$$0 \to (\mathcal{O}_Y(-nE)/\mathcal{O}_Y(-(n+1)E) \cong \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathcal{O}_Y/\mathcal{O}_Y(-(n+1)E) \to \mathcal{O}_Y/\mathcal{O}_Y(-nE) \to 0,$$

so that the induced map $H^0(\mathcal{O}_Y/\mathcal{O}_Y(-(n+1)E)) \to H^0(\mathcal{O}_Y/\mathcal{O}_Y(-nE))$ is surjective for every $n \geq 0$ (because $H^1(\mathcal{O}_{\mathbb{P}^1}(n))$ vanishes). In particular, for $n = 1$ we get the inclusion

$$\mathcal{O}_{\mathbb{P}^1}(1) \cong H^0(\mathcal{O}_Y(-E)/\mathcal{O}_Y(-2E)) \subset H^0(\mathcal{O}_Y/\mathcal{O}_Y(-2E))$$

so there are variables $z_1^{(1)}, z_2^{(2)}$ in $H^0(\mathcal{O}_Y/\mathcal{O}_Y(-2E))$. For all $n \geq 1$, pick variables $z_1^{(n)}, z_2^{(n)}$ mapping onto $z_1^{(n-1)}, z_2^{(n-1)}$. Taking $z_i \coloneqq \varprojlim z_i^{(n)}$, we have $\mathbb{C}[[z_1, z_2]] \cong \varprojlim H^0(\mathcal{O}_Y/\mathcal{O}_Y(-nE))$.

Now we must identify $Y$ with $\mathrm{Bl}_x X$. Claim: there exists $f\colon Y \to \tilde{X} \coloneqq \mathrm{Bl}_x X$ that factors a map $\pi\colon Y \to X$. By the universal property of blowups, it is enough to show $\mathfrak{m}_x\mathcal{O}_Y$ is the ideal sheaf of a Cartier divisor of $Y$. Clearly $\mathfrak{m}_x\mathcal{O}_Y \subset I_E$, but we saw that we can pick $t_1, t_2 \in \mathfrak{m}_x\mathcal{O}_{X,y}$ which generate $\mathcal{O}_Y(-E) = I_E$ mod $\mathcal{O}_Y(-2E)$. By Nakayama, they generate $I_E$. Hence $\mathfrak{m}_x\mathcal{O}_Y = I_E$, and we get a factorization.

Let $\tilde{E} \subset \tilde{X}$ be the exceptional divisor. Clearly $f\colon Y - E \cong X - \{x\} \cong \tilde{X} - \tilde{E}$ is an isomorphism, so $f(E) \subset \tilde{E}$. Suppose $f$ is not an isomorphism, so at some point $y \in \tilde{E}$ it is not an isomorphism. By Zariski's main theorem, there exists a curve $C$ such that $f(C) = y \in \tilde{E}$. But then $C$ must be $E$ itself, because nothing else is mapped to $\tilde{E}$. However $f(E) = \tilde{E}$ since $f$ is surjective, a contradiction. Hence $f$ is an isomorphism. □

**Proposition 2.3.7.** *Suppose $Y, E, X$ are as in Castelnuovo's theorem, with $\pi\colon Y \to X$ the blow-down. Let $f\colon Y \to Z$ be any morphism such that $f(E) = \{x\} \subset Z$. Then there exists $\bar{f}\colon X \to Z$ factoring $f$.*

*Proof.* We have the graph $\Gamma_f \subset Y \times Z$ of $f$, and we also have the image $\Gamma_{\bar{f}} \subset X \times Z$ of $\Gamma_f$ under the projection $\pi\colon Y \to X$. It is also the closure of the graph of $f|_{X-\{x\}}$. We want to show that $\Gamma_{\bar{f}} \cong X$ under $\pi$, so that $X \cong \Gamma_{\bar{f}} \xrightarrow{\pi_Z} Z$ is the desired factorization. If $\Gamma_{\bar{f}} \to X$ is not an isomorphism at $x$, by Zariski's main theorem there exists a curve $C \subset \Gamma_{\bar{f}}$ such that $\pi_1(C) = \{x\}$. On the other hand, $C$ is the image of $E$ from $Y$, because if not, then $C - E$ is a curve in $Y - E \cong X - \{x\}$, which is impossible. But then $\pi_2(C) \subset \pi_2(E) = \{\text{pt}\}$, contradicting that $C \subset X \times Z$ is a curve. ☐

*Remark.* This proposition is also true if $X$ is just assumed normal at $x$. We will see three applications of Castelnuovo's theorem.

**Corollary 2.3.8** (Factorization of morphisms)**.** *Suppose $Y$ and $X$ are surfaces, and $f\colon Y \to X$ is a birational morphism. Then $f$ is a sequence of blow-ups, i.e.*

$$Y = X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \xrightarrow{\pi_1} X_0 = X.$$

*Proof.* One way to show this is to show that $Y \to X$ factors via $\mathrm{Bl}_x X$, and we induct on $\rho(Y) - \rho(X)$. Another way is to show there exists $E$ exceptional in $Y$ such that $f(E) = \{\text{pt}\}$. Then $Y \to X$ factors through the blow-down $Y \to \bar{Y}$. We do the second approach.

Take $f\colon Y \to X$. If $f^{-1}$ is defined at all $x \in X$, then $f$ is an isomorphism and we are done. By Zariski's main theorem, there exists a curve $C = \bigcup_{i=1}^r C_i$ such that $f(C) = \{\text{pt}\}$. There exists $H$ ample on $X$, so $f^*H$ is nef and big on $Y$, but $(f^*H) \cdot C_i = 0$ for all $i$. By Hodge index theorem, $C_i^2 < 0$ for all $i$. Claim: for some $i$, we have $K_Y \cdot C_i < 0$. Then $C_i = E$ is exceptional and we are done.

To prove the claim, note that there is an inclusion $f^*K_X \to K_Y$ which is an isomorphism at the generic point. So $K_Y = f^*K_X + \sum_i r_i C_i + D$ where $r_i > 0$ and $D$ is effective (and the $C_i$ are not in the support of $D$). There exists $i$ such that $C_i \cdot \sum_j r_j C_j < 0$, because otherwise $(\sum_j r_j C_j)^2 \geq 0$, but by Hodge index, the $C_j$ span a negative-definite sublattice. If $r_i = 0$ and $f^*K_X = K_Y$ in a neighborhood of $x$, by the inverse function theorem, $Y \cong X$, a contradiction. Fix such an $x$. Then $K_Y \cdot C_i = (f^*K_X) \cdot C_i + (\sum_j r_j C_j) \cdot C_i + (D \cdot C_i)$, but only the middle term is non-zero (and is negative). ☐

**Definition 2.3.9.** A **rational map** $f\colon X \to Z$ is a map defined on a (non-empty) Zariski-open subset of $X$, and in fact on the complement of a codimension 2 subset of $X$.

**Corollary 2.3.10** (Elimination of indeterminacy)**.** *Let $X$ be a surface, and $Z$ any variety. Let $f\colon X \to Z$ be a rational map. Then there exists a sequence of blow-ups $Y = X_n \to X_{n-1} \to \cdots \to X_0 = X$ and a morphism $\hat{f}\colon Y \to Z$ factoring through $f$.*

*Proof.* Assume $Z = \mathbb{P}^N$, and that $f$ is defined by a linear system $V \subset |D|$. Also, we might as well assume $N \geq 1$. So there exist $D_1, D_2 \in V$ such that $D_1 \cap D_2$ is finite. Then $D^2 = D_1 \cdot D_2 \geq 0$. In fact, $D$ is nef. If $B_S V = \emptyset$, then $f$ is already a morphism. Pick $x \in B_S V$ and let $m_0$ be the minimal multiplicity $\mathrm{mult}_x C$ for $C \in V$. Then $\pi^*C - m_0 E \geq 0$ for all $C \in V$, and there exists $D_0 \in V$ such that $D_1 := \pi^* D_0 - m_0 E$ is the proper transform of $D_0$. Then let $V' := \{\pi^*C - m_0 E : C \in V\}$. Now $(D_1)^2 \geq 0$ because $V'$ has no fixed curves. However $0 \leq D_1^2 = D^2 - m_0^2 < D^2$. Now we have replaced $V$ by $V_1$ on the blow-up, and by construction the map $X_1 \to \mathbb{P}^N$ and $X \to \mathbb{P}^N$ agree when they are defined. Moreover, the self-intersection decreases, and is non-negative. Hence this process terminates in a finite number (less than $D^2$) of steps. So eventually we get a bpf linear system. ☐

**Corollary 2.3.11** (Factorization of birational maps)**.** *Let $f\colon X_1 \to X_2$ be a birational map. Then there exists $Y$ and birational morphisms $g_i\colon Y \to X_i$, where $g_1$ is a composition of blow-ups and $g_2$ is a composition of blow-downs, such that the diagram commutes.*

**Corollary 2.3.12.** *The invariants $p_g$, $P_n$, and $q$ are birational invariants.*

*Remark.* In higher dimensions, these are still birational invariants, but the argument is more complicated.

**Example 2.3.13** (Cremona transformation $\mathbb{P}^2 \to \mathbb{P}^2$)**.** Define the morphism $f$ by $[x_0 : x_1 : x_2] \mapsto [1/x_0 : 1/x_1 : 1/x_2]$. Clearly $f^2 = \mathrm{id}$. But it is not everywhere defined. Rewrite it as $[x_0 : x_1 : x_2] \mapsto [x_1x_2 : x_0x_2 : x_0x_1]$. Hence it is defined by the linear system $V$ of quadrics passing through $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. The base locus of $V$ is exactly these three points: it is easy to see that this is a complete linear system of quadrics passing through these three points. The morphism is therefore undefined at these three points. If we let $E_i$ be the exceptional curves arising from blowing up each of these points, and $L_{ij}$ the proper transforms of lines between the three points, then $f$ is the composition of the blow-ups to get $E_i$, and then blow-downs of $L_{ij}$. This is the factorization of $f$.

**Definition 2.3.14.** Let $|D|$ be a (often bpf) linear system on $X$. Fix a point $x \in X$, and look at the sub-linear system $V \coloneqq \{C \in |D| : x \in C\}$. Then $x \in B_S V$. We say $x$ is an **assigned base point**, and any other base points are called **unassigned**. The linear system $V$ gives a rational map $X \to \mathbb{P}^N$ undefined at $x$. (Often we write $V \coloneqq |D - x|$.) Now if we look at $\tilde{X} \coloneqq \mathrm{Bl}_x X \xrightarrow{\pi} X$, we can look at the pullback $\pi^* V$ of divisors in $V$. Then there is an identification $V \cong |\pi^* D - E|$. This is a linear system on $\tilde{X}$, The good case is when $|\pi^* D - E|$ is bpf on $\tilde{X}$. In particular, there must exist sections which are smooth at $x$. Then $x$ is a **simple base point**. Otherwise, if there are still base points on $E$, they are called **infinitely-near base points**.

**Example 2.3.15.** We can assign infinitely-near base points. Take $|D - 2x| = \{C \in |D| : x \in C, \operatorname{mult}_x C \geq 2\}$.

## 2.4 Minimal models of surfaces

**Definition 2.4.1.** A surface $X$ is **minimal** if there are no exceptional curves on $X$. (Roughly, this means that $X$ can't be blown down to a smooth surface, i.e. there is no birational morphism $X \to X'$ where $X'$ is smooth.) A **minimal model** for a surface $Y$ is a birational morphism $\pi \colon Y \to X$ where $X$ is minimal.

**Proposition 2.4.2.** *For any $Y$, there exists a minimal model.*

*Proof.* If $Y$ is already minimal, take $\pi \colon Y \to Y$ the identity. Otherwise there exists an exceptional curve $E$ on $Y$, so we can contract it to get a surface $Y_1$, and $\rho(Y_1) = \rho(Y) - 1$. ☐

**Example 2.4.3.** Minimal models may not be unique. If $Y$ is a surface with two exceptional divisors $E_1, E_2$, and $E_1 \cap E_2 \neq \emptyset$, we can only contract one of them, and so a choice is involved. More generally, suppose there exists $C$ a smooth rational curve on $Y$ with $C^2 = n \geq 0$. Then if we blow up $C$ at $n+1$ distinct points, the result $C'$ has $(C')^2 = -1$, and again we have a choice.

**Definition 2.4.4.** A surface $X$ is a **strong minimal model** of $Y$ if:

1. $X$ is a minimal model of $Y$;
2. there exists a birational morphism $f \colon Y \to X$ such that if $\tilde{Y} \to Y$ is a blow-up and $g \colon \tilde{Y} \to X'$ is a birational morphism to a smooth surface $X'$, then

$$\begin{array}{ccc} Y & \longleftarrow & \tilde{Y} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xleftarrow{\exists h} & X'. \end{array}$$

We say $X'$ **dominates** $X$ if there exists a morphism $X' \to X$ which makes the diagram commute. This gives us a partial order on the set of models. In other words, a strong minimal model is dominated by every other model.

*Remark.* From our example earlier, two exceptional curves that meet non-trivially are obstructions to the existence of a strong minimal model. If $X$ is a strong minimal model of $Y$ and $X'$ is any minimal model, then in fact $X \cong X'$.

**Theorem 2.4.5.** *Suppose, for some $N \geq 1$, that $P_N(Y) \neq 0$. Then $Y$ has a strong minimal model.*

*Remark.* More deeply, this is an if and only if statement, i.e. $Y$ has a strong minimal model iff $P_N(Y) \neq 0$ for some $N$, iff there does not exist $C \subset Y$ where $C$ is smooth rational and $C^2 \geq 0$.

*Proof.* Let $X$ be a minimal model. Let $\tilde{Y} \to Y'$ be a blow-up. We must show there exists $f\colon Y' \to X$. We can assume $\tilde{Y} = Y$. Then $Y \to X$ is a composition of blow-downs $Y = Y_n \to Y_{n-1} \to \cdots \to Y_0 = X$ (or an isomorphism). Also, we can assume $Y \neq X$, because otherwise $Y$ is minimal, and $Y' = Y$. So we assume $n \geq 1$ and there is some curve $E_n \subset Y$ contracted to a point in $Y_{n-1}$. We induct on $n$, so suppose $Y \to Y'$ is a single blow-down contracting the curve $F \subset Y$.

1. The easiest case is where $E_n = F$. Then $Y_{n-1} = Y'$ and we are done.
2. If $E_n \cap F = \emptyset$, then the image of $F$ in $Y'$ is still exceptional, and we can blow it down to get $\bar{Y}$. But $\bar{Y}$ is the same as the blow-down of $Y_{n-1}$ by contracting $E_n$ (by the universal property of blow-ups).
3. The bad case is $E_n \cap F \neq \emptyset$, but is finite. Claim: this contradicts $P_N(Y) \neq 0$. Since $(E_n + F) \cdot E_n = -1 + E_n \cdot F \geq 0$, we know $E_n + F$ is nef. We have $NK_Y$ effective, so choose some effective $D \in |NK_Y|$. Because $E_n + F$ is nef, $(E_n + F) \cdot D \geq 0$. Then $(E_n + F) \cdot (NK_Y) \geq 0$, and hence $(E_n + F) \cdot K_Y \geq 0$. Clearly this can only happen if $E_n \cdot K_Y \geq 0$ or $F \cdot K_Y \geq 0$, but they are both exceptional so these are in fact both $-1$, a contradiction. □

## 2.5 More general contractions

Let $X, Y$ be smooth complex (connected) manifolds of dimension 2.

**Definition 2.5.1.** Let $\operatorname{Div}^c X$ denote the abelian group generated by compact (holomorphic) curves on $X$ (i.e. reduced, irreducible, dimension 1). An element in $\operatorname{Div}^c X$ is a finite sum $\sum_{i=1}^r n_i C_i$, where $n_i \in \mathbb{Z}$ and $C_i$ are compact irreducible curves. There is a pairing

$$\operatorname{Pic} X \otimes_{\mathbb{Z}} \operatorname{Div}^c X \to \mathbb{Z}, \quad L \cdot \sum n_i C_i := \sum_i n_i \deg L|_{C_i}$$

(where $\operatorname{Pic} X$ is holomorphic line bundles). There is also a natural homomorphism

$$\operatorname{Div}^c X \to \operatorname{Pic} X, \quad \sum n_i C_i \mapsto \mathcal{O}_X\Big(\sum n_i C_i\Big).$$

So in particular there is an induced pairing $\operatorname{Div}^c X \otimes_{\mathbb{Z}} \operatorname{Div}^c X \to \mathbb{Z}$.

*Remark.* We can extend this to holomorphic but not necessarily compact divisors on $X$. The problem with non-compact surfaces is that divisors may have infinitely many components, so taking the free abelian group doesn't work. However in practice we will only intersect divisors with compact curves, so this doesn't matter. In general, let $M_X$ be the sheaf of meromorphic functions on $X$, and define $D_X := M_X^*/\mathcal{O}_X^*$. Then $\operatorname{Div} X := H^0(X, D_X)$. Concretely, there exists an open cover $X = \bigcup U_\alpha$ and there exists a meromorphic function $f_\alpha \in M_X(U_\alpha)$, and we think of $D|_{U_\alpha}$ as the divisor associated to $f_\alpha$ (i.e. zeros minus poles). Of course, two $f_\alpha$ may define the same divisor if locally they differ by an element of $\mathcal{O}_X^*$. From the exact sequence $0 \to \mathcal{O}_X^* \to M_X^* \to D_X \to 0$, we get

$$\operatorname{Div} X = H^0(X, D_X) \to H^1(X, \mathcal{O}_X^*) = \operatorname{Pic} X,$$

which has the usual interpretation. By the exponential sheaf sequence, the Chern class map $c_1$ still takes $H^1(X, \mathcal{O}_X^*)$ into $H^2(X, \mathbb{Z})$.

Now inside $D_X$ there is a semi-group $\mathcal{O}_X \setminus \{0\}$. The image $H^0(X, \mathcal{O}_X \setminus \{0\})/\mathcal{O}_X^*$ inside $D_X$ is the semi-group of effective divisors.

**Definition 2.5.2.** Let $Z = \sum n_i C_i$ be a compact divisor in $\operatorname{Div}^c X$. Assume it is effective and non-empty. There are many ways we could define the **arithmetic genus** $p_a(Z)$ of $Z$:

1. $2p_a(Z) = (K_X + Z) \cdot Z$;
2. $p_a(Z) = 1 - \chi(\mathcal{O}_Z)$.

We proved the equality of these definitions in the projective case using Riemann–Roch, but now we cannot use that.

**Lemma 2.5.3.** *These two are equal.*

*Proof.* Let $\omega_Z$ be the dualizing sheaf of $Z$. By adjunction, $\omega_Z = (K_X \otimes \mathcal{O}_X(Z))|_Z$. Hence $\deg \omega_Z = \sum n_i \deg \omega_Z|_{C_i} = (K_X + Z) \cdot Z$. On the other hand, by Riemann–Roch on $Z$, we get $\chi(\omega_Z) = \deg \omega_Z + \chi(\mathcal{O}_Z)$. By Serre duality, $\deg \omega_Z = -\deg \mathcal{O}_Z$. Hence $\deg \omega_Z = 2 - 2\chi(\mathcal{O}_Z)$. □

**Definition 2.5.4.** Let $X$ be above, and $C = \bigcup_{i=1}^r C_i \subset X$. We say $C$ is **contractible** if there exists a normal analytic space $\bar{X}$ and a point $x \in \bar{X}$ and a holomorphic map $\pi \colon X \to \bar{X}$ such that $\pi|_{X-C} \to \bar{X} - x$ is an isomorphism and $\pi(C) = x$.

*Remark.* If $\pi \colon X \to \bar{X}$ is a birational and proper morphism, then for all $x \in \bar{X}$, the inverse image $\pi^{-1}(x)$ is connected. (This is the analytic version of Zariski's main theorem.) If $C$ is contractible, then $\bar{X}$ is unique.

**Theorem 2.5.5** (Mumford)**.** *Assume $X, C$ as above and $C$ connected. Then the intersection matrix $(C_i \cdot C_j)$ is negative definite.*

*Proof.* After shrinking $\bar{X}$, we can assume there exists a holomorphic function $f \in \Gamma(\mathcal{O}_{\bar{X}})$ such that $f \neq 0$ but $f(x) = 0$. Consider $(f) = H$ a hypersurface on $\bar{X}$. Then $\pi^*(f) = (f \circ \pi) = \pi^* H = H' + \sum_i s_i C_i$ where $H'$ is a component not equal to the $C_i$. We know all $s_i > 0$, and $H'$ is non-empty. Since $\pi(H') = H$, we have $H' \cap C \neq \emptyset$. By assumption, $H' \cdot C_i \geq 0$ and $H' \cdot C_j > 0$ for some $j$.

Claim: for all $i \neq j$, $C_i \cdot C_j \geq 0$ and for all $j$, there exists an $i$ such that $C_i \cdot C_j > 0$. In fact, we cannot write $\{1, \dots, r\} = A \sqcup B$ disjoint and non-empty but mutually orthogonal, i.e. if $i \in A$ and $j \in B$ then $C_i \cdot C_j = 0$. Secondly, $C_i^2 < 0$ for all $i$. Thirdly, assume for all $j$, the sum $\sum_i s_i(C_i \cdot C_j) \leq 0$, and there exists a $j$ such that $\sum_i s_i(C_i \cdot C_j) < 0$.

The first part of the claim follows by connectedness. If there is one $C_i$, the statement is vacuous. Otherwise it is the statement that $\bigcup_i C_i$ is a connected curve. For the second part, $\mathcal{O}_X(\pi^* H) = \pi^* \mathcal{O}_{\bar{X}}(H) = \pi^* \mathcal{O}_{\bar{X}} = \mathcal{O}_X$. Hence $\pi^* H \cdot C_j = 0$ for all $j$. Then $C_j^2 < 0$ by plugging in the expression of $H$. in terms of $H'$. Now write $H' \cdot C_j + \sum_i s_i C_i \cdot C_j = 0$. For some $j$, we know $H' \cdot C_j > 0$, and therefore $\sum_i s_i C_i \cdot C_j < 0$.

The rest of the proof is a formal argument. For simplicity, replace $s_i C_i$ by some vector $v_i$. Say we have a $\mathbb{R}$-vector space $V$ with elements $v_i \in V$ for $i = 1, \dots, r$, and a bilinear form on $V$. Suppose $v_i \cdot v_j \geq 0$ for $i \neq j$ and we cannot divide $\{1, \dots, r\} = A \sqcup B$ where $A, B$ are non-empty and $v_i \cdot v_j = 0$ for $i \in A$ and $j \in B$. Secondly, assume $v_j^2 \leq 0$ for all $j$. Thirdly, assume for all $j$, the sum $\sum_i s_i(v_i \cdot v_j) \leq 0$, and there exists a $j$ such that $\sum_i s_i(v_i \cdot v_j) < 0$. The conclusion is that the intersection matrix $(x, y)$ is negative definite. Equivalently, for all $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, we have $(\sum_i \lambda_i v_i)^2 \leq 0$ with equality iff $\lambda_1, \dots, \lambda_r = 0$.

First suppose that $\lambda_1, \dots, \lambda_r \geq 0$. First we will show the intersection matrix is negative semidefinite. Compute

$$(\sum_i \lambda_i v_i)^2 = \sum_j \lambda_j \sum_i \lambda_i v_i \cdot v_j = \sum_j \lambda_j \bigg( \sum_i \lambda_i v_i \cdot v_j + \sum_{i \neq j} (\lambda_i - \lambda_j) v_i \cdot v_j \bigg).$$

By the third assumption, $\sum_i \lambda_i v_i \cdot v_j \leq 0$. By rearranging, we get the above expression is at most $-(\lambda_i - \lambda_j)^2 v_i \cdot v_j \leq 0$. Now if $\lambda_i \geq 0$ for $i \in A$ and $\lambda_j < 0$ for $j \in B$, then we write $w_1 = \sum_{i \in A} \lambda_i v_i$ and

$w_2 = \sum_{j \in B} \lambda_j v_j$. Then $(w_1 + w_2)^2 = w_1^2 + 2w_1 \cdot w_2 + w_2^2$. The terms $w_1^2, w_2^2$ are $\leq 0$, and $w_1 \cdot w_2 \leq 0$ as well because $A \cap B = \emptyset$.

To show the intersection matrix is actually negative definite, it suffices to show that for any non-empty subset $A \subset \{1, \dots, r\}$ and $\lambda_i > 0$, we have $\sum_{i \in A} \lambda_i v_i^2 < 0$. By looking at the terms in the inequality above, it is enough to show that there is some $j \in A$ such that $(\sum_{i \in A} v_i \cdot v_j < 0$. But in general, if $A = \{1, \dots, r\}$, then this follows from the third assumption. Otherwise $A, B \neq \emptyset$, so there exists a $j \in A$ and $k \in B$ such that $v_j \cdot v_k \neq 0$. (This is the connectedness result.) Then $v_j \cdot \sum_i v_i \leq 0$, but we can write it as $v_j \cdot \sum_{i \in A} v_i + v_j \cdot \sum_{\ell \in B} v_\ell \leq 0$. By assumption, $v_j \cdot \sum_{\ell \in B} v_\ell > 0$. Hence $v_j \cdot \sum_{i \in A} v_i < 0$. □

**Theorem 2.5.6** (Grauert's contraction criterion)**.** *Assume $X, C$ as above, and assume $(C_i, C_j)$ is negative definite. Then there exists a normal analytic surface $\bar{X}$ such that there is a holomorphic $\pi\colon X \to \bar{X}$ of degree 1. More precisely, $\pi\colon X - C \to \bar{X} - \{x\}$ is an isomorphism for some $x \in \bar{X}$.*

*Remark.* The universal property remains the same: for all analytic spaces $Z$, any holomorphic morphism $f\colon X \to Z$ such that $f(C) = \{\text{pt}\}$ factors via a unique morphism $\bar{X} \to C$ through $\pi\colon X \to \bar{X}$.

