The 3-fold K-theoretic DT/PT vertex correspondence holds

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Joint with Nikolas Kuhn and Felix Thimm [arXiv:2311.15697]

Donaldson–Thomas theory

Setting:

X = smooth (quasi-)projective 3-fold over $\mathbb{C}$

$\mathcal{M}^{\mathrm{DT}}(X) = \{$ideal sheaves of curves on $X\}$.

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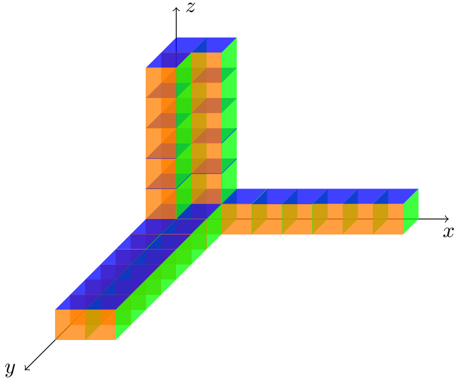

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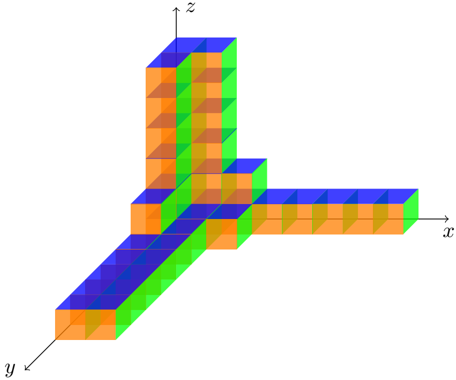

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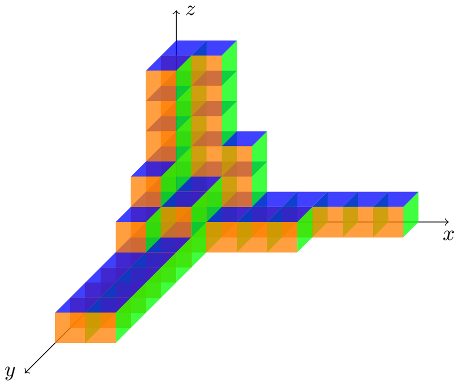

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E.g. $\mathsf{V}^{\mathrm{DT},K}_{\emptyset,\emptyset,\emptyset}(\text{non-equivariant}) = \prod_{n>0}(1 - Q^n)^{-n}$ is MacMahon's famous enumeration of 3d partitions.

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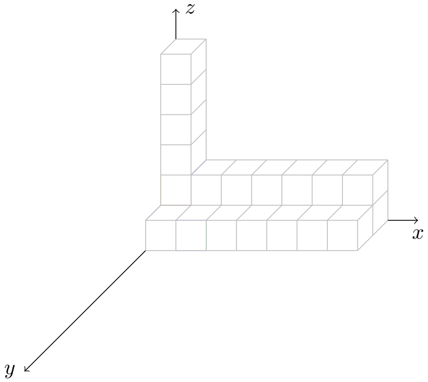

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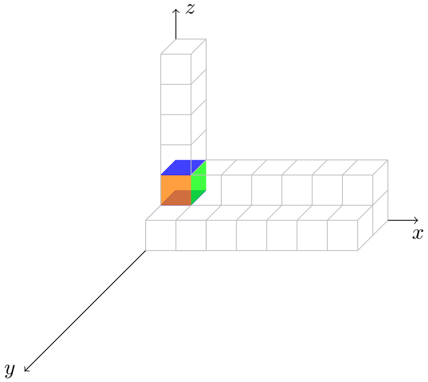

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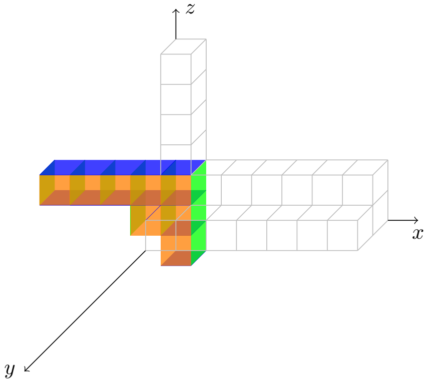