*Remark.* If $X$ is actually projective, when is $\bar{X}$ projective? There may be no non-trivial line bundles on $\bar{X}$. If $\bar{L}$ is a line bundle on $\bar{X}$, then we can pull it back to get $L \coloneqq \pi^*\bar{L}$ a line bundle on $X$, and $\pi_* L = \pi_*\pi^*\bar{L} = \bar{L} \otimes \pi_*\mathcal{O}_X = \bar{L}$. Also, since $\bar{L}$ is trivial in some analytic neighborhood $\bar{U}$ of $x$, that implies $L$ is trivial in $U = \pi^*\bar{U}$, which is some neighborhood of $C$. Conversely, if $L$ is a line bundle on $X$ and there exists an analytic neighborhood $U$ of $C$ such that $L|_U$ is trivial, then $\pi_* L|_U = \pi_*\mathcal{O}_U = \mathcal{O}_{\bar{U}}$. Hence $\pi_* L$ is a line bundle on $\bar{X}$. Facts: we can choose $\bar{U}$ a contractible Stein neighborhood of $x \in \bar{X}$. (Stein means that no coherent sheaves have any higher cohomology on $\bar{U}$.) Let $U = \pi^*\bar{U}$. Then $U$ retracts onto $C$. In particular, $H^i(U, \mathbb{Z}) = H^i(C, \mathbb{Z})$, and if $\mathcal{F}$ is a coherent analytic sheaf, $H^i(U, \mathcal{F})$ is computed by the Leray spectral sequence $E_2^{p,q} = H^p(\bar{U}, R^q\pi_*\mathcal{F})$. (Because $\bar{U}$ is Stein, this SS degenerates.) The exponential sheaf sequence therefore gives

$$\begin{array}{ccccccc}
H^1(U, \mathcal{O}_U) & \longrightarrow & H^1(U, \mathcal{O}_U^*) & \longrightarrow & H^2(U, \mathbb{Z}) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
H^0(\bar{U}, R^1\pi_*\mathcal{O}_U) & & \mathrm{Pic}(U) & & H^2(C, \mathbb{Z}) = \bigoplus_{i=1}^r \mathbb{Z}(C_i). & &
\end{array}$$

If $R^1\pi_*\mathcal{O}_U = 0$, then $\mathrm{Pic}(U) \cong \mathbb{Z}^r$, given by the map $L \mapsto (\deg L|_{C_i})_i$. In this case, $L$ is analytically trivial in $U$ iff $\deg L|_{C_i} = 0$ for all $i$.

**Definition 2.5.7.** In dimension 2, we say $(X, C)$ is **minimal** if there are no exceptional curves in $C$. We can define **strongly minimal** as in the global case. Every minimal resolution is strongly minimal, so we speak of "the" minimal resolution.

## 2.6 Rational singularities

**Definition 2.6.1.** The point $x \in \bar{X}$ is a **rational singularity** if $R^1\pi_*\mathcal{O}_X = 0$. This definition is independent of which $\bar{X}$ we pick by the Leray spectral sequence $R^p f_*(R^q g_*) \Rightarrow R^{p+q}(f \circ g)_*$. In particular, if $f$ is a composition of blow-downs on smooth surfaces, then $R^1 f_*\mathcal{O} = 0$ for $i = 1$, so $R^1\pi_*\mathcal{O}_X$ is independent of the resolution.

**Example 2.6.2.** If $(\bar{X}, x)$ is smooth, then it is rational.

**Example 2.6.3.** If $C = C_1$, $C_1 = \mathbb{P}^1$, and $C_1^2 \leq -2$, then $(\bar{X}, x)$ is rational. We know $R^1\pi_*\mathcal{O}_X = \varprojlim H^1(\mathcal{O}_{nC}) = 0$ by the short exact sequence $0 \to \mathcal{O}(nC) \to \mathcal{O}_{(n+1)C} \to \mathcal{O}_{nC} \to 0$ and by induction $H^1(\mathcal{O}_{nC}) = 0$ for all $n$.

**Lemma 2.6.4.** *A singularity $(\bar{X}, x)$ is rational iff for all $Z = \sum n_i C_i \subset X$, we have $H^1(\mathcal{O}_Z) = 0$.*

*Proof.* By definition, $R^1\pi_*\mathcal{O}_X = \varinjlim_Z H^1(\mathcal{O}_Z) = 0$. Conversely, if $Z' \geq Z$, then there is a natural surjection $\mathcal{O}_{Z'} \to \mathcal{O}_Z$ which induces a surjection $H^1(\mathcal{O}_{Z'}) \to H^1(\mathcal{O}_Z)$. This is by the long exact sequence of $0 \to \mathcal{O}_{Z''}(-Z) \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$, where $Z'' := Z' \cdot Z$ forms the appropriate kernel. Hence $\varprojlim_Z H^1(\mathcal{O}_Z) \twoheadrightarrow H^1(\mathcal{O}_Z)$ surjects for any given $Z$. But $R^1\pi_*\mathcal{O}_X = 0$ by hypothesis, so $H^1(\mathcal{O}_Z) = 0$ for any given $Z$ as well. ☐

**Corollary 2.6.5.** *If $(\bar{X}, x)$ is rational, then:*

1. *every $C_i \cong \mathbb{P}^1$;*
2. *$C_i \cdot C_j$ is either 0 or 1;*
3. *the dual graph $\Gamma$ of $C$ is contractible, i.e. is a tree. (Here the vertices of $\Gamma$ correspond to $C_i$, and $C_i$ and $C_j$ are connected by an edge iff $C_i \cdot C_j \neq 0$.)*

*Remark.* These conditions are not sufficient for the singularity to be rational.

**Theorem 2.6.6** (M. Artin)**.** *$(\bar{X}, x)$ is rational iff for all $Z$ supported on $C$, the arithmetic genus $p_a(Z) \leq 0$.*

*Proof.* Note that $p_a(Z) = 1 - \chi(\mathcal{O}_Z) = 1 - h^0(\mathcal{O}_Z)$ since in the rational case, $H^1(\mathcal{O}_Z) = 0$. Since $h^0(\mathcal{O}_Z) > 0$, we get $p_a(Z) \leq 0$. Conversely, if $\operatorname{supp} Z \subset C$, then $h^1(\mathcal{O}_Z) = 0$. In the special case $Z = C_i$, we have $p_a(C_i) \geq 0$, and therefore $p_a(C_i) = 0$ by the hypothesis $p_a(Z) \leq 0$. Hence $C_i = \mathbb{P}^1$, and $h^1(\mathcal{O}_{\mathbb{P}^1}) = 0$. In general, $Z = \sum n_i C_i$, and we induct on $\sum_i n_i$. Claim: for some $i$, we have $-C_i^2 + Z \cdot C_i \leq 1$. Otherwise $-C_i^2 + Z \cdot C_i \geq 2$ for all $i$, i.e.

$$K_X \cdot C_i + Z \cdot C_i = (-2 - C_i^2) + Z \cdot C_i \geq 0,$$

so that $(K_X + Z) \cdot Z = 2p_a(Z) - 2 \geq 0$, a contradiction. Since $-C_i^2 > 1$, we must have $Z \cdot C_i < 0$, so $C_i \subset \operatorname{supp} Z$. So look at $Z' = Z - C_i \geq 0$. Then $Z' \neq 0$, and by induction $h^1(\mathcal{O}_{Z'}) = 0$. There is an exact sequence

$$0 \to \mathcal{O}_{C_i}(-Z') \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0$$

where $\mathcal{O}_{C_i}(-Z')$ is a line bundle of degree $\geq -1$. Hence $H^1(\mathcal{O}_{C_i}(-Z') = 0$, and therefore $H^1(\mathcal{O}_Z) = 0$. ☐

**Corollary 2.6.7.** *Suppose $C_i^2 = -2$ for all $i$ and $C_i \cong \mathbb{P}^1$. Then $(\bar{X}, x)$ is rational.*

*Proof.* By adjunction, $K_X \cdot C_i = 0$ for all $i$ (since $K_X \cdot C_i + C_i^2 = -2$). Hence $K_X \cdot Z = 0$ for all $Z = \sum n_i C_i$. Then $K_X \cdot Z + Z^2 < 0$ by negative definiteness. So $p_a(Z) \leq 0$, and we are done by Artin's theorem. ☐

**Definition 2.6.8.** $(\bar{X}, x)$ is a **rational double point (RDP)** if for the minimal resolution, $C_i^2 = -2$ for all $C_i \cong \mathbb{P}^1$. An RDP is therefore a rational singularity.

*Remark.* Fact: the dual graph $\Gamma$ must be one of three types: $A_n$, $D_n$, or $E_n$ (for $n = 6, 7, 8$). In fact, the lattice spanned by the $C_i$ is a root lattice (in the sense of root systems) of the type $A_n$, $D_n$, or $E_n$, with the convention that the intersection form is negative definite.

*Remark.* There are many characterizations of RDPs:

1. rational singularities which are hypersurfaces in $(\mathbb{C}^3, 0)$ (embedding dimension = 3);
2. rational singularities of multiplicity 2;
3. rational + Gorenstein ($\omega_{\bar{X}} = \mathcal{O}_{\bar{X}}$) surface singularities;
4. satisfy $K_X = \pi^*\omega_{\bar{X}}$ (du Val's characterization: they "don't affect the conditions of adjunction");
5. (Klein) singularities $(\mathbb{C}^2, 0)/G$ where $G \subset \mathrm{SL}(2, \mathbb{C})$.

**Example 2.6.9.** Let $C = C_1$, $C_1^2 = -2$, $C_1 \cong \mathbb{P}^1$, analytically defined by $x^2 + y^2 + z^2 = 0$ in $(\mathbb{C}^3, 0)$. More generally, $A_n$ is given by $x^2 + y^2 + z^{n+1} = 0$.

*Remark.* RDPs arise in the global theory of projective surfaces. Suppose $X$ is a smooth projective surface, and either $K_X$ or $-K_X$ is nef and big (but not necessarily ample). Consider $C = \bigcup C_i$ where $K_X \cdot C_i = 0$. But then the big-ness implies via Hodge index theorem that there are only finitely many such $C_i$, and $(C_i \cdot C_j)$ is negative definite. If we consider the connected components, these give dual graphs of type $A_n, D_n, E_n$. Given such a curve, $C_i^2 < 0$ but $2p_a(C_i) - 2 = K_X \cdot C_i + C_i^2 < 0$, so the only possibility is $p_a(C_i) = 0$ and $C_i = \mathbb{P}^1$ and $C_i^2 = -2$. By negative definiteness, for $i \neq j$ we have $C_i \cdot C_j \leq 1$ (otherwise $(C_i + C_j)^2 \geq 0$). We will see that either $K_X$ or $K_{X^{-1}}$ induces an ample divisor on $\bar{X}$, which is the contraction of $X$.

Likewise, suppose $K_X \equiv 0$ is numerically trivial. Let $H$ be nef and big on $X$ but not ample, and consider $C_i$ such that $H \cdot C_i = 0$. Then the same argument shows all these $C_i$ are smooth of self-intersection $-2$, with dual graphs of type $A_n, D_n, E_n$.

**Theorem 2.6.10.** *Let $X$ be smooth and projective and $C = \bigcup_{i=1}^r C_i$ with $C_i$ irreducible and $(C_i \cdot C_j)$ negative definite. For simplicity, assume $C$ is connected. Suppose if $\bar{X}$ is an analytic contraction, then all singularities in $\bar{X}$ are rational, i.e. if $\pi\colon \bar{X} \to X$ is a contraction, then $R^1\pi_*\mathcal{O}_X = 0$. Then $\bar{X}$ is a normal projective variety, i.e. this contraction in the analytic category can be done in the algebraic category, and thus $\pi$ is a morphism of algebraic varieties.*

*Proof.* Start with $H$ very ample on $X$, and assume $H^1(\mathcal{O}_X(H)) = 0$. Consider the lattice $\Lambda = \mathbb{Z}^r = \bigoplus_{i=1}^r \mathbb{Z}[C_i] \subset \operatorname{Num} X$. We get a functional $\Lambda \to \mathbb{Z}$ defined on basis vectors by $C_i \mapsto H \cdot C_i$. On the other hand, the intersection form defines a homomorphism $\Lambda \to \Lambda^\vee$, which is injective and hence the image has finite index. After replacing $H$ by $NH$, we can therefore assume $H \cdot C_i = -(\sum n_j C_j) \cdot C_i$ for all $i$, i.e. $(H + \sum n_j C_j) \cdot C_i = 0$ for all $i$. Write $Z \coloneqq \sum n_j C_j$. A previous lemma shows $n_j \geq 0$. In fact, since $H \cdot C_j \geq 0$, we have $n_j > 0$. So $H + Z$ is nef and $(H+Z)^2 = H \cdot (H+Z) + Z \cdot (H+Z) > 0$ so $H + Z$ is also big.

Rationality implies there exists $U$ an analytic neighborhood of $\bigcup C_i$ such that $\mathcal{O}_X(H+Z)|_U \cong \mathcal{O}_U$. In particular, $\mathcal{O}_X(H+Z)|_Z = \mathcal{O}_Z$. Therefore the exact sequence $0 \to \mathcal{O}_X(H) \to \mathcal{O}_X(H+Z) \to (\mathcal{O}_Z(H+Z) = \mathcal{O}_Z) \to 0$ gives

$$\cdots \to H^0(\mathcal{O}_X(H+Z)) \to H^0(\mathcal{O}_Z) \to H^1(\mathcal{O}_X(H)) = 0,$$

i.e. the section $1 \in H^0(\mathcal{O}_Z)$ lifts to some in $H^0(\mathcal{O}_X(H+Z))$. Hence the linear system $|H+Z|$ has no base points on $C$. Now it suffices to mimic the proof of Castelonuovo's criterion: we get $\varphi\colon X \to \mathbb{P}^N$ which separates points and tangent directions in $X - C$ and $\varphi(C) = \{\text{pt}\}$ separated from $\varphi(y)$ for $y \notin C$, so in the Stein factorization $X \xrightarrow{\pi} \bar{X} \to \varphi(X)$, we know $\bar{X}$ is projective and $\pi$ exactly contracts the curve $C$. ☐

*Remark.* In fact, this argument essentially shows $\pi_*\mathcal{O}_X(H+Z)$ is ample on $\bar{X}$: it is a line bundle which is big and meets all curves positively, so Nakai–Moishezon applies.

*Remark.* In the case $K_X$ or $-K_X$ is nef and big, we construct $\bar{X}$, and then by Nakai–Moishezon $\omega_{\bar{X}} = \pi_* K_X$ so either $\omega_{\bar{X}}$ or $\omega_{\bar{X}}^{-1}$ is ample.

## 2.7 Fundamental cycles

Let $\pi\colon (X, C) \to (\tilde{X}, x)$ be a resolution of a normal surface singularity. Write $C = \bigcup_{i=1}^r C_i$. Assume $C$ is connected and $(C_i \cdot C_j)$ is negative definite.

**Proposition 2.7.1.** *There is a unique non-zero effective cycle $Z_0 = \sum n_i C_i$ such that:*

1. *$Z_0 \cdot C_i \leq 0$ for all $i$ and $Z_0 \cdot C_i < 0$ for some $i$;*
2. *$Z_0$ is minimal with respect to all such effective non-zero cycles $Z$, i.e. given another such cycle $Z$, we have $Z_0 \leq Z$.*

**Definition 2.7.2.** Such a $Z_0$ as in the proposition is called the **fundamental cycle** of the resolution.

*Remark.* If $Z \geq 0$ is non-zero and $Z \cdot C_i \leq 0$ for all $i$, then in fact there has to be some $i$ such that $Z \cdot C_i < 0$. Otherwise $Z \cdot C_i = 0$ for all $i$, i.e. $Z^2 = 0$, contradicting negative definiteness. Also, clearly if $Z$ is as above, then $n_i > 0$ for all $i$. Otherwise if $n_i = 0$, then $Z \cdot C_j > 0$ for some $j$. So given $Z \neq 0$, all coefficients must be positive.

*Remark.* For RDPs, $\operatorname{span}\{[C_i]\}$ is (the negative of) a root lattice. The classes of the $C_i$ are the simple roots, and positive roots are effective divisors of square $-2$, and the fundamental cycle corresponds to the highest root.

*Proof.* Suppose $Z_1, Z_2$ have property (1), i.e. $Z_i \cdot C_j \leq 0$ for all $j$ and $Z_i \neq 0$ is effective (which implies $Z_i \cdot C_j < 0$ for some $j$ by a previous remark). Say $Z_1 = \sum n_i C_i$ and $Z_2 = \sum m_i C_i$. Let $\min(Z_1, Z_2) \coloneqq \sum \min\{m_i, n_i\} C_i$. We have seen that $n_i, m_i > 0$, so that $\min\{m_i, n_i\} > 0$. In particular, $\min(Z_1, Z_2)$ is still effective and non-zero. Since $\min(Z_1, Z_2) \cdot C_j \leq \max\{Z_1 \cdot C_j, Z_2 \cdot C_j\} \leq 0$, the minimal cycle $\min(Z_1, Z_2)$ also has the desired property (1). So consider the set

$$\{Z = \sum n_i C_i : Z \geq 0,\ Z \neq 0,\ Z \cdot C_i \leq 0 \, \forall i\}.$$

In the proof of Mumford's theorem, we produced cycles of this type, i.e. this set is non-empty. In fact, given $f \in \mathcal{O}_{\bar{X},x}$ in $\mathfrak{m}_x$, we showed the divisor $(\pi^* f) = H' + \sum s_i C_i$ where $\sum s_i C_i$ satisfies property (1). Now take any minimal element in the set with respect to $\leq$. Given any two such $Z', Z''$, we can take $\min(Z', Z'') \leq Z', Z''$. By the minimality assumption, $\min(Z', Z'') = Z' = Z''$. So there is a unique minimal element. □

*Remark.* If $f \in \mathfrak{m}_x$, then $f$ defines a $Z$. By construction, the fundamental cycle $Z_0 \leq Z$. We can look at all such $Z$ arising from these $f$, and take their minimal cycle. This minimal cycle is sometimes $Z_0$ but not always.

*Remark.* If $\rho\colon \tilde{X} \to X$ is a blowup at $y \in C$, then $\rho^* Z_0 = \tilde{Z}_0$, the fundamental cycle for $\pi \circ \rho\colon \tilde{X} \to \bar{X}$. In particular, this implies $(Z_0)^2$ is independent of the choice of resolution.

**Definition 2.7.3** (Algorithm for finding $Z_0$)**.** Start with any $C_i$ and call it $Z_1$. If $C = C_i$, i.e. $i = r = 1$, then stop. Otherwise there is some $j$ such that $C_i \cdot C_j > 0$. Set $Z_2 = C_i + C_j$. Inductively, suppose we found $Z_1, \ldots, Z_k$. If $Z_k \cdot C_i \leq 0$ for all $i$, stop. Otherwise there exists $\ell$ such that $Z_k \cdot C_\ell > 0$. Set $Z_{k+1} \coloneqq Z_k + C_\ell$.

Note that $Z_k$ is connected. Less obviously (see lemma below), $Z_0 - Z_k \geq 0$, i.e. $Z_k \leq Z_0$. The construction terminates at some point $Z_n$ when we have $0 < Z_n \leq Z_0$ and $Z_n \cdot C_\ell \leq 0$ for all $\ell$. Then $Z_n \leq Z_0$ is satisfies property (1), so by the minimality of $Z_0$ we have $Z_n = Z_0$.

**Lemma 2.7.4.** $Z_k \leq Z_0$.

*Proof.* For $k = 1$, this is obvious. Induct on $k$. We know $Z_{k+1} = Z_k + C_\ell$ where $Z_k \cdot C_\ell > 0$. Then $(Z_0 - Z_k) \cdot C_\ell = Z_0 \cdot C_\ell - Z_k \cdot C_\ell < 0$, i.e. $C_\ell$ is in the support of the effective (by induction) divisor $Z_0 - Z_k$. Hence $Z_0 - Z_{k+1} = Z_0 - Z_k - C_\ell \geq 0$. □

**Definition 2.7.5.** Let $X$ be a complex surface (not necessarily compact) and suppose $Z = \sum n_i C_i \in \operatorname{Div}^c X$ (with $n_i > 0$). Assume $Z$ connected. Then a **computation sequence** for $Z$ is a sequence $Z_1, \ldots, Z_n$ with

1. $Z_1 = C_i$ for some $i$ and $Z_n = Z$, and
2. $Z_{k+1} = Z_k + C_\ell$ where $Z_k \cdot C_\ell > 0$.

**Example 2.7.6.** Not all $Z$ have a computation sequence. For example, let $Z = nC_1$ where $C_1$ irreducible, $C_1^2 \leq 0$, and $n > 1$. But we showed $Z_0$ the fundamental cycle of a singularity has a computational sequence.

*Remark.* In the definition, suppose $C_i^2 < 0$ for all $i$. Start with $Z_1 = C_i$ for any $i$, and keep defining $Z_{k+1}$ as indicated. Then either this terminates and the intersection matrix $(C_i \cdot C_j)$ is negative definite, or the intersection matrix is not negative definite and this procedure never terminates.

**Definition 2.7.7.** Let $Z = \sum n_i C_i > 0$. Then $Z$ is **numerically connected** if whenever $Z = A + B$ where $A, B \geq 0$, we have $A \cdot B \geq 0$ with equality iff $A = 0$ or $B = 0$.

*Remark.* Exercise via the Hodge index theorem: if $Z$ is a nef and big divisor on $X$ projective, then $Z$ is numerically connected.

**Lemma 2.7.8.** *If $Z$ is numerically connected, then a computation sequence for $Z$ exists.*

*Proof.* Again start with $Z_1 = C_i$ for some $i$. If $Z = C_i$, stop. Otherwise let $A = C_i$ and $B = Z - C_i$. Then $A \cdot B \geq 0$, and in fact since $A, B > 0$ we have $A \cdot B > 0$. So there must exist an $\ell$ such that $A \cdot C_\ell > 0$. So set $Z_2 = C_i + C_\ell$. Repeat. $\square$

**Lemma 2.7.9** (Ramanujam's lemma)**.** *Suppose $Z$ connected, and a computation sequence exists. Let $L$ be a line bundle on $Z$ and suppose that $\deg(L|_{C_i}) \leq 0$ for all $i$. Then $H^0(Z, L)$ has dimension 0 or 1. It has dimension 1 iff $L = \mathcal{O}_Z$.*

*Proof.* Choose a computation sequence $Z_1 = C_i, Z_2, \ldots, Z_n = Z$, with $Z_{k+1} = Z_k + C_\ell$ where $Z_k \cdot C_\ell > 0$. We will show inductively that $h^0(Z_k, L|_{Z_k}) \leq 1$ with equality iff $L|_{Z_k} = \mathcal{O}_{Z_k}$.

1. ($k = 1$) $Z_1 = C_i$ is reduced irreducible, and $L|_{C_i}$ is some line bundle of degree $\leq 0$. If there is a section $s \in H^0(C_i, L|_{C_i})$, then we get
$$0 \to \mathcal{O}_{C_i} \xrightarrow{s} L|_{C_i} \to Q \to 0$$
where $Q$ is a skyscraper sheaf. By Riemann–Roch, $\deg L|_{C_i} = \ell(Q)$, the length of $Q$. Hence $Q = 0$ and $L|_{C_i} = \mathcal{O}_{C_i}$.

2. (inductive step) We have the usual exact sequence
$$0 \to \mathcal{O}_{C_i}(-Z_k) \to \mathcal{O}_{Z_{k+1}} \to \mathcal{O}_{Z_k} \to 0.$$
We know $\mathcal{O}_{C_i}(-Z_k)$ has negative degree on $C_\ell$. So tensoring with $L$ and taking $H^0$, we get
$$0 \to H^0(C_\ell(-Z_k) \otimes L) \to H^0(L|_{Z_{k+1}}) \to H^0(L|_{Z_k}).$$
But $H^0(C_\ell(-Z_k) \otimes L)$ has degree $< 0$, and is therefore 0. Hence $H^0(L|_{Z_{k+1}}) \subset H^0(L|_{Z_k})$. By the induction hypothesis, $h^0(L|_{Z_k}) \leq 1$, and therefore the same holds for $h^0(L|_{Z_{k+1}})$. If $h^0(L|_{Z_{k+1}}) = 1$, the inclusions must all be isomorphisms, i.e.
$$H^0(L|_{Z_{k+1}}) = H^0(L|_{Z_k}) = \cdots = H^0(L|_{C_i}),$$
and hence $L|_{C_i} = \mathcal{O}_{C_i}$ for all $i$. If $s \in H^0(L|_{Z_{k+1}})$ is a non-zero section, then $s|_{C_i}$ is everywhere non-zero. Look at $\mathcal{O}_{Z_{k+1}} \xrightarrow{s} L|_{Z_{k+1}}$. Because it is surjective on every $C_i$, it is surjective by Nakayama. Locally, $L|_{Z_{k+1}} \cong \mathcal{O}_{Z_{k+1}}$. It is a general fact that if $R$ is a Noetherian ring, $M$ is a finite $R$-module, and $\varphi \colon M \to M$ is surjective, then $\varphi$ is actually injective and hence an isomorphism. It follows that $L|_{Z_{k+1}} \cong \mathcal{O}_{Z_{k+1}}$ globally. $\square$

**Corollary 2.7.10.** *If $Z_0$ is the fundamental cycle of $\pi \colon (X, C) \to (\bar{X}, x)$, then $q\ h^0(\mathcal{O}_{Z_0}) = 1$ and $p_a(Z_0) = h^1(\mathcal{O}_{Z_0}) \geq 0$. More generally, if $Z_1, \ldots, Z_n = Z_0$ is a computation sequence for $Z_0$, the same is true for all the $Z_i$.*

**Theorem 2.7.11** (M. Artin)**.** *$(\overline{X}, x)$ is a rational singularity iff $p_a(Z_0) = 0$ (iff $h^1(\mathcal{O}_{Z_0}) = 0$ iff $(K_X + Z_0) \cdot Z_0 = -2$).*

**Example 2.7.12.** Take $C_1, \ldots, C_n$ disjoint curves each intersecting $D$ once transversely. Let $C_i^2 = -d_i$ and $D^2 = -e$. Assume $C_i \cong D \cong \mathbb{P}^1$ for all $i$. Then the dual graph is a tree. The singularity is rational iff $e \geq 3$. (It is non-rational iff $e = 2$.)

*Proof.* Rational implies $p_a(Z_0) \leq 0$. But $Z_0$ is the fundamental cycle implies $h^0(\mathcal{O}_Z) = 1$, so $p_a(Z_0) = h^1(\mathcal{O}_Z) \leq 0$. The hard direction is the converse, that $p_a(Z_0) = 0$ implies rationality.

By Ramanujam, if $p_a(Z_0) = 0$, then $h^0(\mathcal{O}_{Z_0}) = 1$ and $h^1(\mathcal{O}_{Z_0}) = 0$. Fix a computation sequence $Z_{k+1} = Z_k + C_\ell$. Then $p_a(Z_k) = 0$. Recall that if $Z' \geq Z$, then $H^1(\mathcal{O}_{Z'}) \twoheadrightarrow H^1(\mathcal{O}_Z)$ is a surjection. So it is enough to show for all $N > 0$, we have $H^1(\mathcal{O}_{NZ_0}) = 0$. Let $L$ be a line bundle on $NZ_0$. Claim: if $\deg L|_{C_i} \geq 0$, then $H^1(NZ_0, L) = 0$. We are done by proving this claim.

Consider the short exact sequence $0 \to \mathcal{O}_{C_\ell}(-MZ_0 - Z_k) \to \mathcal{O}_{MZ_0+Z_{k+1}} \to \mathcal{O}_{MZ_0+Z_k} \to 0$, for $1 \leq M \leq N$. We do a double induction on $k, M$. The base case $M = 0$ and $k = 1$ is obvious since $\deg(L|_{C_i}) \geq 0$. For all $1 \leq k \leq n-1$, we have $Z_k \cdot C_\ell = 1$ by the assumption $p_a(Z_k) = 0$ (and therefore for all rational singularities). This comes from applying $p_a(Z_k) = p_a(Z_{k+1}) = 0$ to get

$$-2 = 2p_a(Z_k) - 2 = 2p_a(Z_{k+1}) - 2 = K_X(Z_k + C_\ell) + (Z_k + C_\ell)^2.$$

Since $C_\ell \cong \mathbb{P}^1$, expanding gives $Z_k \cdot C_\ell = 1$. Then

$$H^0(\mathcal{O}_{MZ_0+Z_{k+1}} \otimes L) \to H^0(\mathcal{O}_{MZ_0+Z_k} \otimes L) \to H^1(\mathcal{O}_{C_\ell}(-MZ_0 - Z_k) \otimes L).$$

By the definition of the fundamental cycle, $Z_0 \cdot C_\ell \leq 0$. So $\mathcal{O}_{C_\ell}(-MZ_0 - Z_k) \otimes L$ has degree at least $-1$. Hence it is $\mathcal{O}_{\mathbb{P}^1}(a)$ for $a \geq -1$, and $H^1 = 0$. This completes the induction step. ☐

## 2.8 Surface singularities

**Definition 2.8.1.** If $H \coloneqq V(f) \subset (\mathbb{C}^n, 0)$ is a hypersurface, $\operatorname{mult}_0 H$ is just $\operatorname{mult}_0 f$. But in general, we can have $Z \subset (\mathbb{C}^n, 0)$ of higher codimension. Then $\operatorname{mult}_0 Z$ is the degree of the projective tangent cone, which is the degree of the subvariety of $\mathbb{P}^{n-1}$ generated by the initial terms of all $f \in I(Z)$. In general, for $x \in Z$, we can form the graded algebra $\bigoplus_{n\geq 0} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ where $\mathfrak{m}_x \subset \mathcal{O}_{Z,x}$ is the maximal ideal. There are two important invariants.

1. The **multiplicity** of $Z$ at $x$ is defined as follows. Fact: $\operatorname{length}(\mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}) = \dim_{\mathbb{C}} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ is a numerical polynomial in $n$, i.e. for all $n \gg 0$, it is a polynomial $mn^{r-1}/(r-1)! +$ lower order. Then $m$ is precisely $\operatorname{mult}_x Z$.

2. The **embedding dimension** of $Z$ at $x$ is $\dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2$, the dimension of the Zariski tangent space.

**Theorem 2.8.2** (M. Artin)**.** *Suppose $(\overline{X}, x)$ has rational surface singularities with $Z_0$ the fundamental cycle. Then $\operatorname{mult}_x \bar{X} = -(Z_0)^2$ and the embedding dimension is $-(Z_0)^2 + 1$.*

*Remark.* We have $Z_0^2 = -1$ iff $\operatorname{mult}_x \bar{X} = 1$ iff the embedding dimension is 2 iff $(\overline{X}, x)$ is smooth. This is the generalization of Castelnuovo's criterion to more than one component.

*Remark.* We have $Z_0^2 = -2$ iff the embedding dimension is 3 iff $\operatorname{mult}_x \bar{X} = 2$. In other words, $(\overline{X}, x)$ is a hypersurface singularity. This implies that it is Gorenstein, i.e. the dualizing sheaf $\omega_{X,x} \cong \mathcal{O}_{X,x}$ is locally free. (Being Gorenstein is a characterization of $(\overline{X}, x)$ being a RDP.)

**Theorem 2.8.3** (More precise version of Artin's theorem)**.** *Let $(\overline{X}, x)$ be a rational singularity with $Z_0$ the fundamental cycle. Then:*

1. *$H^1(\mathcal{O}_{Z_0}(-NZ_0)) = 0$ and $R^1\pi_*\mathcal{O}_X(-NZ_0) = 0$ for all $N \geq 0$;*

2. *$h^0(\mathcal{O}_{Z_0}(-NZ_0)) = -N(Z_0)^2 + 1$;*

3. *$\mathfrak{m}_x\mathcal{O}_X = \mathcal{O}_X(-Z_0)$;*

4. *for all $n \geq 0$, $\mathfrak{m}_x^n/\mathfrak{m}_x^{n+1} \cong H^0(\mathcal{O}_X(-nZ_0)/\mathcal{O}_X(-(n+1)Z_0)) = H^0(\mathcal{O}_{Z_0}(-nZ_0))$ has dimension equal to $-n(Z_0)^2 + 1$ (which implies Artin's theorem).*

*Proof.* We showed $\mathcal{O}_X(-NZ_0)|_{C_0}$ has degree $\geq 0$ for all $i$. By Ramanujam's lemma, $H^1(\mathcal{O}_{Z_0}(-NZ_0)) = 0$, the first part of (1). We showed earlier this implies $H^1(\mathcal{O}_{MZ_0}(-NZ_0)) = 0$ for any $M$. Taking a limit over $M$, we get $R^1\pi_*\mathcal{O}_X(-NZ_0) = 0$, the second part of (1).

(2) is Riemann–Roch on $Z_0$, since $\chi(\mathcal{O}_{Z_0}) = 1$. But $\chi(\mathcal{O}_{Z_0}(-NZ_0)) = h^0(\mathcal{O}_{Z_0}(-NZ_0))$ by (1). This is equal to the degree $-N(Z_0)^2 + 1$.

Claim: if $L$ is a line bundle on $Z_0$ with $\deg(L|_{C_i}) \geq 0$, then $L$ is bpf, i.e. for every $z \in Z_0$, there exists $s \in H^0(L)$ such that $s_z \in \mathcal{O}_{Z_0,z}$ is non-zero. To see this, pick a computation sequence starting with $C_i$, with $Z_{k+1} = Z_k + C_\ell$. As in the rational singularity case, $Z_k \cdot C_\ell = 1$. Consider the usual exact sequence $0 \to \mathcal{O}_{C_\ell}(-Z_k) \to \mathcal{O}_{Z_{k+1}} \to \mathcal{O}_{Z_k} \to 0$ tensored with $L$. The point is that $\mathcal{O}_{C_\ell}(-Z_k) \otimes L$ has degree at least $-1$ by the same argument as before, which implies there is a surjection $H^0(L|_{Z_{k+1}}) \twoheadrightarrow H^0(L|_{Z_k})$ for all $k$, and by induction, $H^0(L) \twoheadrightarrow H^0(L|_{C_i})$. But $L|_{C_i}$ is $\mathcal{O}_{\mathbb{P}^1}(a)$ where $a \geq 0$ by hypothesis. Since $\mathcal{O}_{\mathbb{P}^1}(a)$ is bpf, by lifting sections we are done.

We can look at $H^0(L) \otimes \mathcal{O}_{Z_0} \to L$, which is surjective by Nakayama. Corollary of the claim: there exist sections $t_0, t_1 \in H^0(L)$ such that the induced map $\mathcal{O}_{Z_0} \oplus \mathcal{O}_{Z_0} = \mathcal{O}_{Z_0} \otimes \operatorname{span}_{\mathbb{C}}\{t_0, t_1\} \subset \mathcal{O}_{Z_0} \otimes H^0(L) \to L$is surjective. The argument is as follows. First show there exists a $t_0$ which is not identically zero on any component. If we do this on every component and then take a general linear combination, we get a $t_0$ which does not vanish on any component. Then find $t_1$ such that $t_1$ does not vanish on $(t_0)$ by the same method.

Now we show (3). We always have $\mathfrak{m}_x\mathcal{O}_X \subset \mathcal{O}_X(-Z_0)$. Consider the exact sequence $0 \to \mathcal{O}_X(-2Z_0) \to \mathcal{O}_X(-Z_0) \to \mathcal{O}_{Z_0}(-Z_0) \to 0$. Since $R^1\pi_*\mathcal{O}_X(-2Z_0) = 0$, we have $R^0\pi_*\mathcal{O}_X(-Z_0) \twoheadrightarrow H^0(\mathcal{O}_{Z_0}(-Z_0))$. Note that we always have $R^0\pi_*\mathcal{O}_X(-Z_0) \subset \mathfrak{m}_x$. Since $\mathfrak{m}_x\mathcal{O}_X \subset \mathcal{O}_X(-Z_0)$ implies $\mathfrak{m}_x \subset R^0\pi_*\mathcal{O}_X(-Z_0)$, we get $\mathfrak{m}_x = R^0\pi_*\mathcal{O}_X(-Z_0)$. By restriction, we get maps $\mathfrak{m}_x \to H^0(\mathcal{O}_X(-Z_0)) \to H^0(\mathcal{O}_{Z_0}(-Z_0))$. Since $\mathfrak{m}_x = R^0\pi_*\mathcal{O}_X(-Z_0) \twoheadrightarrow H^0(\mathcal{O}_{Z_0}(-Z_0))$ factors through $H^0(\mathcal{O}_X(-Z_0))$ via this composition, it follows that $\mathfrak{m}_x \twoheadrightarrow H^0(\mathcal{O}_{Z_0}(-Z_0))$ is also surjective. The map $\mathfrak{m}_x\mathcal{O}_X \to \mathcal{O}_X(-Z_0)$ factors through $H^0(\mathcal{O}_{Z_0}(-Z_0)) \otimes \mathcal{O}_X$. Since $\mathcal{O}_{Z_0}(-Z_0)|_{C_i}$ has degree $\geq 0$, we know $H^0(\mathcal{O}_{Z_0}(-Z_0)) \otimes \mathcal{O}_X \to \mathcal{O}_X(-Z_0)$ is also surjective. Conclusion: the natural map $\mathfrak{m}_x\mathcal{O}_X \to \mathcal{O}_X(-Z_0)$ is surjective. Hence $\mathfrak{m}_x\mathcal{O}_X = \mathcal{O}_X(-Z_0)$.

Since $\mathfrak{m}_x\mathcal{O}_X = \mathcal{O}_X(-Z_0)$, we get $\mathfrak{m}_x^n\mathcal{O}_X = \mathcal{O}_X(-nZ_0)$. Hence $\mathfrak{m}_x^n \subset R^0\pi_*\mathcal{O}_X(-nZ_0)$. Claim: $\mathfrak{m}_x^n = R^0\pi_*\mathcal{O}_X(-nZ_0)$. Note that we've checked this for $n = 1$ already. If we can show that the map $\psi$ is surjective in the diagram

$$\begin{array}{ccc} (R^0\pi_*(-Z_0))^{\otimes n} & \xrightarrow{\psi} & R^0\pi_*\mathcal{O}_X(-nZ_0) \\ \| & & \uparrow \\ \mathfrak{m}_x^{\otimes n} & \longrightarrow & \mathfrak{m}_x^n, \end{array}$$

where the maps are the obvious ones, then we are done. Pick $t_0, t_1 \in H^0(\mathcal{O}_{Z_0}(-Z_0))$ with the property that they generate $\mathcal{O}_{Z_0}(-Z_0)$ at every point. After passing to a suitable neighborhood $\bar{U}$ of $x$, with $U = \pi^{-1}(U)$, we can assume $t_1, t_2$ lift to sections of $H^0(\mathcal{O}_X(-Z_0))$. (This is just the statement that $R^0\pi_*\mathcal{O}_X(-Z_0) \twoheadrightarrow H^0(\mathcal{O}_{Z_0}(-Z_0))$.) We can also assume that the lifted sections $\tilde{t}_1, \tilde{t}_2$ generate $\mathcal{O}_X(-Z_0)$, by Nakayama and shrinking $\bar{U}$ appropriately. Now look at the exact sequence

$$0 \to \mathcal{O}_X(Z_0) \to \mathcal{O}_X^2 \xrightarrow{(\tilde{t}_1, \tilde{t}_2)} \mathcal{O}_X(-Z_0) \to 0$$

where we determined the kernel by comparing determinants. Tensoring with $\mathcal{O}_X(-nZ_0)$ gives

$$0 \to \mathcal{O}_X(-(n-1)Z_0) \to \mathcal{O}_X(-nZ_0)^2 \to \mathcal{O}_X(-(n+1)Z_0) \to 0.$$

For $n \geq 1$, we get $(R^0\pi_*\mathcal{O}_X(-nZ_0))^2 \twoheadrightarrow R^0\pi_*\mathcal{O}_X(-(n+1)Z_0) \to 0$. By induction on $n$, we get

$$((R^0\pi_*\mathcal{O}_X(-Z_0))^{\otimes n})^2 \twoheadrightarrow (R^0\pi_*\mathcal{O}_X(-nZ_0))^2 \twoheadrightarrow R^0\pi_*\mathcal{O}_X(-(n+1)Z_0)$$

is still surjective., and the image contains the image of $(R^0\pi_*\mathcal{O}_X(-Z_0))^{\otimes(n+1)}$. This finishes the proof of the claim.

Now apply $R^0\pi_*$ to the exact sequence $0 \to \mathcal{O}_X(-(n+1)Z_0) \to \mathcal{O}_X(-nZ_0) \to \mathcal{O}_{Z_0}(-nZ_0) \to 0$ to get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^0\pi_*\mathcal{O}_X(-(n+1)Z_0) & \longrightarrow & R^0\pi_*\mathcal{O}_X(-nZ_0) & \longrightarrow & H^0(\mathcal{O}_{Z_0}(-nZ_0)) & \longrightarrow & R^1\pi_* = 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{m}_x^{n+1} & \longrightarrow & \mathfrak{m}_x^n & \longrightarrow & \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1} & \longrightarrow & 0. \end{array}$$

Hence the remaining vertical arrow is also an isomorphism. □

## 2.9 Gorenstein condition for normal surface singularities

**Definition 2.9.1.** Let $R$ be a local ring of a scheme of finite type over a field, or a local ring of an analytic space. $R$ is **Cohen–Macaulay (CM)** if $\operatorname{depth} R = \dim R$. (The **depth** of $R$ is the maximal length of a sequence of elements $x_1, \dots, x_d \in \mathfrak{m}$ such that $x_{i+1}$ is not a zero-divisor in $R/(x_1, \dots, x_i)$.)

**Theorem 2.9.2** (Serre)**.** *If $R$ is normal of dimension $\geq 2$, then* $\operatorname{depth} R \geq 2$.

**Corollary 2.9.3.** *If $R$ is normal and* $\dim R = 2$, *then $R$ is CM.*

**Lemma 2.9.4.** *Let $R$ be a local ring of a scheme of finite type over a field, or a local ring of an analytic space. If $R$ is CM, then there exists a dualizing module $\omega$.*

*Remark.* In general, there exists a dualizing complex, but then it becomes harder to state local duality.

**Definition 2.9.5.** A local ring $R$ is **Gorenstein** iff $\omega = R$. Globally, if $Z$ is a CM scheme or analytic space, we get a dualizing sheaf $\omega_Z$, and the Gorenstein condition is equivalent to $\omega_Z$ is locally free (of rank 1).

**Theorem 2.9.6.** *Suppose $Z$ is normal of dimension 2, or $Z$ is normal and CM of dimension $\geq 2$. Let $U = Z_{reg} := Z - Z_{sing}$ (normal means* $\operatorname{codim} Z_{sing} \geq 2$*), and let $i\colon U \hookrightarrow Z$ be the inclusion. Then $\omega_Z$ is reflexive, i.e. $\omega_Z^{\vee\vee} = \omega_Z$, and $\omega_Z = i_*(\omega_Z|_U) = i_*K_U$.*

**Lemma 2.9.7.** *Let $R$ be a local ring of a finite type $k$-algebra or $R = \mathcal{O}_{Z,x}$ be a local ring of an analytic space. Let $d = \dim R$. Suppose $R$ is normal and CM and there exists a dualizing module $\omega$. Then:*

1. *$\omega$ is torsion-free, i.e. there does not exist a sub-module $M \subset \omega$ with $M \neq 0$ and* $\dim \operatorname{supp} M < d$*;*
2. *$\omega$ is reflexive. More precisely, suppose $\omega \subset N$ where $N$ is torsion-free, and* $\dim \operatorname{supp} N/\omega \leq d-2$*. Then $\omega = N$.*

*Proof.* Basic fact (Bourbaki, Alg. comm. chpt.10 p.137): with $R$ and $\omega$ as above and $M$ a finitely generated $R$-module, $\operatorname{Ext}^i_R(M, \omega) = 0$ if $i < \dim R - \dim \operatorname{supp} M$.

By assumption, $\dim \operatorname{supp} M \leq d-1$, so $\operatorname{Hom}_R(M, \omega) = 0$. So $M = 0$, otherwise the inclusion $M \hookrightarrow \omega$ gives a non-trivial element.

Look at the exact sequence $0 \to \omega \to N \to N/\omega \to 0$. Applying $\operatorname{Hom}_R(-, \omega)$, we get

$$\cdots \to \operatorname{Hom}_R(N, \omega) \to \operatorname{Hom}_R(\omega, \omega) \to \operatorname{Ext}^1_R(N/\omega, \omega) = 0$$

using the fact. So the identity $\mathrm{id} \in \operatorname{Hom}_R(\omega, \omega)$ lifts to a homomorphism $r\colon N \to \omega$. Hence $N \cong \omega \oplus N/\omega$. But we assumed $N$ is torsion-free, so $N/\omega = 0$. □

**Lemma 2.9.8.** *Let $R$ be a normal ring, and $M$ a reflexive $R$-module. Let $Y$ be a subscheme of* $\operatorname{Spec} R$ *of codimension $\geq 2$, and $U := \operatorname{Spec} R - Y$ with inclusion $i\colon U \hookrightarrow \operatorname{Spec} R$. Then $\tilde{M} = i_*i^*\tilde{M}$.*

*Remark.* In the language of schemes, if $Z$ is normal and $\mathcal{F}$ on $Z$ is reflexive, then $\mathcal{F} = i_*i^*\mathcal{F}$. More generally, if $M$ is torsion-free (i.e. $M \mapsto M^{\vee\vee}$ is injective), then $\widetilde{M^{\vee\vee}} = i_*i^*\tilde{M}$ (assuming $\tilde{M}$ is reflexive on $U$).

*Proof of theorem.* Since $\omega$ is reflexive on $U = (\operatorname{Spec} R)_{\text{reg}}$, the natural map $\omega \mapsto \omega^{\vee\vee}$ is equal on codimension $\geq 2$ and is an inclusion because $\omega$ is torsion-free, so by the lemma, $\omega = \omega^{\vee\vee} = i_* i^* \omega = i_* K_U$. □

**Theorem 2.9.9.** *Let $(\overline{X}, x)$ be a normal surface singularity with resolution $\pi\colon (X, C) \to (\overline{X}, x)$. Then the following are equivalent:*

1. *there exists a small neighborhood $\overline{U}$ of $x$, with $U := \pi^{-1}(\overline{U})$, such that $(K_X)|_{U-C} = \mathcal{O}_{U-C}$;*
2. *with $\overline{U}$, $U$ as above and $i\colon \overline{U} - \{x\} = U - C \to \overline{U}$ the inclusion, $i_*(K_X|_{U-C}) = \mathcal{O}_{\overline{U}}$;*
3. *$\omega_{\overline{X}}$ is locally free at $x$, i.e. $(\overline{X}, x)$ is Gorenstein.*

*In addition, suppose $(X, C) \xrightarrow{\pi} (\overline{X}, x)$ is minimal. Then (1), (2), (3) are equivalent to:*

4. *$(K_X)|_U = \mathcal{O}_X(-\sum n_i C_i)|_U$ with $n_i \geq 0$.*

*Proof.* The equivalence of (1), (2), (3) are essentially trivial based on what we have already seen. The difficulty is showing (4) is equivalent to any of them. Clearly (4) implies (2), because $(\mathcal{O}_X(-\sum n_i C_i)|_U)|_{U-C} = \mathcal{O}_{U-C}$. Now the converse. Claim: $K_X|_U = \mathcal{O}_X(-\sum n_i C_i)$ for some $n_i \in \mathbb{Z}$, not necessarily positive. (This works without the minimality assumption.) We have

$$\pi_*(K_X|_U) \subset i_* \mathcal{O}_{\overline{U} - \{x\}} = \mathcal{O}_{\overline{U}}.$$

Because $K_X|_U$ is torsion-free and is equal to $\mathcal{O}_{\overline{U}}$ on $U$, that implies $\pi_*(K_X|_U) = I \cdot \mathcal{O}_{\overline{U}}$ for some ideal sheaf $I$ supported on $\{x\}$. Pulling back, we get a morphism $\pi^* \pi_* K_X/\text{tors} \hookrightarrow K_X$ (where we can mod out by torsion since it is torsion-free). But

$$\pi^* \pi_* K_X = \pi^*(I \cdot \mathcal{O}_{\overline{U}}) = (I \cdot \mathcal{O}_X)|_U \supset \mathcal{O}_X(-NC)$$

for some $N \in \mathbb{Z}_{>0}$ (on $U$). So there is an inclusion $\mathcal{O}_X(-NC)|_U \to K_X|_U$. Hence $K_X|_U \cong \mathcal{O}_X(-NC + Z)|_U$ where $Z$ is an effective divisor supported on $C$. This implies $(K_X + \sum n_I C_i) \cdot C_j = 0$ for all $j$.

Now we use minimality to show $n_i \geq 0$. Minimality implies $K_X \cdot C_j \geq 0$ for all $j$, because $C_j^2 < 0$, so if $K_X \cdot C_j < 0$ then $C_j$ would be exceptional. So $(\sum n_i C_i) \cdot C_j = -K_X \cdot C_j \leq 0$. Let $A := \{j : n_j \geq 0\}$ and $B := \{j : n_j < 0\}$. Write $\sum n_i C_i = Z_1 - Z_2$ where $Z_1 := \sum_{j \in A} n_j C_j$ and $Z_2 := \sum_{j \in B} (-n_j) C_j$. Both $Z_1$, $Z_2$ are effective. Suppose $Z_2 \neq 0$. We saw that $Z_1 \cdot C_j \leq Z_2 \cdot C_j$. If there exists $j \in B$, then $0 \leq Z_1 \cdot C_j \leq Z_2 \cdot C_j$. This implies $(Z_2)^2 \geq 0$ by the usual argument. But we assumed $Z_2 \neq 0$, a contradiction. □

*Remark.* From now on, we assume $(X, C)$ is minimal.

**Corollary 2.9.10.** *$(\overline{X}, x)$ is a rational and Gorenstein singularity iff it is an RDP.*

*Proof.* If $x$ is an RDP, then $C_i \cong \mathbb{P}^1$ and $C_i^2 = -2$ for all $i$. We know it is rational. By adjunction, $K_X \cdot C_i = 0$. This implies $K_X$ is numerically trivial on $U$, a small neighborhood of $C$. Remember that the rational condition implies $\operatorname{Pic} U \cong \mathbb{Z}^r$, given by $L \mapsto (L \cdot C_i)_i$. So the condition that $K_X$ is numerically trivial means $K_X|_U = \mathcal{O}_U$. Hence condition (4) of the theorem holds with $n_i = 0$ for all $i$, and we get the Gorenstein condition.

Assume that $(\overline{X}, x)$ is rational and Gorenstein. Then there exists $n_i \geq 0$ such that $(K_X + \sum n_i C_i) \cdot C_j = 0$. We want to show $n_i = 0$ for all $i$, so that $K_X \cdot C_i = 0$ and $C_i^2 < 0$ give $C_i^2 = -2$ and $C_i = \mathbb{P}^1$. If not, we have $Z = \sum n_i C_i \geq 0$ and $Z \neq 0$. Then $p_a(Z) = 1$, contradicting rationality. □

*Remark.* In the Gorenstein case, we can find $Z := \sum n_i C_i$ such that $(K_X + Z) \cdot C_i = 0$ for all $i$. Minimality says $Z$ is effective. Note that $Z = 0$ is equivalent to the RDP condition. This is equivalent to $K_X = \pi^* \omega_{\overline{X}}$. This is the statement "RDPs don't affect the conditions of adjunction." In all other cases, $Z \geq 0$ and $Z \neq 0$, and we saw $Z \cdot C_i \leq 0$ for all $i$. So $Z$ dominates the fundamental cycle, i.e. $Z_0 \leq Z$. Also, $p_a(Z) = 1$.

**Definition 2.9.11.** Look at $\dim_{\mathbb{C}} R^1\pi_*\mathcal{O}_X = 1$, the next case after rationality. In this case, we say $(\overline{X}, x)$ is **elliptic**. Rational singularities are a tractable class, but this generalization is already too complicated. So instead we look at **elliptic Gorenstein singularities**.