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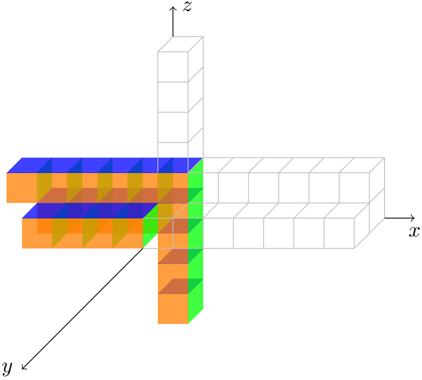

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$$\mathsf{V}^{\mathrm{PT},K}_{\emptyset,\emptyset,\emptyset}(x, y, z) = 1.$$

DT/PT vertex correspondence

Conjecture (PT '07)

$$\mathsf{V}^{\mathrm{DT}}_{\lambda,\mu,\nu} = \mathsf{V}^{\mathrm{PT}}_{\lambda,\mu,\nu}\mathsf{V}^{\mathrm{DT}}_{\emptyset,\emptyset,\emptyset}.$$

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Indeed, we use wall-crossing techniques (which work equally well in equivariant cohomology).

DT/PT (vertex) correspondence

Previous work mostly studied cohomological partition functions

$$\mathsf{Z}^{M}_{X,\beta} := \sum_{n\in\mathbb{Z}} Q^n \int_{[\mathcal{M}^{M}_{\beta_C,n}(X)]^{\mathrm{vir}}} 1, \qquad M \in \{\mathrm{DT}, \mathrm{PT}\}.$$

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Previous work also includes combinatorial approaches to the numerical DT/PT vertex correspondence

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- (Kononov–Okounkov–Osinenko '19) Holds for up to two non-trivial λ, μ, ν. ("Easy")
- (Jenne–Webb–Young '20) Holds for arbitrary λ, μ, ν. (Hard!)

DT/PT vertex correspondence

The full $\mathsf{V}^{M,K}_{\lambda,\mu,\nu}(x,y,z)$ are genuinely equivariant objects, much more sophisticated than $\mathsf{Z}^{M,K}_{X,\beta}$, $\mathsf{Z}^{M}_{X,\beta}$, or $\lim_{xyz\to 1} \mathsf{V}^{M,K}_{\lambda,\mu,\nu}$.

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We always work equivariantly with respect to the obvious action of $\mathsf{T} = (\mathbb{C}^\times)^3$.

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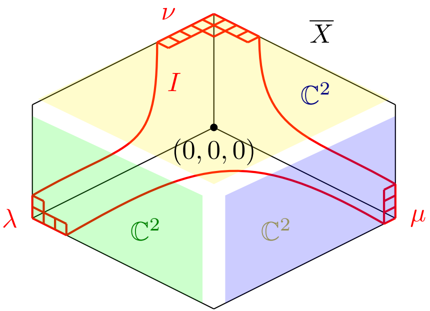

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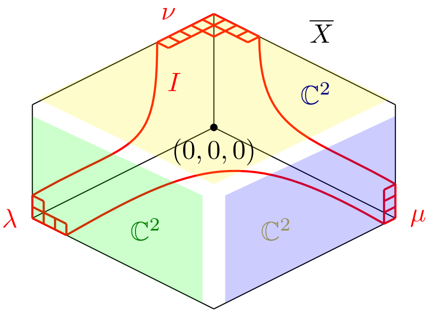

There is a moduli stack $\mathfrak{N}_{(\lambda,\mu,\nu),n}$ of pairs $I = [\mathcal{O}_{\overline{X}} \to \mathcal{E}]$ with this prescribed (derived) restriction to the boundary $D = D_1 \cup D_2 \cup D_3$, and numerical class

$$\mathrm{ch}(I) = (1, 0, -\beta_C, -n), \quad \beta_C := (|\lambda|, |\mu|, |\nu|).$$