**Theorem 2.9.12.** *Let $(\overline{X}, x)$ be any normal surface singularity with minimal resolution $\pi\colon (X, C) \to (\overline{X}, x)$ and fundamental cycle $Z_0$. The following are equivalent:*

1. *$(\overline{X}, x)$ is an elliptic Gorenstein singularity;*

2. *$p_a(Z_0) = 1$ and for all effective connected $Z \subset Z_0$, we have $p_a(Z) = 0$.*

*Remark.* If $C = C_1$ is irreducible, then $C^2 < 0$ and $Z_0 = C$, so that $p_a(C) = 1$ and the condition on subcycles $Z \subset Z_0$ becomes vacuous. So $C$ is either smooth elliptic (**simple elliptic singularities**) or an irreducible nodal or cuspidal rational curve. If $Z_0$ is reduced, then either:

1. (**cusp singularities**) $C$ is a cycle of smooth rational curves with intersection numbers $\geq 2$, with at least one $\geq 3$;

2. (**triangle singularities**) $C$ is either two components meeting along a tacnode (locally $x^2 = y^4$), or three smooth rational curves meeting at a point with the singularity type of three intersecting lines. The case of an irreducible cuspidal curve is included here.

# Chapter 3

# Examples of surfaces

## 3.1 Rational ruled surfaces

**Definition 3.1.1.** Let $H$ be the divisor class of a line in $\mathbb{P}^2$. Then $|H|$ defines the identity $\mathrm{id}\colon \mathbb{P}^2 \to \mathbb{P}^2$, and $|H-p|$ defines a **pencil of lines** through $p$. The blow-up $\mathbb{F}_1 := \mathrm{Bl}_p \mathbb{P}^2$ is $|\pi^*H - E|$, and is bpf. We get a morphism $\mathbb{F}_1 \to \mathbb{P}^1$ with fibers elements of $\pi^*H - E$.

**Definition 3.1.2.** The linear system $|2H|$ corresponds to the Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$, with image the Veronese surface of degree 4. We can show $2H - p$ is very ample, and $|2H-p|$ corresponds to $|2\pi^*H - E|$ in $\mathbb{F}_1$. Compute $(2\pi^*H - E)^2 = 3$, so $\varphi\colon \mathbb{F}_1 \to \mathbb{P}^4$ has image a degree 3 surface in $\mathbb{P}^4$. Since $H \cdot (2\pi^*H - E) = 2$, we see $\varphi(H)$ is a conic. Since $E \cdot (2\pi^*H - E) = 1$, we see $\varphi(E)$ is a line. We call $\varphi(\mathbb{F}_1)$ a **cubic scroll**.

*Remark.* More generally, we can look at $|2H - p_1 - \cdots - p_r|$. We want to assume no three of the $p_i$ are collinear (else there is a fixed component), and also $r \le 4$ because there is a unique conic through five general points, and again there will be a fixed component. Then $\dim |2H - p_1 - \cdots - p_r| = 5 - r$. These correspond to linear systems on $\mathrm{Bl}_{p_1,\ldots,p_r} \mathbb{P}^2$.

**Definition 3.1.3.** Define $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$. Then there is a map $\mathrm{Bl}_p \mathbb{F}_1 \to \mathbb{F}_0$ by contracting a line

**Definition 3.1.4** (All rational ruled surfaces)**.** Suppose $\mathbb{F}_n$ has been constructed inductively, with a morphism $\mathbb{F}_n \xrightarrow{\rho} \mathbb{P}^1$ with all fibers isomorphic to $\mathbb{P}^1$, and there exists a section $\sigma$, i.e. a smooth curve $\sigma$ such that $\sigma \cdot f = 1$, such that $\sigma^2 = -n$. Pick a point $q \in \sigma$, and consider $\mathrm{Bl}_q \mathbb{F}_n$. Contract the proper transform of the fiber $f$ to get $\mathbb{F}_{n+1}$, along with a birational map

$$\mathbb{F}_n \leftarrow \mathrm{Bl}_q \mathbb{F}_n \to \mathbb{F}_{n+1}.$$

This is a potentially non-unique construction of $\mathbb{F}_n$ for every $n \ge 0$.

*Remark.* Up to isomorphism, $\mathbb{F}_n$ is unique, i.e. $\mathbb{F}_n \xrightarrow{\rho} \mathbb{P}^1$ with all fibers are isomorphic to $\mathbb{P}^1$ with a section $\sigma$ where $\sigma^2 = -n$. Also, $\mathbb{F}_n$ are exactly the minimal ruled surfaces over $\mathbb{P}^1$. Also, $\mathbb{F}_n$ with $n \ne 1$ and $\mathbb{P}^2$ are the minimal models of $\mathbb{P}^2$. Hence given a surface $X$ birational to $\mathbb{P}^2$, there is a blow-down $X \to \mathbb{F}_n$ for some $n \ne 1$, or to $\mathbb{P}^2$. (How many different ways does $X$ blow down to $\mathbb{P}^2$ or $\mathbb{F}_n$? This is incredibly complicated.) Finally, the $\mathbb{F}_n$ describe all non-degenerate surfaces in $\mathbb{P}^N$ of minimal degree.

**Proposition 3.1.5.** *Numerical invariants of* $\mathbb{F}_n$*:*

1. $\mathrm{Pic}\,\mathbb{F}_n = \mathrm{Num}\,\mathbb{F}_n = \mathbb{Z}\sigma \oplus \mathbb{Z}f$*;*
2. $q(\mathbb{F}_n) = p_g(\mathbb{F}_n) = 0$*;*
3. $c_1^2 = 8$ *and* $c_2 = 4$*;*

4. $K_{\mathbb{F}_n} = -2\sigma - (n+2)f$.

*Proof.* For Pic and Num, use induction. Clearly it holds for $\mathbb{F}_0$. Remember the relation between $\mathbb{F}_n$ and $\mathbb{F}_{n+1}$ is $\mathbb{F}_n \xleftarrow{\pi} \operatorname{Bl}_q \mathbb{F}_n \to \mathbb{F}_{n+1}$. So a $\mathbb{Z}$-basis for either $\operatorname{Pic}\operatorname{Bl}_q \mathbb{F}_n$ or $\operatorname{Num}\operatorname{Bl}_q \mathbb{F}_n$ is the same: $\{\pi^*\sigma, \pi^* f, E\}$. So we want to look at $(f-E)^\perp$ to get to $\mathbb{F}_{n+1}$. Note that $f \in (f-E)^\perp$ and $\sigma - E \in (f-E)^\perp$. It is easy to check $f$ and $\sigma - E$ are a basis for $\operatorname{Pic}\mathbb{F}_n$ and $\operatorname{Num}\mathbb{F}_n$ by direct computation, or by noting that the intersection form is already unimodular.

We showed that $q$ and $p_g$ are invariant under blow-ups, so $q(\mathbb{F}_n) = q(\operatorname{Bl}_q \mathbb{F}_n) = q(\mathbb{F}_{n+1})$ and likewise for $p_g$.

As an element of Pic, write $K_{\mathbb{F}_n} = a\sigma + bf$. By adjunction, $K_{\mathbb{F}_n} \cdot f + f^2 = -2$. But $f^2 = 0$, so $a = -2$. We also know $K_{\mathbb{F}_n} \cdot \sigma + \sigma^2 = -2$, and $\sigma^2 = -n$, so solving gives $b = -n-2$.

Finally, compute $c_1^2 = (K_{\mathbb{F}_n})^2 = 4(-n) + 4(n+2) = 8$. For $c_2$, note that $c_1^2 + c_2 = 12\chi(\mathcal{O}_{\mathbb{F}_n}) = 12$. Alternatively, keeping track of the Betti numbers, $b_1(\mathbb{F}_n) = b_3(\mathbb{F}_n) = 0$, and $b_2(\mathbb{F}_n) = 2$. □

*Remark.* If $n \equiv 0 \bmod 2$, then the intersection pairing is even, i.e. $\alpha^2 \equiv 0 \bmod 2$ for all $\alpha \in H^2$. (A more complicated way to see this is via the Wu formula.) However if $n \equiv 1 \bmod 2$, then $\sigma^2 \equiv 1 \bmod 2$. So $\mathbb{F}_n$ and $\mathbb{F}_{n+1}$ are never of the same homotopy type. But every $\mathbb{F}_n$ and $\mathbb{F}_{n+2}$ are diffeomorphic, because topologically $\mathbb{F}_n$ is an oriented $S^2$-bundle over $S^2$, and there are only two: $S^2 \times S^2$ ($n$ even), and the twisted sphere bundle over $S^2$ ($n$ odd). An even better complex analytic fact is that $\mathbb{F}_{n+2}$ is deformation-equivalent to $\mathbb{F}_n$, i.e. there exists a complex manifold $X$ of dimension 3 and a proper smooth holomorphic map $\pi\colon X \to \Delta$ ($\Delta$ is the unit disk in $\mathbb{C}$) such that $\pi^{-1}(t) \cong \mathbb{F}_n$ for $t \neq 0$ and $\pi^{-1}(0) = \mathbb{F}_{n+2}$.

**Proposition 3.1.6.** *The following are equivalent:*

1. *the linear system $|a\sigma + bf|$ is bpf;*
2. *the linear system $|a\sigma + bf|$ has no fixed curve (and is non-empty);*
3. *$a\sigma + bf$ is nef;*
4. *$a \geq 0$ and $b \geq na$.*

*Proof.* Clearly (1) implies (2) implies (3). If $a\sigma + bf$ is nef, then $(a\sigma + bf)\cdot f = a \geq 0$ and $(a\sigma + bf)\cdot\sigma = -na + b \geq 0$. Now assume $a \geq 0$ and $b \geq na$. Clearly $|bf| \subset |a\sigma + bf|$, and $|bf|$ is bpf because $|f|$ corresponds to the morphism to $\mathbb{P}^1$. So the only possible base points are on $\sigma$. Let's first consider $a = 1$ and $b \geq n$. Look at the exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_n}(bf) \to \mathcal{O}_{\mathbb{F}_n}(\sigma + bf) \to \mathcal{O}_\sigma(\sigma + bf) = \mathcal{O}_{\mathbb{P}^1}(b-n) \to 0.$$

An easy induction shows $H^1(\mathcal{O}_{\mathbb{F}_n}(bf)) = 0$. So $H^0(\mathcal{O}_{\mathbb{F}_n}(\sigma + bf)) \twoheadrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(b-n))$. But $b - n \geq 0$ so this is bpf on $\sigma$. In the general case, write $a\sigma + bf = a(\sigma + nf) + (b - na)f$. We have seen $\sigma + nf$ is bpf, and $f$ is bpf. The sum of positive multiples of bpf systems is still bpf. □

**Corollary 3.1.7.** *$a\sigma + bf$ is ample iff $a > 0$ and $b > an$, and $a\sigma + bf$ is effective iff $a, b \geq 0$.*

*Proof.* The ample cone is the interior of the nef cone. In fact, we'll show that $a\sigma + bf$ is actually very ample when $a > 0$ and $b > na$, so on $\mathbb{F}_n$, we see ample is equivalent to very ample (since ample implies $a > 0$ and $b > na$).

If $a > 0$ and $b > 0$, then $a\sigma + bf$ is effective because $\sigma$ and $f$ are effective. Conversely, if $a\sigma + bf$ is effective, $(a\sigma + bf)\cdot f = a \geq 0$ since $f$ is nef, and $(a\sigma + bf)\cdot(\sigma + nf) = b \geq 0$ since $\sigma + nf$ is nef. □

*Remark.* Suppose $C \subset \mathbb{F}_n$ irreducible with $C^2 < 0$. Then $C = \sigma$. Likewise, if $C^2 = 0$, then either $C \sim f$ or $n = 0$ and $C \sim \sigma$.

**Definition 3.1.8.** Consider the linear system $|\sigma + kf|$ where $k \geq n$ (equivalently, it is bpf), which gives a morphism $\varphi\colon \mathbb{F}_n \to \mathbb{P}^N$.

**Proposition 3.1.9.** *$N = 2k - n + 1$ and $\deg \varphi(\mathbb{F}_n) = 2k - n$.*

1. *If $k > n$, then $\varphi$ is very ample. The fiber $\varphi(f)$ is a line, and $\varphi(\sigma)$ is a rational normal curve of degree $k - n$. In general, if $\sigma_\infty$ is a smooth element of $|\sigma + nf|$ then $\varphi(\sigma_\infty)$ is a rational normal curve of degree $k$. Given $p \in \sigma$, there exists a $p' \in \sigma_\infty$ such that $p, p'$ are in the same fiber. The image $\varphi(\mathbb{F}_n)$ is the union of lines connecting the two curves $\varphi(\sigma)$ and $\varphi(\sigma_\infty)$.*

2. *If $k = n > 0$, then $\varphi(\sigma)$ is a point, and $\varphi(\sigma_\infty)$ is a rational normal curve in $\mathbb{P}^n$, and $\varphi(\mathbb{F}_n)$ is the cone over $\varphi(\sigma_\infty)$.*

*Proof.* Compute $(\sigma + kf)^2 = -n + 2k$ as stated. Consider the sequence

$$0 \to \mathcal{O}_{\mathbb{F}_n}(kf) \to \mathcal{O}_{\mathbb{F}_n}(\sigma + kf) \to \mathcal{O}_\sigma(\sigma + kf) = \mathcal{O}_{\mathbb{P}^1}(k - n) \to 0.$$

If $k \geq n$, then $H^1(\mathcal{O}_{\mathbb{P}^1}(k - n)) = 0$. Since $\mathcal{O}_{\mathbb{F}_n}(kf) = \pi^*\mathcal{O}_{\mathbb{P}^1}(k)$ and $R^1\pi_*\mathcal{O}_{\mathbb{F}_n}(kf) = 0$, we have $\dim H^0(\mathcal{O}_{\mathbb{F}_n}(\sigma + kf)) = k + 1 + k - n + 1 = 2k - n + 2$. We have already seen that the restriction $H^0(\mathcal{O}_{\mathbb{F}_n}(\sigma + kf)) \to H^0(\mathcal{O}_\sigma(\sigma + kf))$ is surjective, i.e. $\varphi(\sigma)$ is embedded as a rational normal curve. To identify $\varphi(f)$, use

$$0 \to \mathcal{O}_{\mathbb{F}_n}(\sigma + (k-1)f) \to \mathcal{O}_{\mathbb{F}_n}(\sigma + kf) \to \mathcal{O}_f(\sigma + kf) = \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

For $k \geq n$, we have $k - 1 \geq n - 1$, so by previous remarks, $H^1(\mathcal{O}_{\mathbb{F}_n}(\sigma + (k-1)f)) = 0$. Hence $\varphi$ embeds each $f$ as a line. For $\varphi(\sigma_\infty)$ we do the same thing with

$$0 \to \mathcal{O}_{\mathbb{F}_n}((k-n)f) \to \mathcal{O}_{\mathbb{F}_n}(\sigma + kf) \to \mathcal{O}_{\sigma_\infty}(\sigma + kf) = \mathcal{O}_{\mathbb{P}^1}(k) \to 0.$$

Hence $\varphi(\sigma) \subset \mathbb{P}^{k-n}$ and $\varphi(\sigma_\infty) \subset \mathbb{P}^k$. It follows that $\varphi(\mathbb{F}_n) \subset \bigcup_t \langle \varphi(t), \varphi(t') \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the line connecting the two points. This is also the linear span $\langle \varphi(\sigma), \varphi(\sigma_\infty) \rangle$ (i.e. the smallest projective space containing both), so $\mathbb{P}^{k-n}$ and $\mathbb{P}^k$ have to be complementary by dimension reasons.

In the case $k = n > 0$, the same argument shows $\varphi(\sigma)$ is a point, and $\varphi(\mathbb{F}_n)$ is the cone over a rational normal curve in $\mathbb{P}^n$. The fibers $\varphi(f)$ are still lines. □

*Remark.* Let $\bar{\mathbb{F}}_n$ be the normal surface obtained by contracting $\sigma$ (for $n > 0$). So $\varphi$ induces a bijection $\bar{\mathbb{F}}_n \to \varphi(\mathbb{F}_n) \subset \mathbb{P}^{n-1}$ which is in fact an isomorphism. This is because the affine cone over a rational normal curve is a normal variety.

**Corollary 3.1.10.** *For $a\sigma + bf \in \operatorname{Pic} \mathbb{F}_n$, the following are equivalent:*

1. *$a\sigma + bf$ is very ample;*

2. *$a\sigma + bf$ is ample;*

3. *$a > 0$ and $b > an$.*

*Proof.* We have seen that (2) is equivalent to (3), and clearly (1) implies (2). So we show (3) implies (1). Write $a\sigma + bf = (a - 1)(\sigma + nf) + (\sigma + (b - (a - 1)n)f)$. Since $b > an$, we know $b - (a - 1)n > n$. So the second term is very ample, and the first term is bpf. Since bpf + very ample is very ample, we are done. □

*Remark.* What are the lines on $\varphi(\mathbb{F}_n)$, i.e. given $C \subset \mathbb{F}_n$, when is $C$ a line? Answer: $C = f$, or $C = \sigma$ and $k = n + 1$. The latter is unique except for $n = 0$, in which case there is a whole family of $\sigma$'s.

Given a $\mathbb{P}^N$, if $\varphi(\mathbb{F}_n) \subset \mathbb{P}^N$, then $N \equiv n - 1 \bmod 2$, and also $n \leq N - 1$ with equality iff $\varphi(\mathbb{F}_n)$ is a cone. For example, in $\mathbb{P}^4$, we have $\varphi(\mathbb{F}_1)$, which is the rational cubic scroll, or $\varphi(\mathbb{F}_3)$, which is the cone over a rational normal curve.

**Proposition 3.1.11** (Degree/codimension estimate)**.** *Let $X$ be a (irreducible) non-degenerate algebraic variety in $\mathbb{P}^N$. Then $\deg X \geq \operatorname{codim}_{\mathbb{P}^N} X + 1$.*

*Remark.* For curves, $\deg X \geq N$ with equality iff $X$ is a rational normal curve. For surfaces, $\deg X \geq N-1$.

**Theorem 3.1.12.** *Let $X$ be a (irreducible) non-degenerate surface in $\mathbb{P}^N$ and $\deg X = N-1$. Then either $X = \varphi(\mathbb{F}_n)$ or $X \cong \mathbb{P}^2$ embedded by the Veronese embedding in $\mathbb{P}^5$.*

*Proof.* We induct. For $N = 2$ and $N = 3$ this is obvious. Bertini: a general hyperplane section is irreducible. A general hyperplane section is a smooth rational normal curve, so $X$ has only isolated singularities.

If $\operatorname{Sing} X \neq \emptyset$, pick $p \in \operatorname{Sing} X$ and project $\pi_p \colon X \to \mathbb{P}^{N-1}$ (birational). The image $\pi_p(X)$ can't be a surface, else $\deg \pi_p(X) \leq N-3$, violating that $\pi_p(X)$ is non-degenerate. So $\pi_p(X)$ is a curve, and $X$ is a cone over $C$. By counting degrees, $\deg C = N-1$, and $C \subset \mathbb{P}^{N-1}$ is non-degenerate. So $C$ is a rational normal curve, and therefore $X = \varphi(\mathbb{F}_n)$.

If $\operatorname{Sing} X = \emptyset$, pick $p \in X$ (explained later) and project $\pi_p \colon X \to X' \subset \mathbb{P}^{N-1}$ (birational). This induces a morphism $\operatorname{Bl}_p X \to X'$, where the exceptional divisor $E \subset \operatorname{Bl}_p X$ has image $E' \subset X'$ a line. By assumption, $X'$ is non-degenerate in $\mathbb{P}^{N-1}$, so $\deg X' \geq N-2$. But $\deg X' \leq N-2$ since we projected from a point. Hence $\deg X' = N-2$ and $\deg(\pi_p \colon \operatorname{Bl}_p X \to X') = 1$. In particular, this morphism is birational, so all fibers are points or connected, and are contained in lines. Hence all fibers are lines. The connectedness comes from induction: $X' = \varphi(\mathbb{F}_n)$ or is $\mathbb{P}^2$, so $X'$ is normal. (Note: $X'$ is not a curve, else $X$ is singular and $p$ is a vertex.)

Case 1: $X$ has only finitely many lines. Then choose $p$ such that $p$ is not on any line. Then $\operatorname{Bl}_p X \to X'$ is an embedding and $E \mapsto E'$ isomorphically. Since $E^2 = -1$, we get $(E')^2 = -1$ in $X'$. But $X' \cong \mathbb{F}_n$, so $E = \sigma$ and $n = 1$. Hence $X = \mathbb{P}^2$. By the inductive hypothesis, $X'$ is embedded by $E + kf$. $E$ is a line, so $k = 2$, since $E \cdot (E + kf) = 1$. So $X' \subset \mathbb{P}^4$, embedded by $E + kf$, and therefore $X \cong \mathbb{P}^2$ in $\mathbb{P}^5$ of degree $(2k-1) + 1 = 4$. Hence $X$ is the Veronese embedding.

Case 2: $X$ has infinitely many lines, so every point on $X$ lines on a line. (This is because there exists a dominant morphism $Y \to X$ where $Y$ is fibered in lines.) Claim: there exists a point $p \in X$ such that $p$ lies on exactly one line. Note: every point lies on at most finitely many lines. Pick $q \in X$. If $q$ lines on just one line, we are done. Otherwise if $q$ lies on $L_1$ and $L_2$, then $L_1 \cap L_2 = \{q\}$. Since there are only finitely many lines through $q$, there must exist a point $p$ such that $p$ lies on no line through $q$. Consider the projection $\pi_p \colon X \to X' = \varphi(\mathbb{F}_n)$. We know $\pi_p(L_1), \pi_p(L_2)$ are lines in $X'$, and $E'$, the image of $E$ in $\operatorname{Bl}_p X$, is also a line. The only possibilities for configurations of three lines on $\varphi(\mathbb{F}_n)$ such that two of them meet are: a cone, two fibers $f$ and a section $\sigma$, or a fiber $f$ and two sections $\sigma$ (only in he case $n = 0$). A cone is impossible because $\pi_p(q) \notin E'$. The other two are essentially the same case, and in both we have $(E')^2 = 0$. But $(E')^2$ is $-1$ plus the number of lines passing through $p$. Hence there is exactly one line passing through $p$.

TODO: finish. □

## 3.2 More general ruled surfaces

**Definition 3.2.1.** A **ruled surface** $\pi \colon X \to C$ is a surface $X$ and a morphism $\pi$ to a smooth curve $C$ such that the generic fiber is isomorphic to $\mathbb{P}^1$. It is **geometrically ruled** if all fibers are isomorphic to $\mathbb{P}^1$.

**Lemma 3.2.2.** *If $\pi \colon X \to C$ is a ruled surface, there exists a smooth blow-down $X \to \bar{X}$ such that $\bar{\pi} \colon \bar{X} \to C$ is geometrically ruled (and $\bar{\pi}$ is compatible with $\pi$).*

*Proof.* If $f$ is a generic fiber, then $f^2 = 0$ and $K_X \cdot f = -2$. Say we have a reducible fiber $\sum_{i=1}^r n_i C_i$, where $n_i > 0$ and $r > 1$. Then it suffices to find some $C_i$ such that $C_i^2 < 0$ and $K_X \cdot C_i < 0$, because then $C_i$ is exceptional and we can contract it (and then induct on $\operatorname{rank} \operatorname{Pic}$). All fibers are numerically equivalent. So $C_i \cdot f = 0$ and $C_i \cdot \sum n_i C_j = 0$. But $C_i \cdot C_j \geq 0$ for $i \neq j$ and $C_i \cdot C_j > 0$ for some $j$ (by connectedness). So $C_i^2 < 0$ for all $i$. Also, $K_X \cdot f = K_X \cdot \sum n_i C_i$, i.e. $K_X \cdot C_i < 0$ for some $i$. Suppose the fiber is $\pi^*(t)$ where $t$ is a local parameter on $C$, then a priori it is possible that $\pi^*(t) = nf' \equiv f$ where $n > 1$. But $(f')^2 = 0$ since $f^2 = 0$, and $K_X \cdot f' = -2/n$, so $2p_a(f') - 2 = -2/n \geq -2$. Hence the only possibility is $n = 1$, so there aren't actually any multiple fibers. □

*Remark.* If $\pi\colon X \to C$ is a ruled surface and $g(C) \geq 1$, then in fact $\pi$ is unique. This is because if $D \subset X$ is a rational curve, then $\pi(D) = \text{pt}$. So all rational curves are contained in fibers of $\pi$. The blow down $X \to \bar{X}$ is however not unique. Examples are elementary transformations: start with a ruled surface $X$ and a fiber $f$, blow up the fiber, and blow it down the other way. However we can show that all birational maps $X \to X'$ over $C$ are sequences are elementary transformations.

*Remark.* If $\pi\colon X \to C$ is a ruled surface, then $X$ is birational to $C \times \mathbb{P}^1$. This will follow easily from the following.

**Example 3.2.3.** Take $V$ a rank 2 vector bundle over $C$, and consider

$$\mathbb{P}(V) = (V - \{0\})/\mathbb{C}^* = \underline{\text{Proj}}\, \text{Sym}^* V^\vee.$$

where $\{0\}$ is the zero section. In particular, there exists a tautological bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ with $R^0\pi_*\mathcal{O}_{\mathbb{P}(V)}(1) = V^\vee$. In particular, $R^0\pi_*\mathcal{O}_{\mathbb{P}(V)}(d) = \text{Sym}^d V^\vee$.

**Theorem 3.2.4.** *Let $\pi\colon X \to C$ be geometrically ruled. Then:*

1. *there exists a section $\sigma$ of $X$, i.e. a section in $X$ such that $\sigma \cdot f = 1$;*
2. *$X \cong \mathbb{P}(V)$ for some rank 2 vector bundle $V$ over $C$;*
3. *$\mathbb{P}(V) \cong \mathbb{P}(V')$ (as ruled surfaces over $C$) iff $V' \cong V \otimes L$.*

*Analytic proof.* Work in the analytic category. Suppose $T$ is a smooth simply-connected complex curve (but not necessarily compact or complete), and $\pi\colon Y \to T$ is ruled, i.e. $\pi$ is proper holomorphic and all fibers are isomorphic to $\mathbb{P}^1$. Let $\sigma$ be a section. Consider $R^0\pi_*\mathcal{O}_Y(\sigma)$. This is a rank 2 vector bundle $V^\vee$, and $Y \cong \mathbb{P}(V)$ compatible with the natural morphisms to $T$.