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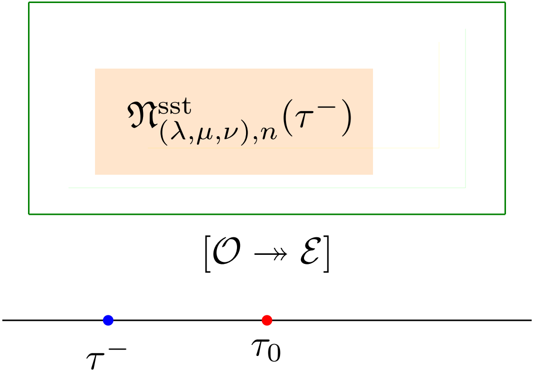

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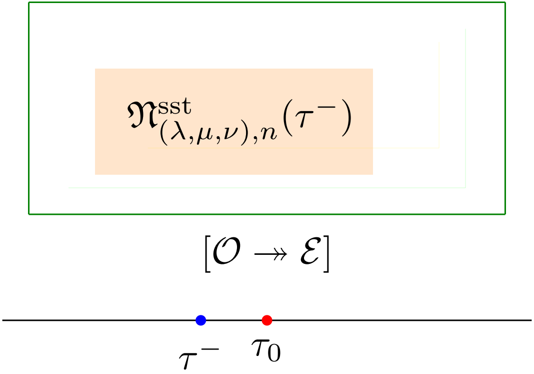

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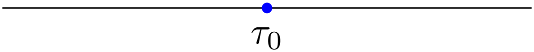
$$\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau_0)$$

$$[\mathcal{O} \to \mathcal{E}] \oplus [0 \to \mathcal{Q}_1] \oplus \cdots \oplus [0 \to \mathcal{Q}_k]$$



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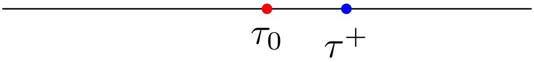

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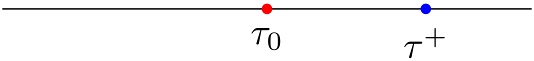

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Proposition

There is a symmetric obstruction theory on $\mathfrak{N}_{(\lambda,\mu,\nu),n}$*, given by* $\mathrm{Ext}_{\overline{X}}(I, I(-D))$*, such that*

$$\mathsf{V}^{\mathrm{DT},K}_{\lambda,\mu,\nu} = \sum_n Q^N \chi(\mathfrak{N}^{\mathrm{sst}}_{(\lambda,\mu,\nu),n}(\tau^-), \widehat{\mathcal{O}}^{\mathrm{vir}})$$

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Nekrasov–Okounkov symmetrization $\widehat{\mathcal{O}}^{\mathrm{vir}} := \mathcal{K}_{\mathrm{vir}}^{1/2} \otimes \mathcal{O}^{\mathrm{vir}}$.

Proof strategy, step 2: add framing data

Problem: master space arguments (later) require stable objects to split into ≤ 2 strictly-semistable pieces at a wall.

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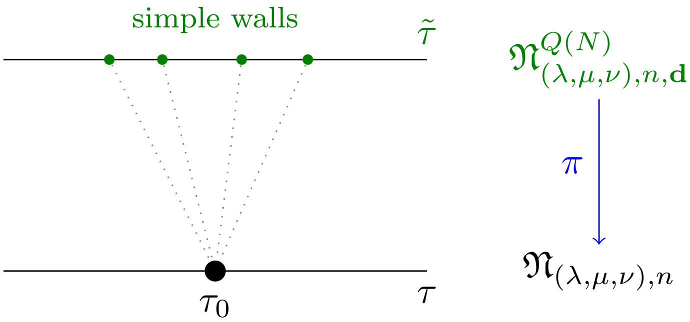

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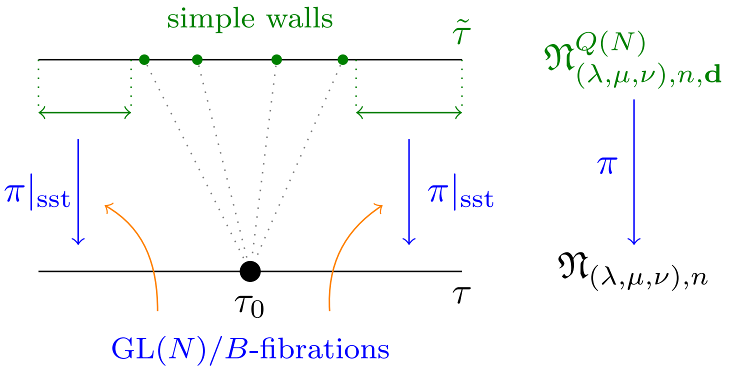