In the global situation $\pi\colon X \to C$ where $\pi$ is smooth and proper, there exist local sections on an open cover $\{U_\alpha\}$ of $C$ (by e.g. disks). Let $\sigma_\alpha$ be a section of $\pi^{-1}(U_\alpha) \to U_\alpha$. Locally, $\pi^{-1}(U_\alpha) \to U_\alpha \cong \mathbb{P}(V_\alpha)$ where $V_\alpha$ is rank 2 holomorphic. Shrink further to assume in fact that $V_\alpha$ is trivial, so that $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{P}^1$. On $\pi^{-1}(U_\alpha \cap U_\beta)$, we have two different representations of the disks as $(U_\alpha \cap U_\beta) \times \mathbb{P}^1$. Comparing them gives components $\bar{A}_{\alpha\beta} \in \text{PGL}_2(\mathcal{O}_{U_\alpha \cap U_\beta})$ of a 1-cocycle. Hence ruled surfaces over $C$ are classified by the cohomology group $H^1(C, \text{PGL}_2(\mathcal{O}_C))$. We can lift $\bar{A}_{\alpha\beta}$ to an element $A_{\alpha\beta} \in \text{GL}_2(\mathcal{O}_{U_\alpha \cap U_\beta})$, but they no longer satisfy the cocycle condition: we only have $A_{\alpha\beta}A_{\beta\gamma}A_{\gamma\alpha} = f_{\alpha\beta\gamma}\,\text{id}$ where it is easy to check that $f_{\alpha\beta\gamma}$ are components of a 2-cocycle. More abstractly, there is an exact sequence of groups $1 \to \mathcal{O}_C^* \to \text{GL}_2(\mathcal{O}_C) \to \text{PGL}_2(\mathcal{O}_C) \to 1$ which gives

$$\cdots \to \text{Pic}(C) = H^1(C, \mathcal{O}_C^*) \to H^1(C, \text{GL}_2(\mathcal{O}_C)) \to H^1(C, \text{PGL}_2(\mathcal{O}_C)) \xrightarrow{\delta} H^2(C, \mathcal{O}_C^*) \to \cdots$$

where $\delta$ is the map we just computed explicitly. Note the second term classifies rank 2 vector bundles, and the third term classifies geometrically ruled surfaces over $C$. Using the exponential sheaf sequence, $H^2(C, \mathcal{O}_C^*) = 0$. ☐

*Remark.* We can try to mimic the above proof algebraically. In étale cohomology, we have the exact sequence

$$\cdots \to H^1(C, \mathbb{G}_m) \to H^1(C, \text{GL}_2) \to H^1(C, \text{PGL}_2) \to H^2(C, \mathbb{G}_m) \to \cdots,$$

but the proof that $H^2(C, \mathbb{G}_m) = 0$ will essentially prove the theorem anyway.

*Algebraic proof.* The key step here (which works in all characteristics) is Tsen's theorem. There is a generic point $\eta = \text{Spec}\, k(C) \in C$, and a generic fiber $X_\eta \to \eta$ which is a smooth curve. We can compute $g(X_\eta) = 0$, so the canonical bundle $K_{X_\eta/\eta}$ is degree 2, and defines an embedding $X_\eta \to \mathbb{P}^2_\eta$. Tsen's theorem says if $K = k(C)$ is a function field in one variable over an algebraically closed field and $F$ is a homogeneous form of degree $d$ in $n \geq d$ variables $x_1, \ldots, x_n$, then there exists a non-trivial zero of $F$ in $K^n$. Equivalently, $V(F) \in \mathbb{P}^{n-1}_K$ is non-empty. In our case ($d = 2$ and $n = 3$), this says $X_\eta$ has a $k(C)$-rational point, which we think of as a section of $X_\eta \to \eta$. This extends, by taking its closure, to some section $\sigma$ of $X$ which we can assume is irreducible. Hence $\sigma\dot{f} = 1$ for generic $f$, and hence for all $f$. This gives $X \cong \mathbb{P}(V)$ where $V^\vee = R^0\pi_*\mathcal{O}_X(\sigma)$. ☐

**Proposition 3.2.5.** *Let $\pi\colon X \to C$ be geometrically ruled and $\sigma$ be a section. Let $D_1$ and $D_2$ be two divisors on $X$ such that $D_1 \cdot f = D_2 \cdot f$. Then there exists a unique line bundle $\lambda \in \operatorname{Pic} C$ such that $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \otimes \pi^*\lambda$. Consequently,* $\operatorname{Pic} X \cong \operatorname{Pic} C \oplus \mathbb{Z}[\sigma]$ *given by $L \cong \pi^*\lambda \otimes \mathcal{O}_X(n\sigma)$.*

*Proof.* After replacing $D_1$ and $D_2$ with $D := D_1 - D_2$, it is enough to show $\mathcal{O}_X(D) = \pi^*\lambda$ for some $\lambda \in \operatorname{Pic} C$ if $D \cdot f = 0$. Look at $R^0\pi_*\mathcal{O}_X(D) = \lambda$. For the trivial bundle on $\mathbb{P}^1$, clearly $H^0(\mathcal{O}_{\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}$. Hence $\pi^*\pi_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ is an isomorphism. So $\mathcal{O}_X(D) = \pi^*\lambda$. Also, by the projection formula, $\lambda = \pi_*\pi^*\lambda = (\pi_*\mathcal{O}_X) \otimes \lambda$. In particular, $\lambda$ is determined by $\mathcal{O}_X(D)$. If $L \in \operatorname{Pic} X$ with $L \cdot f = n \in \mathbb{Z}$, then $L \otimes \mathcal{O}_X(-n\sigma) \cong \pi^*\lambda$, so $L = \mathcal{O}_X(n\sigma) \otimes \pi^*\lambda$. □

## 3.3 Numerical invariants

**Lemma 3.3.1.** *$q(X) = g(C)$ and $p_g(X) = 0$. If $\sigma$ is a section, $\mathcal{O}_\sigma(\sigma) \cong \mathcal{O}_C(\underline{d})$ where $\pi$ identifies $\sigma$ with $C$. Then $K_X = -2\sigma + \pi^*(K_C \otimes \mathcal{O}_C(d))$. In* $\operatorname{Num} X$, *we therefore have $[K_X] = -2\sigma + (2g - 2 + d)f$. Hence $K_X^2 = 8(1 - g)$.*

*Proof.* We already know $R^0\pi_*\mathcal{O}_X = \mathcal{O}_C$, and $R^1\pi_*\mathcal{O}_X = 0$. By the Leray spectral sequence, $H^1(\mathcal{O}_X) = H^1(\mathcal{O}_C)$. So $q(X) = g(C)$. Again by Leray, $p_g(X) = h^{0,2}(X) = \dim H^2(\mathcal{O}_X) = 0$. By adjunction, $K_X \otimes \mathcal{O}_X(f)|_f = K_f = \mathcal{O}_{\mathbb{P}^1}(-2)$. If we write $K_X = \mathcal{O}_X(n\sigma) \otimes \pi^*\lambda$, then restricting to $f$ shows $n = -2$. To compute $\lambda$, we restrict to $\sigma$, where $(K_X \otimes \mathcal{O}_X(\sigma))|_\sigma = K_\sigma$, which is identified with $K_C$ by $\pi^*$. Hence $K_X = \mathcal{O}_X(-2\sigma) \otimes \pi^*\mathcal{O}_C(K_C + \underline{d})$. Now compute

$$K_X^2 = 4\sigma^2 - 4(2g - 2 + d) = 4d + 4d + 8(1 - g) + 4d = 8(1 - g).$$

□

**Definition 3.3.2.** Let $\pi\colon X \to C$ be geometrically ruled. Given $\sigma$ a section, look at

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(\sigma) \to \mathcal{O}_X(\sigma)|_\sigma \to 0.$$

Applying $R^1\pi_*$, we get

$$0 \to \mathcal{O}_C \to V^\vee := R^1\pi_*\mathcal{O}_X(\sigma) \to \lambda \to 0$$

where $\lambda$ is a line bundle on $C$ with $\deg \lambda = \sigma^2$. Hence $\sigma^2 = \deg(\lambda) = -\deg(\det V)$.

*Remark.* If $X = \mathbb{P}(W) = \underline{\operatorname{Proj}}_C(W^\vee)$ where $W$ is a rank 2 vector bundle on $C$, then there is a $\mathcal{O}_X(1)$. General theory says there is a correspondence between sections of $X$ and surjections $W^\vee \to L^{-1,0}$ where $L$ is a line bundle on $C$. Explicitly, given a section $z$ of $X$, we have a surjection $\pi^*W^\vee \to \mathcal{O}_X(1) \to 0$. Restricting to $\tau$, we get $W^\vee = \pi^*W^\vee|_\tau \to \mathcal{O}_X(1)|_\tau = L$. Conversely, given a surjection $W^\vee \to L \to 0$, we have $L^{-1} \hookrightarrow W$ so that $\tau = \mathbb{P}(L^{-1}) \subset \mathbb{P}(W)$ is a section. In general, $\mathcal{O}_X(1)$ is *not* of the form $\mathcal{O}_X(z)$; this is true only up to tensoring with $\pi^*L$ where $L$ is a line bundle on the base.

**Lemma 3.3.3.** *$\mathcal{O}_X(1) = \mathcal{O}_X(\tau)$ iff the kernel of the surjection $W^\vee \to L \to 0$ is trivial, i.e. $\mathcal{O}_L$.*

*Proof.* In one direction, this is the usual exact sequence in the above definition. Note that if $W^\vee = R^0\pi_*\mathcal{O}_X(\tau)$, then $\pi^*\pi_*\mathcal{O}_X(\tau) = \pi^*W \to \mathcal{O}_X(\tau) \to 0$ by checking on fibers. This identifies $\mathcal{O}_X(\tau)$ with $\mathcal{O}_X(1)$. The kernel is therefore $\mathcal{O}_C$.

Conversely, if $0 \to \mathcal{O}_C \to W^\vee \to L \to 0$, then we get a non-zero section $s$ of $H^0(C, \mathcal{O}_C)$ and hence a non-zero section of $W^\vee$. Note that $H^0(C, W^\vee) = H^0(\operatorname{Sym}^1 W^\vee)$. Consider $\tau := D_+(s) \subset \mathbb{P}(W)$. The assumption that $s \neq 0$ means $D_+(s)$ contains no fibers. □

**Lemma 3.3.4.** *We saw that if there exists a section $\sigma$, then $X = \mathbb{P}(V)$. Conversely, if $X = \mathbb{P}(W)$, then there exists a section $\tau$ and $R^0\pi_*\mathcal{O}_X(\tau) = W^\vee \otimes L$ for some line bundle $L$.*

*Proof.* After twisting $W$ by some large power of an ample line bundle, assume there exists a non-zero section of $W$, i.e. a mapping $0 \to \mathcal{O}_C \to W$ (suppressing the fact that we twisted). This section may vanish on some fibers, but it factors through $\mathcal{O}_C(D)$ where $D$ is effective. Hence there is an exact sequence $0 \to \mathcal{O}_C(D) \to W \to L' \to 0$, and therefore there is a section. □

*Remark.* In particular, if $\sigma$ and $\sigma'$ are two sections, then $\mathcal{O}_X(\sigma) = \mathcal{O}_X(\sigma') \otimes \pi^* L$.

*Remark* (Alternate proof of computation of $K_X$)*.* Choose $\sigma$ and write $X = \mathbb{P}(V)$ where $V^\vee = R^0\pi_*\mathcal{O}_X(\sigma)$. Then $\mathcal{O}_X(1) = \mathcal{O}_X(\sigma)$. There is a general formula

$$K_{X/C} = \pi^*(\det V^\vee) \otimes \mathcal{O}_X(-2).$$

Recall that $\det V = \mathcal{O}_C(-\underline{d})$. Note that $\mathcal{O}_X(-2) = \mathcal{O}_X(-2\sigma)$. Putting this together gives the formula for $K_X$ we had earlier.

## 3.4 The invariant $e(V)$

**Definition 3.4.1.** Motivation: we want $e(V)$ to be the largest degree of $L$ such that $0 \to L \to V$. This is not well-defined. So instead take

$$e(V) \coloneqq \max\{2 \deg L - \deg \det V : \exists L \to V\}.$$

Note that $L \to V$ exists iff $L \otimes \lambda \to V \otimes \lambda$ exists, where $\lambda$ is a line bundle. So $e(V \otimes \lambda) = e(V)$, and so $e(V)$ is well-defined if $X = \mathbb{P}(V)$.

*Remark.* If $0 \to L \xrightarrow{s} V$, we may as well assume that $L$ is a sub-bundle, i.e. $V/L$ is a (torsion-free) line bundle as well. If $s$ vanishes at some point, then remove the vanishing by taking $0 \to L \otimes \mathcal{O}_C(D) \to V$ is a line sub-bundle. But $\deg(L \otimes \mathcal{O}_C(D)) = \deg L + \deg D$, and we are looking for the largest degree, so this assumption is valid.

*Remark.* Suppose there is an exact sequence $0 \to L_1 \to V \to L_2 \to 0$. Then $e(V) \le |\deg L_1 - \deg L_2|$. This is because $\det V = \deg L_1 + \deg L_2$. Given $0 \to L \to V$, we have $\deg L \le \max\{\deg L_1, \deg L_2\}$, because otherwise $L^{-1} \otimes L_i = \mathrm{Hom}(L, L_i)$ has degree $< 0$, and therefore has no sections. So $2 \deg L - \deg V = 2 \deg L_i - (\deg L_1 + \deg L_2)$. In particular, if $\deg L_1 \ge \deg L_2$, then $e(V) = \deg(L_1) - \deg(L_2)$. This happens iff $\deg(L_1) \ge (1/2) \deg \det V$.

*Remark.* $e(V) = e(V^\vee)$. This is because $V^\vee = V \otimes (\det V)^{-1}$; there is a non-degenerate pairing $V \otimes V \xrightarrow{\det} \det V$.

**Lemma 3.4.2.** *$e(V) < 0$ (resp. $\le$) iff for all $0 \to L \to V$ we have $\deg L \le (1/2) \deg \det V$ (resp. $\le$).*

*Proof.* Equivalently, $e(V) > 0$ iff there exists $0 \to L \to V$ such that $\deg L > (1/2) \deg \det V$. We may as well assume $0 \to L \to V \to L' \to 0$ where $L'$ is torsion-free. Then $e(V) = \deg L - \deg L' > 0$. (In this case, $L$ is actually unique.) ☐

**Proposition 3.4.3.** *For $X = \mathbb{P}(V)$, we have $e(V) = \max\{-\sigma^2 : \sigma \text{ is a section of } X\}$.*

*Proof.* Given a section $\sigma$, we get $) \to \mathcal{O}_C \to V^\vee \to \lambda \to 0$. Equivalently, $0 \to \lambda^{-1} \to V \to \mathcal{O}_C \to 0$. Since $\lambda^{-1}$ is a sub-bundle of $E$, we have $\det V = -\deg \lambda$ and by definition $e(V) = 2 \deg \lambda^{-1} + \deg \lambda = -\deg \lambda = -\sigma^2$. Conversely, if $0 \to L \to V \to L' \to 0$ (where as usual we assume $L'$ is torsion-free), then we get $0 \to L \otimes (L')^{-1} \to V \otimes L' \to \mathcal{O}_C \to 0$. Dualizing, we get $0 \to \mathcal{O}_C \to V^\vee \otimes (L')^{-1} \to L^{-1} \otimes L' \to 0$. This gives a section $\tau$ with $e(V) \le -\tau^2$. ☐

**Proposition 3.4.4.** *If $e(X) > 0$, then there exists a unique section $\sigma$ such that $\sigma^2 < 0$. In fact $\sigma^2 = -e$. So for all $C$ irreducible on $X$, if $C^2 < 0$ then $C = \sigma$, and if $C^2 = 0$ then $C = 0$ or $C = f$.*

*Remark.* These are the analogues of the corresponding statements for $\mathbb{F}_n$.

*Remark.* For all sections $\sigma$, we have $\sigma^2 \equiv e \bmod 2$. This is because there exists some section $\sigma_0$ with $\sigma_0^2 = -e$ and every section is of the form $\sigma_0 + nf$, so $(\sigma_0 + nf)^2 \equiv \sigma_0 \bmod 2$. In particular, $\mathrm{Num}\, X = \mathbb{Z}[\sigma] \oplus \mathbb{Z}[f]$ is even if $e \equiv 0 \bmod 2$ and odd if $e \equiv 1 \bmod 2$. Topologically, these are the only two types up to diffeomorphism, and in fact deformation type. So $X$ is classified in this sense by $q = g(C)$ and the type of the intersection form. (In fact $e$ is upper semi-continuous in families.)

*Remark.* If we want to classify rank 2 vector bundles $V$ on $C$ up to $V \sim V \otimes \lambda$, we can always assume $\deg \det V = 0$ (and in fact that $\det V \equiv \mathcal{O}_C$ because Pic is divisible), or $\deg \det V = \pm 1$.

**Theorem 3.4.5** (Riemann–Roch for vector bundles on $C$)**.** *$\chi(C, V) = \deg \det V + 2(1-g)$ in the rank 2 case.*

**Example 3.4.6.** Let $C = \mathbb{P}^1$. Normalize so that $\deg \det V$ is 0 or $\pm 1$. In the 0 case, by Riemann–Roch $\chi(V) = 2$. In particular, there exists a non-zero section $0 \to \mathcal{O}_{\mathbb{P}^1} \to V$. This section may vanish on some fibers, but we can twist by $n$ to fix that. Since $\deg \det V = 0$, we know $\det V = \mathcal{O}_{\mathbb{P}^1}$. So the exact sequence is $0 \to \mathcal{O}_{\mathbb{P}^1}(n) \to V \to \mathcal{O}_{\mathbb{P}^1}(-n) \to 0$. But $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-n), \mathcal{O}_{\mathbb{P}^1}(n)) = H^1(\mathcal{O}_{\mathbb{P}^1}(2n)) = 0$ if $n \geq 0$. Hence $V = \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. In particular, $V$ is unstable if $n > 0$, and semistable but not stable if $n = 0$. We can write $V \otimes \mathcal{O}_{\mathbb{P}^1}(n) = \mathcal{O}_{\mathbb{P}^1}(2n) \oplus \mathcal{O}_{\mathbb{P}^1}$. So there exists a section $\sigma$ such that $\sigma = -2n$. Hence $\mathbb{P}(V) = \mathbb{F}_{2n}$.

On the other hand, if $\deg \det V = -1$ (the $+1$ case is more or less the same argument). Then $\det V = \mathcal{O}_{\mathbb{P}^1}(-1)$. Applying Riemann–Roch, $\chi(V) = 1$, so there exists a non-zero section $0 \to \mathcal{O}_{\mathbb{P}^1} \to V$. Twisting again, we get $0 \to \mathcal{O}_{\mathbb{P}^1}(n) \to V \to \mathcal{O}_{\mathbb{P}^1}(-n-1) \to 0$, and $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-n-1), \mathcal{O}_{\mathbb{P}^1}(n)) = H^1(\mathcal{O}_{\mathbb{P}^1}(2n+1)) = 0$ if $n \geq 0$. Hence $V = \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n-1)$. In particular, $V$ is unstable for all $n \geq 0$. We can write $V \otimes \mathcal{O}_{\mathbb{P}^1}(n+1) = \mathcal{O}_{\mathbb{P}^n}(2n+1) \oplus \mathcal{O}_{\mathbb{P}^1}$. Then $\mathbb{P}(V) = \mathbb{F}_{2n+1}$.

We know $\mathbb{F}_n$ has to be $\mathbb{P}(V)$ for some $V$. This example shows $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$. As a corollary, every ruled surface over $\mathbb{P}^1$ is isomorphic to $\mathbb{F}_n$ for a unique $n \geq 0$. Also, as a corollary, every rank 2 vector bundle over $\mathbb{P}^1$ is a direct sum of line bundles, but this is just a special case of a theorem of Grothendieck which is true for any rank.

**Example 3.4.7.** Suppose $C = E$ is an elliptic curve (where $g(E) = 1$). Normalize so that when $\deg V = 0$ we have $\det V = \mathcal{O}_E$, and when $\deg V = 1$ we have $\det V = \mathcal{O}_E(p)$ for a fixed $p$. In the case $\deg V = 0$, i.e. $\det V = \mathcal{O}_E$, either

1. $V \cong L \oplus L^{-1}$ with $\deg L > 0$ (unstable),
2. $V \cong L \oplus L^{-1}$ with $\deg L = 0$, or
3. $V \cong L \otimes \mathcal{E}$ where $\mathcal{E}$ is a rank 2 vector bundle given as $0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E \to 0$ and $L^{\otimes 2} = \mathcal{O}_E$. (Compute $\mathrm{Ext}^1(\mathcal{O}_E, \mathcal{O}_E) = H^1(\mathcal{O}_E) = \mathbb{C}$; the non-split extension is what we call $\mathcal{E}$.) This case is semistable but not stable.

In the case $\deg V = 1$, i.e. $\det V = \mathcal{O}_E(p)$, either

1. $V = L \oplus L^{-1} \otimes \mathcal{O}_E(p)$ with $\deg L \geq 1$ (unstable), or
2. $V$ is a non-split extension $0 \to \mathcal{O}_E \to V \to \mathcal{O}_E(p) \to 0$. (Compute $\mathrm{Ext}^1(\mathcal{O}_E(p), \mathcal{O}_E) = H^1(\mathcal{O}_E(-p)) = H^0(\mathcal{O}_E(p)) = \mathbb{C}$ by Serre duality.) This case is stable.

**Example 3.4.8.** Suppose $g(C) \geq 2$. We want to "classify" all rank-2 vector bundles over $C$ up to $V \cong V \otimes L$.

1. The unstable bundles cannot be parametrized by a scheme of finite type, but they are elementary to describe. Namely, pick $L$ of degree $d$ and $L'$ of degree $d' < d$, and study $\mathrm{Ext}^1(L', L)$.
2. The stable bundles form a good moduli space which is compactified by adding **strictly semistable** bundles, i.e. semistable but not stable. In general, this moduli space is hard to describe.

**Theorem 3.4.9** (Segre–Nagata)**.** *For all rank 2 vector bundles $V$, we have $e(V) \geq -g$. In particular, for the stable case, $-g \leq e(V) < 0$.*

**Example 3.4.10.** If $g = 1$, a "generic" bundle of degree 0 (with $\det = 0$) is $L \oplus L^{-1}$ with $L \neq L^{-1}$. There are therefore exactly two choices for $\lambda$ in $0 \to \lambda \to V \to \lambda' \to 0$.

## 3.5 Ample and nef cones

Fix a section $\sigma$ with $\sigma^2 = -e(X)$, i.e. a section of minimal degree.

**Proposition 3.5.1.** *1. If $e \geq 0$, then $a\sigma + bf$ is ample (resp. nef) iff $a > 0$ (resp. $a \geq 0$) and $b > ae$ (resp. $b \geq ae$). If $\Sigma$ is irreducible and $\Sigma \neq \sigma, f$, then $\Sigma^2 > 0$. We have $\Sigma^2 = 0$ iff $\Sigma = f$ or $e = 0$ and $\Sigma = \sigma$.*

*2. If $e \leq 0$, then $a\sigma + bf$ is ample (resp. nef) iff $a > 0$ (resp. $a \geq 0$) and $b > ae/2$ (resp. $b \geq ae/2$), which is iff $(a\sigma + bf)^2 > 0$ (resp. $(a\sigma + bf)^2 \geq 0$) and $a > 0$ (resp. $a \geq 0$ with $a = 0$ implying $b \geq 0$). If $\Sigma = n\sigma + mf$ and $\Sigma \neq \sigma, f$, then $\Sigma$ is the class of an effective curve iff $n > 0$ and either $n = 1$ and $m \geq 0$, or $n \geq 2$ and $m \geq ne/2$.*

*Proof.* For $e \geq 0$, if $a\sigma + bf$ is ample, then $(a\sigma + bf)f = a > 0$, and $(a\sigma + bf)\sigma = -ae + b > 0$. Conversely, if $a > 0$ and $b > ae$, then these intersections are always positive. If we write $\Sigma = n\sigma + mf$, we know $n \geq 0$ because $\Sigma \cdot f \geq 0$, and $\Sigma \cdot \sigma \geq 0$ implies $ne + m \geq 0$. Note $\Sigma^2 = -n^2e + 2nm \geq -n^2e + nm \geq 0$. It is easy to examine the cases of equality. Under the assumption $a > 0$ and $b > ae$, then $(a\sigma + bf)(n\sigma + mf) = -ane + bn + am > 0$. Hence $a\sigma + bf$ is ample (by Nakai–Moishezon).

For the case $e < 0$, we still have the condition $(a\sigma + bf) \cdot f = a > 0$, and $(a\sigma + bf) \cdot \sigma = -ae + b > 0$. In fact $(a\sigma + bf)^2 = -a^2e + 2ab > 0$ gives $b > ae/2$ (which is more restrictive than the previous bounds). Assume $\Sigma$ is irreducible and $\Sigma \neq \sigma, f$. If $n = 1$, then $\Sigma = \sigma + mf$ is a section. But $-e \leq \Sigma^2 \leq -e + 2m$. Hence $m \geq 0$. If $n \geq 2$, then $\pi\colon \Sigma \to C$ is a finite covering of degree $n$. In characteristic 0, this is a separable morphism, so $2g(\Sigma) - 2 \geq n(2g - 2)$ (by Riemann–Hurwitz). But we can compute $2g(\Sigma) - 2$ using adjunction. Rearranging gives $2m(n-1) \geq ne(n-1)$, so $m \geq ne/2$ because $n > 1$. Now calculate $(a\sigma + bf) \cdot \Sigma > 0$. ☐

## 3.6 del Pezzo surfaces

del Pezzo surfaces are intrinsically interesting and their ample cones have beautiful geometry. They give examples of surfaces of almost minimal degree in the following sense. An embedded surface $X \subset \mathbb{P}^N$, has $\deg X \geq N - 1$ with equality iff $X = \varphi(\mathbb{F}_n)$. What if $\deg X = N$?

1. A trivial case: take $X$ to be the cone over an elliptic normal curve, i.e. $E$ embedded into $\mathbb{P}^{N-1}$ by the complete linear series of degree $N$.
2. If $X$ is smooth, then $N \leq 9$ and $X$ is either a blow-up of $\mathbb{P}^2$ at most 6 points in general position or $X = \mathbb{P}^1 \times \mathbb{P}^1$.
3. If $X$ has RDPs, then a slight modification of this statement is also true.