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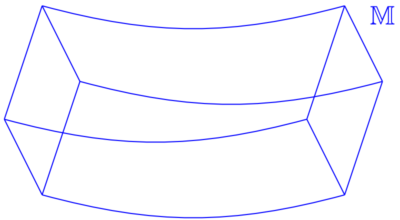

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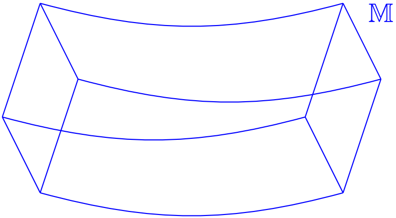

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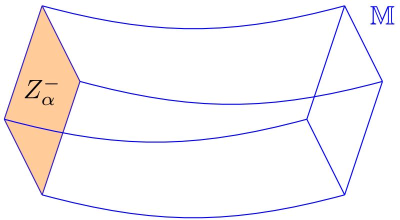

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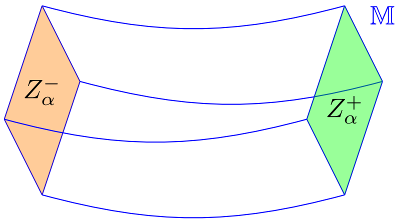

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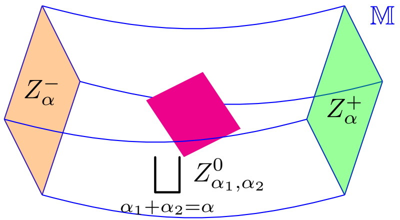

with a $\mathbb{C}_u^\times$-action whose fixed loci are:

- divisors Z_α^- and Z_α^+;
- components $Z^0_{\alpha_1,\alpha_2} \cong Z^-_{\alpha_1} \times Z^-_{\alpha_2}$, for strictly semistable splittings $\alpha_1 + \alpha_2 = \alpha$.

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$$\chi(\mathbb{M}, \mathcal{O}^{\mathrm{vir}}) \quad = \quad \sum_{F \subset \mathbb{M}^{\mathbb{C}^\times}} \chi\left(F, \frac{\mathcal{O}_F^{\mathrm{vir}}}{\wedge^\bullet_{-1}(\mathcal{N}^{\mathrm{vir}})^\vee}\right)$$

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where $\Bbbk := \mathbb{Z}[x^\pm, y^\pm, z^\pm, (xyz)^{\pm 1/2}, u^{\pm 1/2}]$. Apply K-theoretic residue map:

$$0 = \sum_{F \subset \mathbb{M}^{\mathbb{C}^\times}} \mathrm{Res}^K_u \, \chi\left(F, \frac{\widehat{\mathcal{O}}^{\mathrm{vir}}_F}{\widehat{\wedge}^\bullet_{-1}(\mathcal{N}^{\mathrm{vir}})^\vee}\right).$$

where $\mathrm{Res}^K_u(f) := (\mathrm{res}_{u=0} + \mathrm{res}_{u=\infty})(f\, u^{-1}\, du)$.

Proof strategy, step 3: master space

$$0 \quad = \quad \sum_{F \subset \mathbb{M}^{\mathbb{C}^\times}} \mathrm{Res}_u^K \, \chi\left(F, \frac{\widehat{\mathcal{O}}_F^{\mathrm{vir}}}{\widehat{\wedge}_{-1}^{\bullet}(\mathcal{N}^{\mathrm{vir}})^\vee}\right).$$

The goal: put an obstruction theory on $\mathbb{M}$ such that these residues are understandable and explicit.

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$$\mathbb{E} \simeq \kappa \otimes \mathbb{E}^{\vee}[1]$$

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(Restriction to any semistable = stable locus is automatically perfect!)

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Key observation: if $\mathcal{N}^{\mathrm{vir}} = \mathcal{F} - \kappa^{-1} \otimes \mathcal{F}^\vee$ is symmetric, then

$$\frac{1}{\widehat{\wedge}^\bullet_{-1}(\mathcal{N}^{\mathrm{vir}})^\vee} = \prod_{w \in \mathcal{F}} -\frac{(\kappa w)^{1/2} - (\kappa w)^{-1/2}}{w^{1/2} - w^{-1/2}}.$$

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Let $\mathcal{F} = \mathcal{F}_{>0} + \mathcal{F}_{<0}$ be the decomposition into positive and negative $\mathbb{C}_u^\times$-weight. Then

$$\mathrm{Res}_u^K \frac{1}{\widehat{\wedge}^\bullet_{-1}(\mathcal{N}^{\mathrm{vir}})^\vee} = (-1)^{\mathrm{ind}}(\kappa^{\mathrm{ind}/2} - \kappa^{-\mathrm{ind}/2})$$

where $\mathrm{ind} := \mathrm{rank}\,\mathcal{F}_{>0} - \mathrm{rank}\,\mathcal{F}_{<0}$ is a kind of Morse–Bott index of each $\mathbb{C}^\times$-fixed component.

Interlude: symmetric obstruction theories

Our situation: smooth morphisms of Artin stacks

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Technical heart of our work: symmetrized pullback of symmetric obstruction theories.

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Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of Artin stacks, and $\phi : \mathbb{E}_{\mathfrak{Y}} \to \mathbb{L}_{\mathfrak{Y}}$ be a symmetric obstruction theory.

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Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of Artin stacks, and $\phi\colon \mathbb{E}_{\mathfrak{Y}} \to \mathbb{L}_{\mathfrak{Y}}$ be a symmetric obstruction theory. Naive attempt:

$$\begin{array}{ccccccc} & & f^*\mathbb{E}_{\mathfrak{Y}} & & & & \\ & & \big\downarrow{\scriptstyle f^*\phi} & & & & \\ \mathbb{L}_f[-1] & \longrightarrow & f^*\mathbb{L}_{\mathfrak{Y}} & \longrightarrow & \mathbb{L}_{\mathfrak{X}} & \xrightarrow{+1} & \end{array}$$

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(a *non-symmetric* obstruction theory for $\mathfrak{X}$)

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\mathbb{L}_f[-1] & \xrightarrow{\delta} & f^*\mathbb{E}_{\mathfrak{Y}} & \longrightarrow & \mathrm{cone}(\delta) & \xrightarrow{+1} & \\
\downarrow & & \downarrow{\scriptstyle \delta^\vee[1]} & & \downarrow{\scriptstyle \xi} & & \\
0 & \longrightarrow & \kappa \otimes \mathbb{L}_f^\vee[2] & = & \kappa \otimes \mathbb{L}_f^\vee[2] & \xrightarrow{+1} & \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & &
\end{array}
$$

Interlude: symmetric obstruction theories

Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of Artin stacks, and $\phi\colon \mathbb{E}_{\mathfrak{Y}} \to \mathbb{L}_{\mathfrak{Y}}$ be a symmetric obstruction theory. Naive attempt:

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\mathbb{L}_f[-1] & \xrightarrow{\zeta} & \mathrm{cone}(\delta)^\vee[1] & \longrightarrow & \mathbb{E}_{\mathfrak{X}} & \xrightarrow{+1} & \\
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the desired (symmetric?) obstruction theory for $\mathfrak{X}$

Interlude: symmetric obstruction theories

$$
\begin{array}{ccccccc}
\mathbb{L}_f[-1] & \xrightarrow{\zeta} & \mathrm{cone}(\delta)^\vee[1] & \longrightarrow & {\color{red}\mathbb{E}_{\mathfrak{X}}} & \xrightarrow{+1} & \\
\| & & \downarrow & & \downarrow & & \\
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\end{array}
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Required:

Interlude: symmetric obstruction theories

$$\begin{array}{ccccccc}
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Required: the lift δ must exist;

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None of these hold in general, but they do hold on any affine chart for degree reasons.