Consider linear systems of cubics on $\mathbb{P}^2$ with assigned base-points.

**Theorem 3.6.1.** *Let $p_1, \ldots, p_n \in \mathbb{P}^2$ and let $X := \mathrm{Bl}_{p_1,\ldots,p_n} \mathbb{P}^2$ with exceptional divisor $E_i$ corresponding to $p_i$. Let $D := 3\pi^*H - \sum E_i$ where $H$ is the hyperplane class.*

*1. $D$ is ample iff $n \leq 8$ and no three of the $p_i$ are collinear and no six lie on a conic, and, if $n = 8$, the $p_i$ do not lie on an irreducible singular cubic with one of them the singular point. Also, $K_X = \mathcal{O}_X(-d)$, so $-K_X$ is ample.*

*2. If $D$ is ample, it is very ample if $n \leq 6$. It embeds $X \subset \mathbb{P}^d$ where $d = D^2 = 9 - n$.*

*3. If $D$ is ample and $n = 7$, then $D$ is bpf and defines a morphism $\varphi\colon X \to \mathbb{P}^2$ which is finite degree-2 with branch divisor a smooth quartic curve.*

*4. If $D$ is ample and $n = 8$, then there is a unique base point.*

*Remark.* Note that $D^2 = 9 - n$, so if $D$ is ample then $n \leq 8$. That is why the theorem looks only at $n \leq 8$.

**Lemma 3.6.2.** *Assume $p_1, \dots, p_n \in \mathbb{P}^2$ are distinct points, and suppose there exists a reduced irreducible cubic $D_0 \subset \mathbb{P}^2$ such that all $p_i \in (D_0)_{reg}$. Let $X := \mathrm{Bl}_{p_1,\dots,p_n} \mathbb{P}^2$ and $D$ be the proper transform of $D_0$.*

1. *If $n \le 8$, then $D$ is nef and big and $\mathcal{O}_X(D) = K_X^{-1}$.*
2. *If $C \subset X$ and $C$ is irreducible and $C \cdot D = 0$, then $C \cong \mathbb{P}^1$ and $C^2 = -2$.*
3. *If $C \subset X$ and $C$ is irreducible and $C^2 \le 0$, then either $C$ is as in (2), or $C$ is an exceptional curve.*
4. *The linear system $|D|$ is bpf if $n \le 7$, and there is exactly one base point if $n = 8$.*
5. *For $n \le 6$, the associated morphism $\varphi$ is a birational morphism $X \to \varphi(X) \subset \mathbb{P}^N$, and for $n = 7$, $\varphi$ is generically* 2*-to-*1*. In all cases, the positive-dimensional fibers are the curves $C$ in (2).*

*Proof.* Note that $D$ irreducible and $D^2 = 9 - n > 0$ implies $D$ nef and big. We know $K_X = 3\pi^*H + \sum E_i$, so that $\mathcal{O}_X(D) = K_X^{-1}$. If $C \cdot K_X = 0$, then $C^2 < 0$ (by Hodge index), then $0 > 2p_a(C) - 2 \ge -2$ so that $p_a(C) = 0$, i.e. $C \cong \mathbb{P}^1$, and conversely. If $C^2 < 0$, then because $C \cdot D \ge 0$ by $D$ being nef, $C \cdot K_X \le 0$. If $C \cdot K_X = 0$, then we are in case (2). If $C \cdot K_X < 0$, we saw this implies $C$ is exceptional. Now suppose all base points are on $D$. Then look at

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)|_D \to 0.$$

We know $H^1(\mathcal{O}_X) = 0$ ($X$ is birational to $\mathbb{P}^2$). So we get a surjection $H^0(\mathcal{O}_X(D)) \twoheadrightarrow H^0(\mathcal{O}_X(D)|_D)$. But $\mathcal{O}_X(D)|_D$ is a line bundle on $D$ of degree $D^2 = 9 - n$. Note that if $D$ is reduced irreducible with $p_a(X) = 1$ and $L$ is a line bundle on $D$ with $\deg L > 0$, then Serre duality implies $H^1(L) = H^0(L^{-1}) = 0$, so Riemann–Roch implies $\chi(L) = h^0(L) = d$. In particular we have seen $h^0(\mathcal{O}_X(D)) = d + 1$. So the image $\varphi(X) \subset \mathbb{P}^d$, and if it is birational then the degree is also $d$. If $d \ge 2$, then $L$ is bpf, and if $d \ge 3$, then $L$ defines an embedding. This is well-known if $D$ is smooth. To show this, if $x \in D$, there is a short exact sequence $0 \to L \otimes \mathfrak{m}_x \to L \to \mathbb{C}_x \to 0$. To show $L$ is bpf, we want $H^1(L \otimes \mathfrak{m}_x)$. Serre duality still works for singular $x$, giving that

$$H^1(L \otimes \mathfrak{m}_x) = \mathrm{Hom}(L \otimes \mathfrak{m}_x, \omega_D) = H^0(L^{-1} \otimes \mathrm{Hom}(\mathfrak{m}_x, \mathcal{O}_D)) = 0.$$

Alternatively, look at $0 \to L \to \nu_*\nu^*L \to \mathbb{C}_x \to 0$ where $\nu\colon \tilde{D} \to D$ is the normalization map and $x$ is a singular point. For $d = 3$, we get an embedding since $D \to \mathbb{P}^2$ is birational onto its image, which is a plane cubic. But $p_a = 1$ for $D$ and $p_a = 1$ for its image as well. For $n = 8$, given $L$ on $D$ with $\deg L = 1$, then by Riemann–Roch $h^0(L) = 1$ so there exists a section $0 \to \mathcal{O}_D \to L \to \mathbb{C}_y \to 0$ (by counting degrees, the cokernel must be a skyscraper). But $y$ is not the singular point, since $\mathfrak{m}_x$ is not Cartier for $x$ is the singular point. Hence $L = \mathcal{O}_D(y)$ for $y$ a non-singular point. ☐

*Remark.* This proof shows the following classical fact. Suppose $p_1, \dots, p_8$ are eight points of $\mathbb{P}^2$ lying on $(D_0)_{\mathrm{reg}}$ where $D_0$ is an irreducible cubic. Then there exists a unique point $p_9 \in (D_0)_{\mathrm{reg}}$ such that every cubic passing through $p_1, \dots, p_8$ also passes through $p_9$.

**Lemma 3.6.3.** *Let $n \le 8$ and $p_1, \dots, p_n \in \mathbb{P}^2$ such that no three are collinear and no six lie on a conic. Then there exists an irreducible cubic $D_0$ containing $p_1, \dots, p_n$. If $n \le 7$, then we can assume $p_i \in (D_0)_{reg}$.*

*Proof.* There are only finitely many lines containing two points $p_i$. Because of the assumption that no three $p_i$ are collinear, there are only finitely many conics containing five points $p_i$. So we can complete $p_1, \dots, p_n$ to $p_1, \dots, p_8$ with the same hypotheses by choosing the remaining points generally. So wlog assume $n = 8$. By a dimension count, $\dim |\mathcal{O}_{\mathbb{P}^2}(3)| = 9$, so there exists some cubic $D_0$ containing all the $p_i$. In fact every such $D_0$ is irreducible, because otherwise $D_0$ is either $L_1 + L_2 + L_3$ or $L + C$ (where $L$ are lines and $C$ is a conic), but neither contain enough points.

If $n \le 7$, enlarge $n$ to be 7 in the same way. The first part shows there exists $D_0$ irreducible containing $p_1, \dots, p_7$. In other words, a generic element of $|\mathcal{O}_{\mathbb{P}^2}(3) - \sum p_i|$ is irreducible. But in fact we can show there exists $D_0$ (reducible) such that $p_1, \dots, p_7 \in (D_0)_{\mathrm{reg}}$. So the generic element of the linear system has this property as well and we are done. To show this, pick the unique $L$ containing $p_1, p_2$, and the unique smooth conic $C$ containing $p_3, \dots, p_7$. Take $D_0 = L + C$. Here $(D_0)_{\mathrm{sing}} = L \cap C$, so we must show $D \cap C = \emptyset$. But if $p_i \in L \cap C$ and $i > 2$, then $p_1, p_2, p_i$ are collinear, and if $p_1 \in L \cap C$, then there are six points on a conic. ☐

**Lemma 3.6.4.** *Take $p_1, \dots, p_n$ as above, and if $n = 8$, if all 8 lie on an irreducible $D_0$ then all $p_i \in (D_0)_{reg}$. Then there does not exist $C$ on $X$ with $C \cdot D = 0$ and $C \cong \mathbb{P}^1$.*

*Proof.* Start with $p_i \in D_0$ irreducible and not a singular point. Let $X$ and $D$ be as above, and suppose there exists a $C$ as above. Since $C$ is not $E_i$ (exceptional divisors), $C$ is the proper transform of a plane curve of degree $d$, so $C = \pi^* dH - \sum a_i E_i$ where the $a_i \geq 0$ are the multiplicities of the plane curve at the points $p_i$. We know $C^2 = -2$ and $C \cdot D = 0$. Then $d^2 - \sum a_i^2 = -2$, and $3d - \sum a_i = 0$. Plugging $d = (1/3) \sum a_i$ into the first equation, we get

$$\frac{1}{9}\left(\left(\sum a_i\right)^2 - 9 \sum a_i^2\right) = -2.$$

Cauchy–Schwarz gives $(\sum a_i)^2 \leq n \sum a_i^2$. In fact, instead of $n$, we can use $r := \#\{a_i : a_i \neq 0\}$. So

$$-2 \leq \frac{1}{9}(r-9) \sum a_i^2.$$

First, let's assume all $a_i \in \{0, 1\}$. Then $3d = r$ and $d^2 = r - 2$, so that $d^2 - 3d + 2 = 0$, i.e. $d = 1$ and $r = 3$ (proper transform of line) or $d = 2$ and $r = 6$ (proper transform of conic). The remaining possibility is that some $a_i \geq 2$, so that $a_i^2 \geq 4$. Then

$$\frac{1}{9}(9-r)(r+3) \leq \frac{1}{9}(9-r) \sum a_i^2 \leq 2.$$

Hence $(9-r)(r+3) \leq 18$. For $r \leq 7$ this does not happen. For $r = 8$, we have $d = 3$ and all $a_i = 1$ except one, which is 2. ☐

*Remark.* This proves (1) of the theorem, because $D$ is nef an big, with $D \cdot C > 0$ for all $C$. So $D$ is ample by Nakai–Moishezon. The only remaining point is why $D$ is very ample for $n \leq 6$.

*Proof of theorem.* Consider the morphism $\varphi\colon D \to \mathbb{P}^d$. Take $D \subset X$ smooth in $|-K_X|$ (which is bpf, so apply Bertini's theorem). Since $p_a(D) = 1$, $D$ is a smooth elliptic curve, so $\varphi(D) \subset \mathbb{P}^{d-1}$ is a elliptic curve embedded by a complete linear system of degree $d \geq 3$. (In fact, this is the definition of an **elliptic normal curve**.) Fact: $\varphi(D)$ is projectively normal, In other words, $\operatorname{Sym}^d H^0(\mathcal{O}_D(D)) \to H^0(\mathcal{O}_D(kD))$ is surjective. Equivalently, $|\mathcal{O}_{\mathbb{P}^{d-1}}(k)| \to |kD|$ is surjective. General fact: if $X \subset \mathbb{P}^d$ and $D = H \cap X \subset H = \mathbb{P}^{d-1}$, then we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^d}(k-1)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^d}(k)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{d-1}}(k)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^0(\mathcal{O}_X((k-1)D)) & \longrightarrow & H^0(\mathcal{O}_X(kD)) & \longrightarrow & H^0(\mathcal{O}_D(kD)). & &
\end{array}$$

Projective normality says the third vertical arrow is surjective. Then the first vertical arrow is surjective. Hence the middle arrow is surjective. Hence $X$ is projectively normal. Fact: projective normality implies normality. But $\varphi\colon X \to \varphi(X)$ is finite birational and $\varphi(X)$ is normal, and hence $\varphi$ is an isomorphism. ☐

*Remark.* Consider the case $D$ nef and big but not ample. Then there exists some $C$ such that $C \cdot D = 0$. The same argument essentially shows $\varphi(X)$ is the normal surface obtained by contracting all such $C$. In particular, $\varphi(X)$ has only RDPs. It is interesting to ask what RDP configurations of $(-2)$ curves are possible on $X$.

*Remark.* The cases $n = 7, 8$ seem to be exceptional. But for $n = 7$, it is more natural to embed $X$ in a weighted projective space $\mathbb{P}(1, 1, 1, 2)$, and for $n = 8$, embed in $\mathbb{P}(1, 1, 2, 3)$.

**Definition 3.6.5.** $X$ is a **del Pezzo surface** if $-K_X$ is ample.

*Remark.* A priori this seems more general than what we have discussed. Fact: $X$ del Pezzo implies $X = \mathrm{Bl}_{p_1,\dots,p_n}\mathbb{P}^2$ for $n \le 8$, with the $p_i$ as in the theorem, or $X = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. If $X$ is a del Pezzo surface and there exists a birational morphism $X \to \mathbb{F}_n$, then $n = 0, 1$ because there exists $\bar{D}$ smooth in $\mathbb{F}_n$. Note that $\mathrm{Bl}_p \mathbb{F}_0 = \mathrm{Bl}_{p,p_1}\mathbb{P}^2$, so either $X$ is $\mathbb{F}_0$ or it is a blow-up of $\mathbb{P}^2$.

*Remark.* Suppose $D = -K_X$ is very ample. (This is in some sense the main case.) Then we have a morphism $\varphi\colon X \to \mathbb{P}^d$ where $d = D^2$. We can assume $D$ is smooth. Then $2p_a(D) - 2 = K_X \cdot D + D^2 = 0$ (since $D = -K_X$). So $D$ is an elliptic curve. Also, from $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$ and that $H^1(\mathcal{O}_X) = H^1(\mathcal{O}_X(K_X)) = H^1(\mathcal{O}_X(-D)) = 0$ (by Kodaira vanishing, since $D$ is nef and big), we get $\varphi(D)$ is an elliptic normal curve as the same argument we discussed earlier. Note also that $E \subset X$ is exceptional iff $\varphi(E)$ is a line. So exceptional curves on $X$ correspond to lines contained in $\varphi(X)$.

**Lemma 3.6.6.** *Let $X$ be any surface with $D \in |-K_X|$ irreducible, and $C$ be an irreducible curve not equal to $D$ with $C^2 < 0$. Then either*

1. *$C^2 = -2$ and $C \cong \mathbb{P}^1$ with $C \cdot D = 0$, or*
2. *$C^2 = -1 = C \cdot K_X$, and $C \cong \mathbb{P}^1$ with $C$ exceptional.*

*Now assume $D^2 \ge 0$, and $\alpha \in \mathrm{Num}\, X$ with $\alpha^2 = -1$ and $\alpha \cdot K_X = -1$ and $\chi(\mathcal{O}_X) = 1$ (equivalently, $q = h^1(\mathcal{O}_X) = 0$). Furthermore assume there does not exist $C \cong \mathbb{P}^1$ with $C^2 = -2$. Then there exists an exceptional curve $E$ such that $\alpha = [E]$.*

*Remark.* Note that such an exceptional curve $E$ is unique, because $E^2 = -1$. Also, there seems to be no good characterization in the case $D^2 < 0$. For example, if we take 11 points on a smooth plane cubic and blow up, then we get a surface as described with $D^2 \le -2$. Question: is there a closed characterization of exceptional curves?

*Proof.* The first part of the lemma is clear. For the second part, let $L$ be the line bundle corresponding to $\alpha$ (since $\mathrm{Num}\, X = \mathrm{Pic}\, X$). Riemann–Roch says $\chi(L) = (\alpha^2 - \alpha \cdot K_X)/2 + 1 = 1$. Hence either $h^0(L)$ or $h^2(L) > 0$. But $h^2(L) = h^0(L^{-1} \otimes K_X)$, corresponding to $-\alpha - D$, and $(-\alpha - D) \cdot D < 0$. Since $D$ is nef, $L^{-1} \otimes K_X$ is not the class of an effective divisor. Hence $h^0(L) > 0$, so $L = \mathcal{O}_X(E)$ where $E = \sum a_i C_i$ is effective. By assumption, take $C$ is any curve such that $D \cdot C = 0$.

1. If $D^2 > 0$ then $D \cdot E = 1$, so $E = C$. Since $C^2 = -1$ and $C \cdot K_X = -1$, we get $C$ is exceptional.
2. If $D^2 = 0$, then $D$ is primitive since $D \cdot E = 1$. Then $E = C + mD$ where $m \in \mathbb{Z}_{>0}$, and $C \cdot D = 1$. But $-1 = E^2 = C^2 + 2m$, and $C^2 \ge -1$. So $m = 0$, and $E = C$. ☐

**Corollary 3.6.7.** *If $X$ is a del Pezzo surface, the walls of $A(X)$ are classes of exceptional curves. There are only finitely many exceptional curves on $X$.*

## 3.7 Lines on a cubic and del Pezzos

**Theorem 3.7.1.** *If $X$ is a cubic surface in $\mathbb{P}^3$, then there are exactly 27 lines on $X$.*

*First proof.* If we know $X = \mathrm{Bl}_{p_1,\dots,p_6}\mathbb{P}^2$, then we can enumerate the lines:

1. lines $E_1, \dots, E_6$;
2. proper transforms $H = E_i - E_j$ of the lines connecting $p_i$ and $p_j$;
3. proper transforms of conics $2H - \sum_5 E_i$ passing through five points.

So $27 = 6 + \binom{6}{2} + \binom{6}{5}$. Say $L$ is the class of an exceptional curve on $X$, with $L = dH - \sum b_i E_i$. Knowing $L^2 = L \cdot K_X = -1$ gives $d^2 - \sum b_i^2 = -1$ and $3d - \sum b_i = 0$. It follows by Cauchy–Schwarz that $(\sum b_i)^2 \le r \sum b_i^2$ where $r := \#\{i : b_i \neq 0\}$. Hence $(3d-1)^2 \le 6(d^2+1)$. So $d \in \{0, 1, 2\}$. Also, $d^2 - 3d + 2 = \sum b_i(b_i - 1)$. Using these constraints, we can show that the 27 lines above are the only cases. ☐

*Remark.* Because $\operatorname{rank}\operatorname{Num}(\operatorname{Bl}_{p_1,\dots,p_6}\mathbb{P}^2) = 7$, there are at most 6 disjoint exceptional curves. In fact there are two possibilities for maximal subsets of disjoint exceptional curves:

1. there are 6 and they are exceptional curves for some blow-down to $\mathbb{P}^2$;
2. there are 5 and they are exceptional curves for some blow-down to $\mathbb{F}_0$.

**Proposition 3.7.2.** *Every smooth cubic in $\mathbb{P}^3$ is the blow-up of $\mathbb{P}^2$ at six points.*

*Proof.* The main point: there exists some line $L \subset X$. Let $U \subset |\mathcal{O}_{\mathbb{P}^3}(3)|$ correspond to all smooth cubic surfaces. If $X \in U$, then all lines in $X$ are exceptional, and there are only finitely many. Consider the incidence correspondence

$$I \coloneqq \{(L, X) : L \subset X\} \subset \operatorname{Gr}(2,4) \times U.$$

Let $\pi_1, \pi_2$ be the projections from $\operatorname{Gr}(2,4) \times U$. Note that $\pi_2$ is proper, and $\pi_2^{-1}(X) = \{L : L \subset X\}$ is finite (and possibly zero). We will show that $\operatorname{codim}_{\operatorname{Gr}(2,4)\times U} I = 4$, so the image $\pi_2(I)$ is dense. But $\pi_2$ is proper, so the image is also closed. Hence $\pi_2(I) = U$.

Look at $\pi_1^{-1}(L) = \{X : L \subset X\} \subset U$. The codimension of $\pi_1^{-1}(L)$ is at most 4 in $U$. This is because we can pick $p_1, p_2, p_3, p_4 \in L$, and then $X \supset L$ iff $p_1, \dots, p_4 \in X$ by Bezout's theorem. (Actually it is easy to see the dimension is exactly 4, since the 4 conditions are independent.)

Choose $L \subset X$. Consider the linear system $|H - L|$ of hyperplanes in $\mathbb{P}^3$ containing $L$. Then $L$ is a fixed curve. The general element of $H \cap X$ is $L +$ smooth conic. But the linear system has no fixed curve, and has no base points either. So we get a morphism $X \to \mathbb{P}^1$ with the generic fiber $C \cong \mathbb{P}^1$. That implies $X$ is a ruled surface. Hence $X$ is a blow-up of some $\mathbb{F}_n$. By the lemma below, $\mathbb{F}_n$ also has to be a del Pezzo surface. Hence $n = 0, 1$. We also know that $K_{\mathbb{F}_n}^2 = 8$. So it must have been blown up five times. Hence it is a blow-up of $\mathbb{P}^2$ six times. ☐

**Lemma 3.7.3.** *If $X$ is a del Pezzo surface and $X = \operatorname{Bl}_x X'$, then $X'$ is also a del Pezzo surface, by checking $-K'_X$ is big and $-K'_X \cdot C > 0$ for all irreducible $C$.*

*Proof.* Omitted. ☐

*Second proof.* We can also do a Chern class computation. If we fix $X$ smooth, we want to count the number of lines $L \in \operatorname{Gr}(2,4)$ such that $L \subset X$. We can get a virtual count, but then we have to check that none of the lines occur with multiplicity bigger than one. One way to check is a local calculation that shows $I \to U$ is étale and proper. ☐

*Third proof.* Pick $p \in X$ and $p$ not on any line. Then $\operatorname{Bl}_p X \to \mathbb{P}^2$ is a double cover. By standard results about double covers, this is branched along a smooth quartic in $\mathbb{P}^2$. Fact: a smooth quartic has 28 bitangents. (This can be computed by Plücker coordinates.) Their inverse images in $X$ split as line + curve. However one of the bitangents contains the exceptional curve. ☐

**Example 3.7.4** (Lines on a general del Pezzo surface)**.** Let $X = \operatorname{Bl}_{p_1,\dots,p_n}\mathbb{P}^2$ where $n \le 8$. Let $d \coloneqq 9 - n = (K_X)^2$. Famous fact: if $3 \le n \le 8$, i.e. $1 \le d \le 6$, it turns out $(K_X)^\perp$ is an even lattice in $\operatorname{Num} X$ and is a root lattice of type $E_n$ (with the convention that

$$E_5 = D_5, \quad E_4 = A_4, \quad E_3 = A_2 \oplus A_1$$

by erasing nodes in the Dynkin diagram.) To "see" the root lattice in the geometry, note that the (integral) symmetries of $A(X)$ are given by the Weyl group. Lines on $X$ correspond to weights of an interesting representation of the Lie algebra of type $E_n$. For example, $E_6$ has a famous 27-dimensional representation. Fact: for $E_8$, the roots are equal to the weights. So there is a correspondence between the roots of $E_8$ and the "lines" (i.e. exceptional curves) of $X = \operatorname{Bl}_{p_1,\dots,p_8}\mathbb{P}^2$, given by

$$(\alpha \in \operatorname{Num} X \text{ with } \alpha^2 = \alpha K_X = -1) \mapsto \beta \coloneqq \alpha - K_X$$

since now $d = 1$. In fact, $K_X^\perp \cong -E_8$ is an even unimodular lattice. Conversely, given $\beta^2 = -2$ and $\beta \cdot K_X = 0$, let $\alpha \coloneqq \beta + K_X$. Using this, we can enumerate the lines of $X$:

1. $E_i$ for $i = 1, \dots, 8$, giving 8;
2. $H = E_i - E_j$, giving $\binom{8}{2}$;
3. $2H - \sum_5 E_i$, giving $\binom{8}{5}$;
4. $3H - 2E_i - \sum_6 E_j$;
5. $4H - 2\sum_3 E_i - \sum_5 E_j$;
6. $5H - 2\sum_6 E_i - \sum_2 E_j$;
7. $6H - 3E_i - 2\sum_7 E_j$.

Adding up all these, we get 240. In fact, this gives all exceptional curves in fewer blow-ups as well:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Curves | 1 | 3 | 6 | 10 | 16 | 27 | 56 | 240. |

**Example 3.7.5** (Kodaira)**.** Let $X$ be the blow-up of $\mathbb{P}^2$ at 9 points, which are the base locus of a pencil of cubic curves, i.e. take two cubic curves $C_0$ and $C_1$ with $C_0 \cap C_1 = \{p_1, \dots, p_9\}$. Put $\pi \colon X \to \mathbb{P}^1$ given by the pencil. Assume all elements of the pencil are irreducible, which is equivalent to all fibers of $\pi$ being irreducible. In particular, we can take a Lefschetz pencil (all elements are smooth with one ordinary double point). All fibers are $-K_X$.

Claim: there are no curves $C$ on $X$ with $C = \mathbb{P}^1$ and $C^2 = -2$. Otherwise given $C$ with $C \cdot f = 0$, then $\pi(C)$ is a point and therefore $C$ is a component of a fiber and is reducible. By a previous lemma, it follows that $\alpha$ is the class of an exceptional curve iff $\alpha^2 = -1$ and $\alpha \cdot K_X = -1$.

Claim: there is a bijection between the set of exceptional curves on $X$ and $\mathbb{Z}^8 = (f)^\perp/\mathbb{Z}f$, where we view $(f)^\perp \subset \operatorname{Num} X \cong \mathbb{Z}^{10}$ and note that $f \in (f)^\perp$. In particular, there exists infinitely many exceptional curves. The bijection is as follows. Fix an $\alpha_0 = [E_0]$, the class of some exceptional curve $E_i$. Given $\alpha = [E]$, map $\alpha \mapsto \alpha - \alpha_0$. Because they have the same intersection with $K_X$, we get $(\alpha - \alpha_0) \cdot K_X = 0$, i.e. $\alpha - \alpha_0 \in (f)^\perp$. Take its image in $(f)^\perp/\mathbb{Z}f$. Conversely, given $\beta \in (f)^\perp/\mathbb{Z}f$, lift it to $\tilde{\beta} \in (f)^\perp$. Consider $\alpha_0 + \tilde{\beta}$. Then

$$(\alpha_0 + \tilde{\beta})^2 = \alpha_0^2 + 2\alpha_0 \cdot \beta + \beta^2 = -1 + \text{even} \equiv 1 \bmod 2$$

by the Wu formula, and $(\alpha_0 + \tilde{\beta}) \cdot K_X = -1$. Claim: there is a unique $n \in \mathbb{Z}$ such that $(\alpha_0 + \tilde{\beta} + nf)^2 = -1$. Then this class $\alpha \coloneqq \alpha_0 + \tilde{\beta} + nf$ is an exceptional curve, and this construction is inverse to the previous one. To see uniqueness of $n$, note that

$$(\alpha_0 + \tilde{\beta})^2 = -1 + 2k, \quad (\alpha_0 + \beta^2 + kf)^2 = -1 + 2k - 2k(\alpha_0 \cdot f) = -1.$$

*Remark.* In this construction, the choice of points $p_1, \dots, p_9$ is not general. If we choose $p_1, \dots, p_9$ in general, then $|-K_X| = D$ is a single curve, and we can assume $D$ is a smooth cubic. We can check directly that there do not exist $C \subset X$ with $C^2 = -2$ and $C \cong \mathbb{P}^1$, but this involves slightly different methods.

*Remark.* What is $(f)^\perp/\mathbb{Z}f$? It is easy to see that if $\alpha_0$ is the class of an exceptional curve, then $(f)^\perp/\mathbb{Z}f \cong (\mathbb{Z}f \oplus \mathbb{Z}\alpha_0)^\perp$. In particular, this is unimodular, rank 8, negative-definite and even. Hence this is isomorphic to $-E_8$ (which is the unique positive such lattice). In fact, this is equal to the Mordell–Weil group of the elliptic surface $X/\mathbb{P}^1$. The intersection pairing is the same as the height pairing. (So there exist cases with only finitely many exceptional curves.)

*Remark.* We can blow up $\geq 10$ points. If they are sufficiently general, then we will always have infinitely many exceptional curves. Open problem: describe them. In particular, is there a closed-form description analogous to the cohomological description we gave for 9 points?

## 3.8 Characterization of del Pezzo surfaces

**Theorem 3.8.1.** *$X$ is a del Pezzo surface implies $X$ is a blow-up of $\mathbb{P}^2$ at $\leq 8$ points, or $X = \mathbb{F}_0$.*

*Remark.* Recall that if $X$ is del Pezzo and $X = \mathrm{Bl}_p X'$, then $X'$ is also del Pezzo. In fact, more generally, if $X$ is any surface and $X = \mathrm{Bl}_p X' \xrightarrow{\rho} X'$ and $D$ is ample on $X$, then $\rho_* D$ is ample on $X'$. (This is an easy Nakai–Moishezon argument.)

**Lemma 3.8.2.** *If $X$ is del Pezzo and $D := -K_X$ and $d := D^2$, then*

1. *$H^i(X, \mathcal{O}_X(D)) = 0$ for $i > 0$,*
2. *$\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) = d + 1$.*

*Proof.* Write $H^i(\mathcal{O}_X(D)) = H^i(\mathcal{O}_X(K_X + 2D))$. Since $D$ is ample, this is zero by Kodaira vanishing. (Actually this also holds if $D$ is only nef and big, by Ramanujam vanishing.) By Riemann–Roch, $\chi(\mathcal{O}_X(D)) = \frac{2D^2}{2} + \chi(\mathcal{O}_X) = d + 1$ since $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$ again by Kodaira vanishing. ☐

**Corollary 3.8.3.** *$D = -K_X > 0$ and $d + 1 > 0$.*

*Remark.* If $D \in |-K_X|$, then $p_a(D) = 1$, since $2p_a(D) - 2 = K_X \cdot D + D^2 = 0$.

**Lemma 3.8.4.** *Suppose there exists a reducible element of $|D|$. Then either $X$ is a blow-up of $X'$ (which is necessarily del Pezzo) or $X = \mathbb{P}^2$ or $X = \mathbb{F}_0$.*

*Proof.* If there exists a reducible section, then write $D = A + B$ where $A$ is irreducible and $B > 0$. If $A^2 < 0$, then $A \cdot K_X < 0$. Hence $A$ is exceptional, so we can blow it down, and therefore $X$ is a blow-up. So assume $A^2 \geq 0$. We know $D = A + B$ is nef and big, so it is numerically connected. So $A \cdot B > 0$. But $2p_a(A) - 2 = K_X \cdot A + A^2 = -A^2 - A \cdot B + A^2 < 0$. In particular, this means $p_a(A) = 0$ and $A \cong \mathbb{P}^1$ by the usual argument. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(A) \to \mathcal{O}_A(A) = \mathcal{O}_{\mathbb{P}^1}(a) \to 0$$

has $a = A^2 \geq 0$, so $H^0(\mathcal{O}_X(A)) \twoheadrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(a))$ is surjective. If $a = 0$, then we get a morphism $X \to \mathbb{P}^1$ with generic fiber $A \cong \mathbb{P}^1$. So $X$ is ruled. We may as well assume $X$ is in fact minimal, so $X = \mathbb{F}_n$. But necessarily $n = 0$, because it is minimal and del Pezzo. If $a > 0$, then $|A|$ is bpf and birational, since $|\mathcal{O}_{\mathbb{P}^1}(a)|$ is. Then we get a morphism $\varphi \colon X \to \mathbb{P}^{a+1}$ of degree $a$. It is also easy to see $\varphi$ is finite, because if there exists $C$ with $\varphi(C) = \{\mathrm{pt}\}$, then $A \cdot C = 0$. Then $C^2 < 0$ and $C \cdot K_X < 0$, and $C$ is exceptional and we assumed there are no exceptional curves. Hence the only possibility for $X$ (because of the degree and codimension estimate) is that $\varphi$ is an embedding, so $X = \varphi(X) = \mathbb{F}_n$ with $n = 0$ necessarily, or $X = \mathbb{P}^2$. ☐

**Lemma 3.8.5.** *Let $D = -K_X$ be very ample and $\varphi \colon X \to \mathbb{P}^d$ be the corresponding morphism. Identify $X$ with its image $\varphi(X)$. Then there are only finitely many lines on $X$.*

*Proof.* First of all, lines correspond to exceptional curves and hence are rigid, in the sense that there does not exist a connected scheme $T$ of dimension $\geq 1$ and a cycle $\mathcal{C} \subset X \times T$ (over $T$) with $t \mapsto C_t \subset X$ a line for all $t \in T$ such that $C_{t_1} \neq C_{t_2}$ for some $t_1 \neq t_2$. Else $C_{t_1} \equiv C_{t_2}$ are numerically equivalent, but $C_{t_1} \cdot C_{t_2} \geq 0$, but on the other hand $C_{t_i}^2 = -1$.

Let $G(2, d+1)$ be the Grassmannian of lines in $\mathbb{P}^d$. Then there exists a closed subscheme $J \subset G(2, d+1)$ such that $J = \{L : L \subset X\}$. Then $\dim J = 0$ and $J$ is therefore finite. This is because any variety $X$ in projective space is an intersection $H_0 \cap \cdots \cap H_N$ of hypersurfaces $H_i$ of degree $r_i$. If $J_i := \{L \in G(2, d+1) : L \subset H_i\}$, then $J = J_1 \cap \cdots \cap J_n$. We can describe $J_i$ as $\{L : \exists p_1, \ldots, p_{r_i+1} \in H_i \cap L\}$, and then we see $\dim J = 0$. ☐

*Proof of theorem.* Start with $X$ a del Pezzo surface of degree $d$. Claim: if $d \geq 3$, then $D$ is very ample, and if $d = 2$, then $D$ is bpf and defines a 2-to-1 morphism $X \to \mathbb{P}^2$. First suppose there exists an irreducible $D \in |-K_X|$. Then look at $\mathcal{O}_X(D)|_D$. This is a line bundle on $D$ of degree $d \geq 2$. We saw earlier that if $p_a(D) = 1$, then $D$ is bpf for $d = 2$ and very ample for $d \geq 3$. For $d \geq 3$, we get $\varphi \colon X \to \mathbb{P}^d$ birational of degree one. The general hyperplane section is smooth and hence elliptic normal. Hence $\varphi(X)$ is normal, and $X \to \varphi(X)$ is birational, and hence an isomorphism.

If there does not exist an irreducible $D$, then $X = \mathrm{Bl}_p X'$ where $D' = -K_{X'}$ and $(D')^2 = d + 1$. We can induct on the rank of $\mathrm{Num}\, X$. The key inductive step is the case where $X' \subset \mathbb{P}^{d+1}$ and $X = \mathrm{Bl}_p X'$. Then there does not exist a line $L$ on $X'$ with $p \in L$, because the proper transform would be a curve $C \subset X$ with $C^2 = -2$, which do not exist because $D$ is ample. So $p \in X' \subset \mathbb{P}^{d+1}$ and we can project onto $\mathbb{P}^d$. This is in fact a morphism on $X$. In particular, this says that $D = \pi^*(D') - E$ is bpf. So there exists a smooth (and hence irreducible) $D \subset X$, so we have reached the previous case.

Assume $d \geq 3$. We have shown $X \subset \mathbb{P}^d$, and there are only finitely many lines on $X$. So choose $p \in X$ with $p$ not on any line, and consider the morphism $\mathrm{Bl}_p X = X_1 \to \mathbb{P}^{d-1}$ defined by $\pi^* D - E = -K_{X_1}$. This has no positive-dimensional fibers, because otherwise we have a line passing through $p$. Therefore $-K_{X_1}$ is ample, and $X_1 \subset \mathbb{P}^{d-1}$ is still del Pezzo. So we can repeat until we get $X_n \subset \mathbb{P}^3$, where the exceptional curves $E_{i+1} \subset X_{i+1} = \mathrm{Bl}_{p_i} X_i$ are all disjoint in $X_n$. So $X_n$ is a cubic surface, but every cubic surface is a blow-up of $\mathbb{P}^2$ at $p_1, \ldots, p_6$. By enumerating them, there are at most 6 disjoint exceptional curves in $X_n$. (In fact any maximal set of disjoint exceptional curves has either 5 or 6 elements.) Hence $n \leq 6$. But $3 = d - n$, so $d \leq 9$. We can complete $E_1, \ldots, E_n$ to a maximal set of disjoint exceptional curves $E_1, \ldots, E_n, E_{n+1}, \ldots, E_6$ (or $E_5$), so the remaining ones live on $X$, the original surface. Blowing them down too, we get $\mathbb{F}_0$ or $\mathbb{P}^2$ (depending on whether we had 5 or 6 exceptional curves). Hence $X = \mathbb{F}_0$, or $X$ is a blow-up of $\mathbb{P}^2$ at $\leq 6$ points.

For the case $d = 2$, we have seen that $X$ is a double cover of $\mathbb{P}^2$. General formulas for double covers show that th branch locus is a smooth quartic, so there exist bitangent lines. The inverse image of any bitangent line is a reducible section of $|-K_X|$. Hence there exists an exceptional curve on $X$, and $X$ is a blow-up of a cubic surface. So $X$ is the blow-up of $\mathbb{P}^2$ at 7 points.

For the case $d = 1$, the linear system $|-K_X|$ has a simple base point. Again we can assume that all elements are in fact irreducible (because if we had a reducible one, we would be in a previous situation). Take $D_1, D_2 \in |-K_X|$ such that $D_1 \cdot D_2 = 1 = d$. Say the simple base point is at $p$. Blow it up to get $\tilde{X} \to \mathbb{P}^1$. We want to find an exceptional curve in $\tilde{X}$ disjoint from $E$. Then there exists an exceptional curve on $X$, so $X$ is the blow-up of $\mathbb{P}^2$ at 8 points. We can do this by arguing that the rank of $\mathrm{Num}\, X$ is 10, since $c_1^2(\tilde{X}) + c_2(\tilde{X}) = 12\chi(\mathcal{O}_{\tilde{X}}) = 12$. But $c_1^2(\tilde{X}) = 0$, so $\operatorname{rank} H_2(\tilde{X}) = 10$. As in Kodaira's example, exceptional curves on $\tilde{X}$ correspond to $(f)^\perp/\mathbb{Z}f = -E_8$. Take $\alpha_0 = [E]$. The condition that $E'$ and $E$ are disjoint is $(\alpha - \alpha_0)^2 = -2$, where $\alpha = [E']$. There are exactly 8 classes in $-E_8$ of square $-2$, so we can indeed locate an exceptional curve $E'$ disjoint from $E$. □

*Remark.* Let $X$ be smooth of degree $d$ in $\mathbb{P}^d$. Assume $X$ is non-degenerate, and linearly normal (so it is not a projection). By Clifford's theorem or otherwise, smooth hyperplane sections $D$ of $X$ are elliptic normal curves. By adjunction, $(K_X + D)|_D = \mathcal{O}_D$. In other words, $K_X \otimes \mathcal{O}_X(1)$ has trivial restriction to all smooth $D \in |D|$. In fact this shows $K_X = \mathcal{O}_X(-1)$, so $X$ is del Pezzo. If there exists an irreducible pencil (i.e. a pencil such that all $D_t$ are irreducible), then this is an easy argument.

Minor variations show that if $X \subset \mathbb{P}^d$ is degree $d$, normal, non-degenerate, and linearly normal, then either $X$ is a cone over an elliptic normal curve, or $X$ is a "generalized del Pezzo surface," i.e. the only singularities are RDPs, and $\omega_X^{-1}$ is very ample.

## 3.9 K3 surfaces

**Definition 3.9.1.** A surface $X$ is a **K3 surface** if $K_X = \mathcal{O}_X$ and $q(X) = 0$.

*Remark.* Fact: if $X$ is any algebraic surface and $K_X$ is trivial, then either $X$ is K3 or $X$ is an abelian surface (in which case $q(X) = 2$). For compact complex surfaces, either $X$ is K3, $X$ is a torus ($q = 2$), or

$X$ is Kodaira's surface, which is a fiber bundle over an elliptic curve with fiber an elliptic curve. If $X$ is an algebraic surface (compact complex) and $K_X \equiv 0$, then in fact $X$ has an étale cover with $K_X \cong \mathcal{O}_X$ and degree 2, 4, or 6.

*Remark.* "Most" complex torii are not abelian varieties. Likewise, "most" complex analytic K3 surfaces are not algebraic.

*Remark.* Over $\mathbb{C}$, all K3 surfaces are in fact simply connected and are all diffeomorphic.

**Example 3.9.2.** The easiest example is a smooth quartic surface in $\mathbb{P}^3$. By adjunction, this is a K3 surface. We also have the complete intersection examples: $(2,3)$ in $\mathbb{P}^4$, or $(2,2,2)$ in $\mathbb{P}^5$.

**Example 3.9.3** (Kummer surfaces)**.** Let $A$ be an abelian surface. Then we have the involution $i\colon A \to A$ given by $a \mapsto -a$. Then $B := A/i$ is singular at the 16 fixed points of $i$. These singularities have local type ordinary double points $x^2 + y^2 + z^2 = 0$. Blow up to get $\tilde{A}$, with 16 smooth rational exceptional curves $E_i$. By the functoriality of blowing up, $i$ extends to an involution on $\tilde{A}$. The fixed locus is now a disjoint union of smooth codimension-1 subvarieties. The quotient $\tilde{B} := \tilde{A}/i$ is a resolution of singularities of $B$. Let $C_i \subset \tilde{B}$ be the images of $E_i$. Let $\pi\colon \tilde{A} \to \tilde{B}$. Then $\pi^* C_i = 2E_i$, so $\pi^*(C_i)^2 = -4$. Hence $C_i \cong \mathbb{P}^1$ with self-intersection $-2$, i.e. the singular points of $B$ are rational double points.

Claim: $\omega_B = \mathcal{O}_B$. This is because $\omega_{B^{\mathrm{reg}}} = K_{B^{\mathrm{reg}}}$, and $K_{B^{\mathrm{reg}}} = \mathcal{O}_{B^{\mathrm{reg}}}$ because if $\omega$ is a generating section of $K_A$, pulling back by $i^*$ leaves it invariant. From what we saw about singularities, it follows that $\omega_B = i_*\omega_{B^{\mathrm{reg}}} = \mathcal{O}_B$. Hence $K_{\tilde{B}} = \mathcal{O}_{\tilde{B}}$ (Gorenstein singularities don't affect adjunction). Claim: $q(\tilde{B}) = 0$. This is because $H^0(\Omega_{\tilde{B}}) = H^0(\Omega_{\tilde{A}})^i = H^i(\Omega_A)^i = \{0\}$.

*Remark.* Via this construction, $A$ and $A^\vee$ give the same $\tilde{B}$. But this is essentially the only ambiguity.

*Remark.* The case that Kummer studied was the case where $A = J(C)$, the Jacobian of $C$, where $C$ is a genus 2 curve. It turns out that $|2\theta|$ (where $\theta$ is the theta divisor) is bpf, and defines an embedding $B := A/i \hookrightarrow \mathbb{P}^3$. These are what was classically meant by "Kummer surfaces."

*Remark.* Note that we automatically get $p_g = 1 = h^0(K_X) = h^0(\mathcal{O}_X)$. Moreover, all the higher plurigenera are 1 for the same reason. Clearly also $c_1^2 = 0$. By Noether's formula,

$$c_1^2 + c_2 = 12\chi(\mathcal{O}_X) = 12(1 - 0 + 1) = 24,$$

so $c_2 = 24 = \chi_{\mathrm{top}}(X)$. We have $b_0, b_4 = 1$ and $b_1, b_3 = 0$. Hence $b_2 = 22$. Fact: $H_1(X;\mathbb{Z}) = 0$ and $H^2(X,\mathbb{Z})_{\mathrm{tors}} = 0$. Let $\Lambda := H^2(X,\mathbb{Z})$, called the **K3 lattice**; its rank is therefore 22. It is even by the Wu formula, because $c_1 = 0$, and unimodular by Poincaré duality. By the Hodge index, its type is $(3,19)$; the positive part is $2p_g + 1$. By the classification of lattices, $\Lambda \cong U^3 \oplus (-E_8)^2$, where $U$ is the hyperbolic plane.

*Remark.* Suppose $C \subset X$ is an irreducible curve. Adjunction implies $\omega_C = \mathcal{O}_X(C)|_C$, so in particular $-2 \le 2p_a(C) - 2 = C^2$. We know already this holds with equality iff $C \cong \mathbb{P}^1$. We can use this to describe the ample cone $A(X) \subset \operatorname{Num} X \otimes \mathbb{R}$. The walls are correspond to curves $C$ such that $C^2 = -2$ and $C \cong \mathbb{P}^1$. Given such a $C$, we can look at $s_C := \alpha \mapsto \alpha + \alpha \cdot [C]$, the reflection about $[C]$. They generate a reflection group $W$ which acts properly discontinuously on $\mathcal{C}_+$, and $A(X)$ is a fundamental domain for this action. However, there can be infinitely many such $C$, and so $W$ can be complicated.

**Example 3.9.4.** Let $Y \to \mathbb{P}^1$ be Kodaira's example of a blow-up of $\mathbb{P}^2$ at 9 points which are the base points of a cubic pencil but are otherwise general. Take the double cover $X \to Y$ branched along two smooth fibers $f_1 + f_2$. Then $K_X \cong \mathcal{O}_X$ and $q(X) = 0$.

*Remark.* The short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_C \to 0$$

shows that $H^0(\mathcal{O}_X(C)) \twoheadrightarrow H^0(K_X)$ is a surjection. In general, $\dim H^0(\mathcal{O}_X(C)) = g + 1$. More generally, if $X \subset \mathbb{P}^3$ is smooth, non-degenerate and linearly normal, and the hyperplane sections are canonical curves in $\mathbb{P}^{g-1}$, then in fact $X$ is a K3 surface. If $C^2 = 0$, then $C$ is elliptic. So $|C|$ is bpf and we get a pencil

$X \to \mathbb{P}^1$ whose generic fiber is elliptic. If $C^2 > 0$, i.e. $g(C) \geq 2$, then $C$ is nef and big, and by looking at $0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to K_C \to 0$, standard facts tell us that on $C$, $K_C$ is bpf, and is very ample unless $C$ is hyperelliptic, in which case it is 2-to-1. Then $|C|$ is bpf on $X$, and we get a morphism $\varphi\colon X \to \mathbb{P}^g$. If $C$ is not hyperelliptic, $\varphi$ is birational and $\varphi(X)$ has hyperplane sections which are canonical curves. By Noether's theorem, these are projectively normal. Hence $\varphi(X)$ is projectively normal, and therefore normal (by the same kind of argument that we saw for del Pezzo surfaces). So $\varphi(X)$ is a contraction of $X$ at curves $D$ such that $C \cdot D = 0$, so $D^2 < 0$, so $D \cong \mathbb{P}^1$ and $D^2 = -2$. So $\varphi(X)$ has RDPs. If $C$ is hyperelliptic, then $\deg \varphi = 2$, so $\varphi(X)$ is a scroll. In fact, all curves $C' \in |C|$ are hyperelliptic in this case. The linear system $|2C|$ is almost always birational, and $|3C|$ is always birational. This embeds $\bar{X}$ (the contraction of $X$).

We want to construct moduli spaces of pairs $(X, H)$ where $X$ is K3 and $H$ is a primitive ample divisor, and ideally compactify. The natural way to compactify is to extend to $(X, H)$ where $H$ is nef and big. This motivates the question: what can we say about nef and big divisors on a K3 surface?

*Remark.* If $D$ is a divisor on $X$ and $D^2 = -2$, then either $D$ or $-D$ is effective. (By Riemann–Roch, $\chi(\mathcal{O}_X(D)) = D^2/2 + 2 \geq 1$, so either $h^0(\mathcal{O}_X(D)) > 0$ or $h^2(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(-D)) > 0$.)

*Remark.* If $D$ is nef and big, then $h^0(\mathcal{O}_X(D)) = D^2/2 + 2$. This is by Ramanujam vanishing: $h^1(\mathcal{O}_X(D)) = h^2(\mathcal{O}_X(-D)) = 0$. So $h^0(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D))$.

*Remark.* If $D$ is irreducible and $D^2 > 0$, then $|D|$ is bpf. In fact, if $D = C$ is smooth, we have a surjection $H^0(X, \mathcal{O}_X(C)) \twoheadrightarrow H^0(X, K_C)$, so $D^2 \geq 0$ iff $g(C) \geq 1$. In general, the short exact sequence is

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \omega_D \to 0,$$

and we have the following fact for $D$ irreducible and Gorenstein: if $p_a(D) \geq 1$, then $\omega_D$ is bpf.

Goal: produce a quasi-projective moduli space of **polarized K3 surfaces**, i.e. pairs $(X, H)$ where $H$ is nef and big, $H^2 = 2k > 0$ and $H$ is primitive. Let $\mathcal{F}_{2k}$ denote the set of all such pairs. In general, we want all of these to be embedded in some fixed projective space. Then we can use Hilbert scheme methods.

**Theorem 3.9.5.** *Let $D$ be a nef and big divisor on $X$ a K3 surface. Then $|D|$ has a base point iff $|D|$ has a fixed component iff $D \equiv aE + R$ where $E$ is a smooth elliptic curve and $R$ is smooth rational (so $E^2 = 0$ and $R^2 = -2$). In all cases, $|2D|$ is bpf and $|3D|$ is birational.*

*Proof.* Any linear system on any surface $X$ can be written as $D = D_m + D_f$ where $D_m$ is "moving" and $D_f$ is "fixed." Here $D_m$ has no fixed curves. Hence $h^0(\mathcal{O}_X(D_m)) = h^0(\mathcal{O}_X(D))$. Also, $h^0(\mathcal{O}_X(D_f)) = 1$.

Case 1: $D_m^2 > 0$. Since $D_m$ has no fixed curves, it is automatically nef, and big. So $h^0(\mathcal{O}_X(D_m)) = D_m^2/2 + 2$. But since we assumed $D$ is also nef and big, we get $D_m^2/2 + 2h^0(\mathcal{O}_X(D)) = D^2/2 + 2$. Hence $D_m^2 = D^2 = (D_m + D_f)^2 = D_m^2 + 2D_m \cdot D_f + D_f^2$. Hence $2D_m \cdot D_f + D_f^2 = 0$. On the other hand, write

$$2D_m \cdot D_f + D_f^2 = D_m \cdot D_f + (D_m + D_f) \cdot D_f = D_m \cdot D_f + D \cdot D_f > 0$$

since $D_m$ and $D$ are nef. Hence $D_m \cdot D_f = D_f^2 = 0$. If $D_f > 0$ then $h^0(\mathcal{O}_X(D_f)) \geq 2 + D_f^2/2 \geq 2$. This contradicts $h^0(\mathcal{O}_X(D_f)) = 1$. Hence $D_f = 0$, and $D_m = D$. If $|D_m|$ contains an irreducible curve, then $|D_m|$ is bpf, since $2p_a(D) - 2 = D^2 \geq 0$. In general, a modified Bertini theorem says if not, then $D = \sum D_i$ where $D_i \equiv D_j$ are smooth. So $|D_i|$ is bpf, and hence so is $|D|$. In fact it contains an irreducible element.

Case 2: $D_m^2 = 0$. In this case, there cannot be a base locus, because $D_1 \cdot D_2 = 0$ if $D_1, D_2 \in |D_m|$. So the general element is smooth. If $D_m = E$ irreducible, then $E^2 = 0$, so $E$ is smooth elliptic. If $D_m = aE$ with $E$ emooth elliptic, then $a \geq 2$. This is because if $a = 1$, then $h^0(\mathcal{O}_X(D)) = D^2/2 + 2 = h^0(\mathcal{O}_X(E))$. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to K_E = \mathcal{O}_E \to 0$$

shows that $h^0(\mathcal{O}_X(E)) = 2$. Hence $D^2 = 0$, contradicting that $D$ is big. We have $0 < D^2 = (D_m + D_f)^2 = (aE + D_f)^2 = 0 + 2a(E \cdot D_f) + D_f^2$. We know $D_f^2 < 0$ because otherwise $h^0(\mathcal{O}_X(D_f)) = D_f^2/2 + 2 \geq 2$. Hence $E \cdot D_f \neq 0$. So there exists some component $R$ of $D_f$ such that $R \cdot E \neq 0$. All components of $D_f$ must be isomorphic to $\mathbb{P}^1$ (else they move in a bpf series). So $R \cong \mathbb{P}^1$ and $R^2 = -2$. Claim: $E \cdot R = 1$.