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Better attempt: do the naive thing, but using Kiem–Savvas' étale-local notion of almost-perfect obstruction theory (APOT) assuming $\mathfrak{X}$ is DM. This is:

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Theorem (Kiem–Savvas '20, '21)

APOTs induce virtual structure sheaves $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir}}$. They satisfy equivariant localization assuming $\mathcal{N}^{\mathrm{vir}}$ also exists globally.

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Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of Artin stacks, and $\phi\colon \mathbb{E}_{\mathfrak{Y}} \to \mathbb{L}_{\mathfrak{Y}}$ be a symmetric obstruction theory.
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Proof strategy, step 4: put it all together

Resulting simple wall-crossing formula:

$$
\begin{aligned}
0 = {} & \chi(Z^-_\alpha, \widehat{\mathcal{O}}^{\mathrm{vir}}_{Z^-_\alpha})(\kappa^{-\frac{1}{2}} - \kappa^{\frac{1}{2}}) + \chi(Z^+_\alpha, \widehat{\mathcal{O}}^{\mathrm{vir}}_{Z^+_\alpha})(\kappa^{\frac{1}{2}} - \kappa^{-\frac{1}{2}}) \\
& + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \text{s.t. } \cdots}} \chi(Z^-_{\alpha_1}, \widehat{\mathcal{O}}^{\mathrm{vir}}_{Z^-_{\alpha_1}})\chi(Z^-_{\alpha_2}, \widehat{\mathcal{O}}^{\mathrm{vir}}_{Z^-_{\alpha_2}}) \\
& \qquad \cdot (-1)^{\mathrm{ind}(\alpha_1,\alpha_2)}(\kappa^{\frac{\mathrm{ind}(\alpha_1,\alpha_2)}{2}} - \kappa^{-\frac{\mathrm{ind}(\alpha_1,\alpha_2)}{2}}).
\end{aligned}
$$

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$$\begin{aligned} 0 = \chi(Z_\alpha^-, \widehat{\mathcal{O}}^{\mathrm{vir}}_{Z_\alpha^-}) \; - \; \chi(Z_\alpha^+, \widehat{\mathcal{O}}^{\mathrm{vir}}_{Z_\alpha^+}) \\ + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \text{s.t. } \cdots}} \chi(Z_{\alpha_1}^-, \widehat{\mathcal{O}}^{\mathrm{vir}}_{Z_{\alpha_1}^-}) \chi(Z_{\alpha_2}^-, \widehat{\mathcal{O}}^{\mathrm{vir}}_{Z_{\alpha_2}^-}) \cdot [\mathrm{ind}(\alpha_1, \alpha_2)]_\kappa \end{aligned}$$

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where $[N]_\kappa$ are (symmetric) quantum integers

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Can quantize many formulas in Joyce–Song this way.

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Can iterate the simple wall-crossing formula.

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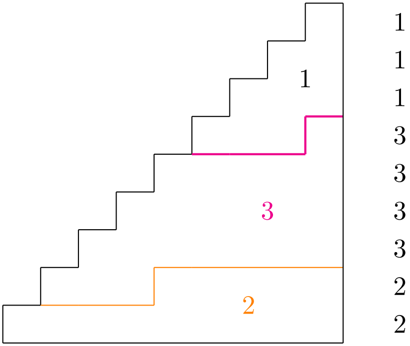

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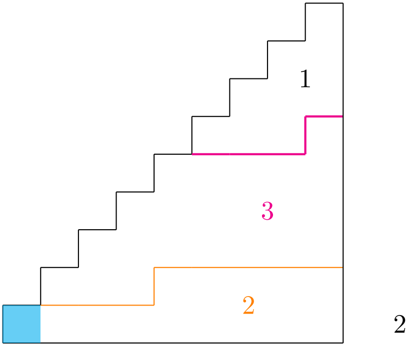

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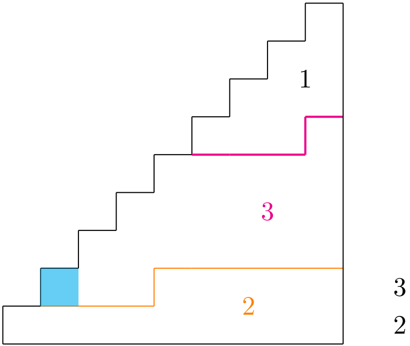