Otherwise $E \cdot R \geq 2$, so $(E+R) \cdot R \geq 0$. Hence $E+R$ is nef. Since $(E+R)^2 = 2(E \cdot R) + R^2 \geq 4-2 = 2$. Hence $E+R$ is big. Then $h^0(\mathcal{O}_X(E+R)) > 2 = h^0(\mathcal{O}_X(E))$. Hence $R$ is not a fixed component in $E+R$, a contradiction. Finally, claim: $D = aE + R$. Write $D = D_1 + D_2$ where $D_1 = aE + R$ and $D_2 = D - D_1 \geq 0$. Apply the argument of case 1 with $D_m = D_1$ and $D_f = D_2$. Note that $D_1$ is nef since $D_1 \cdot E = 1$ and $D_1 \cdot R = a - 2 \geq 0$. By the same argument as case 1 again, we get $(D_1 \cdot D_2) + (D_1 + D_2) \cdot D_2 = 0$. Hence $D_2 = 0$, else $h^0(\mathcal{O}_X(D_2)) > 1$. But $D_2 \leq D_f$. Therefore $D_2 = 0$.

Finally, check that $2D$ is bpf. We have

$$0 \to \mathcal{O}_X(2aE + R) \to \mathcal{O}_X(2aE + 2R) \to \mathcal{O}_R(2a-4) \to 0.$$

Since $a \geq 2$, we get $\mathcal{O}_{\mathbb{P}^1}(2a-4)$ is bpf. Hence we must show this sequence is exact on $H^0$, i.e. $H^1(\mathcal{O}_X(2aE + R)) = 0$. This is true by Ramanujam vanishing, since $aE + R$ is nef and big so $2aE + R$ is even more nef and big. By a slight modification of this argument, $3D$ is birational. □

## 3.10 Period map

Let $X$ be a complex analytic K3 surface (not necessarily algebraic). Let $\Lambda \coloneqq H^2(X, \mathbb{Z})$ and $\Lambda_{\mathbb{C}} \coloneqq \Lambda \otimes \mathbb{C} = H^2(X, \mathbb{C})$. Then there is a Hodge decomposition

$$\Lambda_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.$$

The $H^{2,0} = H^0(\Omega_X^2)$ is the complex line determined by $\mathbb{C} \cdot \omega$ where $\omega$ is a non-vanishing holomorphic 2-form. Note that $H^{0,2} = \overline{H^{2,0}}$, and $H^{1,1} = (\mathbb{C}\omega \oplus \mathbb{C}\bar{\omega})^\perp$. So the entire Hodge structure on $H^2$ is determined by $\mathbb{C}\omega$. In particular, it is a line in $\Lambda_{\mathbb{C}} \cong \mathbb{C}^{22}$, and therefore a point in $\mathbb{P}(\Lambda_{\mathbb{C}}) = \mathbb{P}^{21}$. It satisfies two conditions coming from Hodge theory:

1. $\omega \cdot \omega = 0$, which says $\mathbb{C} \cdot \omega$ lives in a quadric $Q \subset \mathbb{P}^{21}$;
2. $\omega \cdot \bar{\omega} = \int_X \omega \wedge \bar{\omega} > 0$, which says $\mathbb{C} \cdot \omega$ lives in an open (not Zariski open) subset $D$ of $Q$.

We call $D$ the **period domain**. Consider families $p \colon \mathcal{X} \to S$, where $S$ is a complex manifold or more generally a reduced complex space, and $p$ is a proper smooth and holomorphic map with all fibers being K3 surfaces. Assume there is a local system $R^2 p_* \mathbb{Z} \subset R^2 p_* \mathbb{C}$. These correspond to representations $\pi_1(S, s_0) \to (\Lambda = \mathrm{Aut}(H^2(X_{s_0}, \mathbb{Z})))$ (or $\mathbb{C}$) where the action is by integral isometries. The assumption we want to make is that if the action is trivial, e.g. $S$ is simply connected, then $R^2 p_* \mathbb{Z} \cong \underline{\Lambda}$, the constant sheaf, and $R^2 p_* \mathbb{C} \cong \underline{\Lambda_{\mathbb{C}}}$. Of course, we can always achieve this by replacing $S$ by $\tilde{S}$, the universal cover. Then $\Lambda_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_S$ is a holomorphic vector bundle, and is filtered by Hodge sub-bundles. In particular, $R^0 p_* \Omega^2_{\mathcal{X}}$ is a holomorphic line sub-bundle of $\Lambda_{\mathbb{C}} \otimes \mathcal{O}_S$. This corresponds to a holomorphic map $S \to \mathbb{P}(\Lambda_{\mathbb{C}})$. Its image lies in the period domain $D$, because we can check this point-by-point. The **period map** is this map $S \to D$. If $S$ is not simply-connected, the best we can hope for is $S \to \Gamma \backslash D$. Unfortunately, $\Gamma$ does not act properly discontinuously on $D$, so $\Gamma \backslash D$ is not Hausdorff! In fact, we can check $\mathrm{SO}_0(3,19)$ acts transitively on $D$. So $D = \mathrm{SO}_0(3,19)/H$ where $H$ is the isotropy group is a point. The subgroup $H$ is closed, but is not compact.

Look at the period map for $\tilde{S}$. It is a holomorphic map, so we can ask: what is its derivative? First, the tangent space of $\mathbb{P}(\Lambda_{\mathbb{C}})$ at any point $\mathbb{C}\omega = H^{2,0} \subset \Lambda_{\mathbb{C}}$ is given by $\mathrm{Hom}(H^{2,0}, \Lambda_{\mathbb{C}}/H^{2,0})$. Since $\omega^2 = 0$, we can replace by $\mathrm{Hom}(H^{2,0}, (H^{2,0})^\perp / H^{2,0})$. But $(H^{2,0})^\perp = H^{2,0} \oplus H^{1,1}$, so $(H^{2,0})^\perp / H^{2,0} = H^{1,1}$. Hence the tangent space is $\mathrm{Hom}(H^{2,0}, H^{1,1}) = \mathrm{Hom}(H^1(\Omega^2_X), H^1(\Omega^1_X))$. It is hard to say something about $T_{S,s}$. Kodaira–Spencer theory gives a map $\theta_s \colon T_{S,s} \to H^1(X_s, T_{X_s})$, coming from the relative tangent bundle sequence:

$$0 \to T_{X/S} \to T_{\mathcal{X}} \to p^* T_S \to 0.$$

This has the property that $T_{\mathcal{X}/S}|_{X_s} = T_{X_s}$. In fact, taking $R^i p_*$, we get

$$0 \to R^0 p_* T_{\mathcal{X}/S} \to R^0 p_* T_{\mathcal{X}} \to R^0 p_* p^* T_S \to R^1 p_* T_{\mathcal{X}/S}$$

Note that since $T_X = \Omega_X^1$, the deformation theory of K3 surfaces is unobstructed; by Kodaira–Spencer–Kuranishi theory, obstructions lie in $H^2(X, T_X) = H^2(X, \Omega_X^1) = 0$. Fact: (Kodaira–Griffith) the differential of the period map is the cup product $H^1(X_s, T_{X_s}) \to \operatorname{Hom}(H^0(\Omega_{X_s}^2), H^1(\Omega_{X_s})^1) = H^1(\Omega_{X_s}^1)$. The local Torelli theorem says if $\tilde{S} = U$ is universal, then the differential of the period map is injective, and the image of $U \to D$ is an open set. The period map is locally an immersion.

**Theorem 3.10.1** (Global Torelli)**.** *Let $X, X'$ be two K3 surfaces, and suppose $\varphi\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ is a map such that:*

1. *$\varphi$ is an isometry of lattices;*
2. *$\varphi \otimes \mathbb{C}\colon H^2(X, \mathbb{C}) \to H^2(X', \mathbb{C})$ is an isomorphism of Hodge structures (i.e. $\varphi(\mathbb{C} \cdot w) = \mathbb{C} \cdot w'$);*
3. *for all $\alpha \in H^2(X, \mathbb{Z})$ such that $\alpha = [C]$ where $C \cong \mathbb{P}^1$, i.e. $\alpha^2 = -2$, we have $\varphi(\alpha) = [C']$ where the same holds.*

*Then there is a unique isomorphism $\psi\colon X' \to X$ such that $\psi^* = \varphi$.*

*Remark.* The ample cone $A(X)$ has walls given by the $[C]$ above. Define the **Kähler cone** $\mathcal{K}(X) \subset H^{1,1}(X, \mathbb{R})$ to be the open convex cone containing cohomology classes associated to Kähler metrics. In fact for all K3 surfaces $X$, we have $\mathcal{K}(X) \neq \emptyset$; we say K3 surfaces are "K"ahlerian." It is easy to see that $\mathcal{K}(X) \subset \{x \in \mathcal{C}^+ : x \cdot C > 0\ \forall C\}$, where $\mathcal{C}^+$ is the component of $\{x \in H^{1,1}(X, \mathbb{R}) : x^2 > 0\}$. In fact, these are equal. (So condition (3) in the global Torelli theorem is equivalent to $\varphi(K(X)) = K(X')$.) The Kähler cone $\mathcal{K}(X)$ is a fundamental domain for the group $W_X \coloneqq \{s_\alpha\}$ where $s_\alpha(x) \coloneqq x + (x \cdot \alpha)\alpha$. These reflections $s_\alpha$ are Hodge isometries of $H^2(X, \mathbb{Z})$, so $s_\alpha(\mathbb{C} \cdot w) = \mathbb{C}w$. So $s_\alpha$ satisfies conditions (1) and (2) of global Torelli, but not (3) because $s_\alpha([C]) = -[C]$. However, if $\varphi$ is any Hodge isometry (i.e. satisfies (1) and (2)), then $\varphi(\mathcal{K}(X)) = w\mathcal{K}(X')$ where $w \in W_{X'}$, the Weyl group of $X'$. So $w^{-1}\varphi$ satisfies (3). Hence $\varphi = w\psi^*$ for a unique choice of $w$.

**Theorem 3.10.2** (Surjectivity of the period map)**.** *All points of the period domain $D$ are in the image of the period map.*

Fix $k \in \mathbb{Z}_{>0}$ and consider the coarse moduli space

$$\mathcal{F}_{2k} \coloneqq \{(X, H) : X \text{ K3 surface, } H \text{ nef and big, primitive, } H^2 = 2k\}.$$

It is normal, and in fact is an orbifold of dimension 19 for all $k$. Also, $D_{2k} \coloneqq \{\mathbb{C}w \in D : w \cdot h = 0\} \subset D$, where $h \in \Lambda$ is a class with $h^2 = 2k$ and $h \in \mathcal{C}^+$. All possible choices of $h$ are equivalent under the action of $\Gamma \coloneqq \operatorname{Aut}^+(\Lambda)$. The space $D_{2k}$ is nicer than $D$, because it is a homogeneous space $\mathrm{SO}^+(2, 19)$, and is in fact a Hermitian symmetric space (independent of $k$). Now consider $\Gamma_{2k} \subset \Gamma$, which is an arithmetic subgroup of $\mathrm{SO}^+(2, 19)$, and acts properly discontinuously on $D_{2k}$. Then we get a period map $\mathcal{F}_{2k} \to \Gamma_{2k} \backslash D_{2k}$.

**Theorem 3.10.3** (Global Torelli theorem (algebraic version))**.** *The period map is an isomorphism. So $\mathcal{F}_{2k}$ is irreducible.*

*Remark.* Arithmetic quotients of Hermitian symmetric spaces are studied by Baily–Borel. Facts:

1. $\Gamma_{2k} \backslash D_{2k}$ is quasi-projective, therefore so is $\mathcal{F}_{2k}$;
2. $\Gamma_{2k} \backslash D_{2k}$ has a "minimal" compactification called the **Baily–Borel compactification**, but it is very singular.

There are other compactifications, e.g. the infinitely many toroidal compactifications (Mumford et al), which in some sense are blow-ups of the Baily–Borel compactification.

**Example 3.10.4.** Here are some applications of the local and global Torelli theorems.

1. All K3 surfaces are diffeomorphic to a smooth quartic surface in $\mathbb{P}^3$, and hence to each other. (Hence they are all simply-connected.)

2. If $X$ is a K3 surface, then in principle we know $\mathrm{Aut}(X)$. Let $\mathrm{Hodge}(X)$ be the group of Hodge isometries of $X$. Then $\mathrm{Hodge}(X) = W_X \rtimes \mathrm{Aut}(X)$, which follows directly from global Torelli.

3. We can construct K3 surfaces with given configurations of curves. This is because the period map is surjective.

There exist complex (algebraic) surfaces which are homotopy equivalent to K3 surfaces but are not K3 surfaces themselves. However, by M. Freedman, they are all homeomorphic to K3 surfaces. If $X$ is a complex surface which is diffeomorphic to a K3 surface, then $X$ is a K3 surface. Donaldson showed the image of the map $\mathrm{Diff}(X) \to \mathrm{Aut}(H^2(X,\mathbb{Z}))$ is equal to $\mathrm{Aut}^+(H^2(X,\mathbb{Z}))$, the index 2 subgroup which preserves $\mathcal{C}^+$.

## 3.11 Elliptic surfaces

Let $\pi\colon X \to C$ be a proper morphism (with connected fibers) from a smooth surface to a smooth curve. In particular, $\pi$ is flat. Let $f$ denote the general fiber, with genus $g = g(f)$. As divisors, $\pi^*(t) = \sum n_i C_i$ with $n_i > 0$. We may also want to work locally, where $\pi\colon X \to \Delta$ with $\Delta$ the unit disk, or $\operatorname{Spec} R$ where $R$ is a DVR, and $\mathcal{O}_X(f) = \mathcal{O}_X$, e.g. because $\mathcal{O}_X(f) = \pi^*\mathcal{O}_\Delta(t) = \mathcal{O}_X$. Hence all fibers are numerically equivalent. In particular, $f \equiv \sum_{i=1}^r n_i C_i$. Fix a singular fiber $\sum n_i C_i$. Let $\Lambda \coloneqq \bigoplus_i \mathbb{Z}[C_i]$ be the rank $r$ lattice generated by the components $C_i$. On $\Lambda$ there is an intersection pairing.

**Lemma 3.11.1.** *$\Lambda$ is negative semi-definite, with radical of rank 1 generated over $\mathbb{Q}$ by $\sum n_i C_i$. In fact, a primitive generator is $\sum a_i C_i$ where $a_i > 0$ and $\gcd(a_1, \dots, a_r) = 1$, and if $m = \gcd(n_1, \dots, n_r)$ then $n_i = m a_i$.*

*Proof.* If $r = 1$, then the fiber is of the form $mC$ where $C^2 = 0$. If $r > 1$, then for all $i$ there exists $j$ such that $C_i \cdot C_j \neq 0$. Since these are distinct curves, $C_i \cdot C_j > 0$. Also, $(\sum n_i C_i)^2 = f^2 = 0$, so $n_i C_i^2 + \sum_{j \neq i} n_j (C_i, C_j) = 0$. Hence $(C_i, C_j) \geq 0$ with $> 0$ for at least one $j$. The argument for a contractible configuration implies $\Lambda$ is negative semi-definite with radical generated over $\mathbb{Q}$ by $\sum n_i C_i$. ☐

**Corollary 3.11.2.** *If $A$ is a proper subset of $\{1, \dots, r\}$, then $\operatorname{span}\{C_i : i \in A\}$ is negative definite.*

**Corollary 3.11.3.** *If $E = \sum a_i C_i$ as before, then $E$ is numerically connected.*

*Proof.* Say $E = D_1 + D_2$ with $D_1, D_2 > 0$. Write $0 = E \cdot D_1 = D_1^2 + D_1 \cdot D_2$. Since $D_1$ is not a multiple of $E$, and $E$ is a primitive generator, we have $D_1^2 < 0$. Hence $D_1 \cdot D_2 > 0$. ☐

**Corollary 3.11.4.** *$H^0(\mathcal{O}_E) = \mathbb{C}$. In fact if $\lambda$ is any line bundle on $E$ and $\deg(\lambda|_{C_i}) \leq 0$, then $h^0(\lambda) \leq 1$ with equality iff $\lambda = \mathcal{O}_E$.*

*Proof.* This is Ramanujam's lemma. ☐

**Definition 3.11.5.** Let $m \coloneqq \gcd(n_1, \dots, n_r)$ and $E \coloneqq \sum a_i C_i$ with $m a_i = n_i$. We say $\sum n_i C_i$ is a **multiple fiber** if $m > 1$, i.e. $\sum n_i C_i = mE$ for $m > 1$.

**Proposition 3.11.6.** *If $mE$ is a multiple fiber, then $\mathcal{O}_E(E)$ is a torsion line bundle of order $m$, and there exists a connected étale cover of $E_{red}$ of degree $m$. (Equivalently, $H_1(E_{red}, \mathbb{Z}/m) \neq 0$.)*

*Proof.* Locally, we have $\pi\colon X \to \Delta$, and assume $0 \in \Delta$ corresponds to $\sum n_i C_i$ with all other fibers smooth. Locally on $X$, there exists analytic coordinates $x, y$ such that $\pi$ is given by $t = g(x,y)$. The statement that it is a multiple fiber means that, possibly after shrinking, $g = h^m$. Consider the Cartesian diagram

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & X' & \longrightarrow & X \\ & & \big\downarrow & & \big\downarrow \\ & & \Delta & \xrightarrow{t = w^m} & \Delta \end{array}$$

where $X'$ is the Cartesian product and $\tilde{X}$ is the normalization. Locally on $X$, we have $t = g = h^m$. So on $X'$, we get $0 = w^m - h^m = \prod_{\zeta \in \mu_m}(w - \zeta h)$. Each factor gives a branch of $\tilde{X}$. Locally, the fiber in $\tilde{X}$ is given by $\sum a_k \tilde{C}_k$, where the $\tilde{C}_k$ cover $C_k$, though individually they might be reducible. All coefficients $a_i = n_i/m$ appear as the coefficient of some $\tilde{C}_k$. In particular, $\gcd(a_k) = 1$. Also, the general fiber over $w \neq 0$ is a smooth fiber over $w^m$, and in particular it is connected of genus $g$. Upshot: $\tilde{E}$, which is the actual fiber over 0, is connected, is a non-multiple, and $\tilde{E}_{\text{red}} \to E_{\text{red}}$ is an étale map. Hence $H^1(E_{\text{red}}, \mathbb{Z}/m) \neq 0$. Finally, we want to show $\mathcal{O}_E(E)$ has order $m$. Let $\varphi\colon \tilde{X} \to X$ be the induced (étale) map. Clearly $\varphi^* \mathcal{O}_E(E) = \mathcal{O}_{\tilde{E}}(\tilde{E})$. On the other hand, $\mathcal{O}_{\tilde{E}}(-\tilde{E}) = \mathcal{O}_{\tilde{E}}$ because $w \bmod w^2$ is a generator of the ideal sheaf, and there is a $\mu_m$-action on $\mathcal{O}_{\tilde{E}}(-\tilde{E})$ given by $w \mapsto \tau \cdot w$. Since $mE \equiv f$, we get $\mathcal{O}_E(mE) = \mathcal{O}_E$, so $\mathcal{O}_E(E)$ is torsion of order dividing $m$. Say $\mathcal{O}_E(kE)$ is trivial. Then $\mathcal{O}_E(-kE)$ is trivial, If $s$ is a section, we can pull it up to a section of $\mathcal{O}_{\tilde{E}}(-k\tilde{E})$ which is $\mu_m$-invariant. Hence $m \mid k$. So the order of $\mathcal{O}_E(E)$ is exactly $m$. ☐

**Corollary 3.11.7.** *Let $f = mE$. Then $H^0(\mathcal{O}_f) = \mathbb{C}$, and hence $H^1(\mathcal{O}_f)$ has dimension $g$.*

*Proof.* We know this already for $E$ because we saw $H^0(\mathcal{O}_E) = \mathbb{C}$. Consider the exact sequence

$$0 \to \mathcal{O}_E(-kE) \to \mathcal{O}_{(k+1)E} \to \mathcal{O}_{kE} \to 0.$$

For $0 < k < m$, we know $\mathcal{O}_E(kE)$ is torsion but not trivial. So $H^0(\mathcal{O}_{(k+1)E}) \subset H^0(\mathcal{O}_{kE})$. By induction, starting at $k = 1$, we get $\mathbb{C} = H^0(\mathcal{O}_{mE}) \subset \cdots \subset H^0(\mathcal{O}_E) = \mathbb{C}$. By flatness, $\chi(\mathcal{O}_{f_t})$ is constant, where $f_t := \pi^*(t)$. For $t \neq 0$, we get $1 - g$, and for $t = 0$, we get $h^0(\mathcal{O}_{f_0}) - h^1(\mathcal{O}_{f_0})$. We just showed $h^0(\mathcal{O}_{f_0}) = 1$. Hence $h^1(\mathcal{O}_{f_0}) = g$. ☐

**Definition 3.11.8.** $i\colon X \to C$ is **relatively minimal** if there are no exceptional curves in the fibers of $i$. Clearly relatively minimal models exist, by the usual argument we do in the global case. Similarly, we can define **strongly minimal models**.

**Proposition 3.11.9.** *If $g \geq 1$, then strongly minimal models exist and are unique up to isomorphism over $C$.*

*Proof.* This is analogous to previous arguments. The only distinction is if there exists two exceptional curves $E \neq F$ in a fiber with $E \cdot F \neq 0$. Let $n := E \cdot F \geq 1$. If $n > 1$, the intersection form $\begin{pmatrix} -1 & n \\ n & -1 \end{pmatrix}$ of the lattice spanned by $E$ and $F$ is not negative semi-definite. Hence $n = 1$. Now look at $E + F$. Compute $(E + F)^2 = -2 + 2 = 0$. Hence $E + F$ is a primitive generator of the radical of $\Lambda$. So the fiber $f$ is $E + F$. (Note $H_1(E + F, \mathbb{Z}) = 0$.) By direct computation, $g(f) = 0$. ☐

**Definition 3.11.10.** Let $L$ be a line bundle on $X$. We say $L$ is **$\pi$-nef** if for all $C$ a component of a fiber (as opposed to all $C$), $\deg(L|_C) \geq 0$.

**Proposition 3.11.11.** *$K_X$ is $\pi$-nef iff $X \to C$ is strongly minimal.*

*Proof.* We'll show that if $g \geq 1$, then $K_X$ is $\pi$-nef iff $X \to C$ is minimal. But if strongly minimal models exist, then minimal is the same as strongly minimal.

If there exists an exceptional curve $C$ in a fiber, then $K_X \cdot C = -1$, so that $K_X$ is not $\pi$-nef. Conversely, say $C$ is a component of a fiber such that $C \cdot K_X < 0$. We know $C^2 \leq 0$, and $C^2 = 0$ iff $mC$ is a fiber for some $m \geq 1$. If $C^2 < 0$, then $C$ is exceptional, and so $\pi$ is not relatively minimal. If $C^2 = 0$, then $K_X \cdot C + C^2 = K_X \cdot C < 0$. Hence $C \cong \mathbb{P}^1$. Since $H_1(\mathbb{P}^1, \mathbb{Z}/m) = 0$, we see $m = 1$, and $g(C) = g(f) = 0$. ☐

*Remark.* For all fibrations with $g \geq 1$, we might as well assume strongly minimal, i.e. $K_X$ is $\pi$-nef. When $g = 1$, $X \to C$ is an elliptic surface.

**Example 3.11.12.** A trivial example is $E \times C \xrightarrow{\pi_2} C$. A less trivial example is if $E$ has an automorphism $\sigma$ (of order 2, 3, 4, or 6). Suppose also that $C$ also has an automorphism $\sigma_2$ of the same order. Then look at $(E \times C)/(\sigma_1, \sigma_2) \to C/(\sigma_2)$. Away from the fixed points of $\sigma_2$, the fibers are $E$. In general, this will have singularities.

**Example 3.11.13.** If we blow up $\mathbb{P}^2$ at 9 points which are the base locus of a pencil, then we get a rational elliptic surface. For a generic pencil, all fibers are irreducible, and we can assume modal. There are infinitely many exceptional curves, but none of them are contained in a fiber.

**Example 3.11.14.** Some K3 surfaces are elliptic, in which case they are automatically relatively minimal. In fact if $X$ is a smooth K3, then $X$ is an elliptic surface iff there exists $E$ on $X$ with $E$ smooth and $g(E) = 1$.

**Lemma 3.11.15.** *If $X$ is an elliptic surface and $C$ is a component of a fiber, then either $C^2 = 0$ and the fiber is $mC$, or $C^2 = -2$ and $C \cong \mathbb{P}^1$.*

*Proof.* Let $f = \sum a_i C_i$ be a fiber. Then $f^2 = 0$ and $K_X \cdot f = 0$ since $K_X \cdot f + f^2 = 2p_a(f) - 2 = 0$ if $f$ is smooth. Then $K_X$ is $\pi$-nef, so $K_X \cdot C_i \geq 0$, so $K_X \cdot C_i = 0$ for all $i$. If $C_i^2 = 0$, then $f = mC_i$ and $r = 1$. Else $C_i^2 < 0$, and $K_X \cdot C_i = 0$, so $C_i^2 = -2$ and it follows that $p_a(C_i) = 0$ and $C_i \cong \mathbb{P}^1$. ☐

*Remark.* Suppose we have a (possibly multiple) fiber $f$ with $f_{\text{red}}$ irreducible. Then $p_a(f_{\text{red}}) = 1$. The only possibilities are: smooth elliptic curve, nodal, or cuspidal. So we can have $m > 1$ in the smooth or nodal case, but not in the cuspidal case. This is because a cuspidal curve is homeomorphic to $S^2$, so $H_1(C, \mathbb{Z}/m) = 0$.

Now suppose $f = \sum n_i C_i$ is reducible. Then all components are isomorphic to $\mathbb{P}^1$, and $C_i^2 = -2$. The lattice $\Lambda$ is negative semi-definite with a 1-dimensional radical. We can classify all such lattices: $\tilde{A}_n$, $\tilde{D}_n$ (for $n \geq 4$), $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$ (the affine ADE diagrams).