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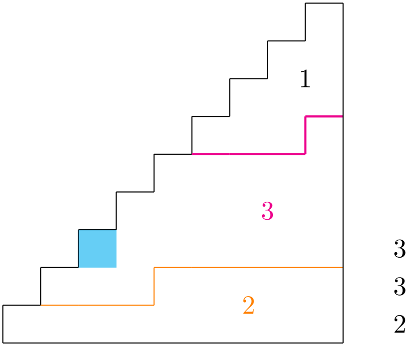

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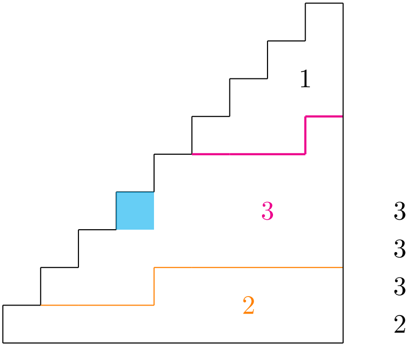

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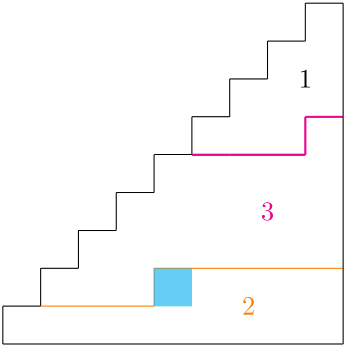

2
3
3
3
2

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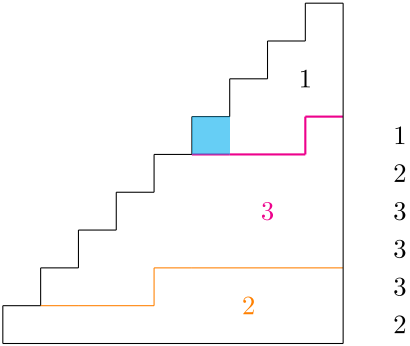

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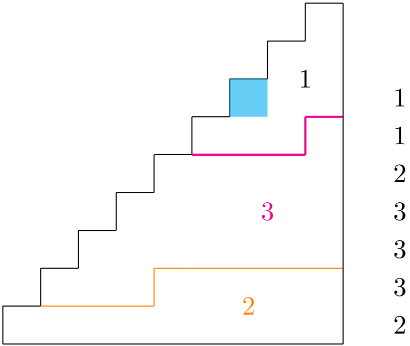

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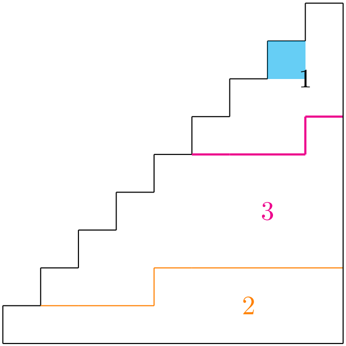

1
1
1
2
3
3
3
2

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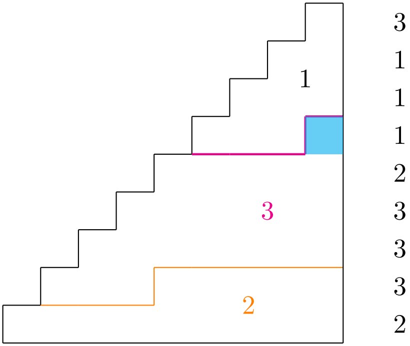

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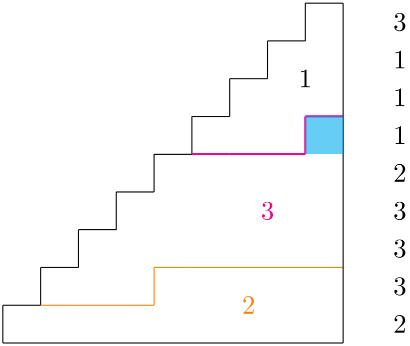

$R(m_1, \ldots, m_k) := \{\text{rearrangements of } 1^{m_1} 2^{m_2} \cdots k^{m_k}\}.$

Proof strategy, step 4: put it all together

Iterated simple wall-crossings produce complicated combinatorics.

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Proposition (Kuhn–L.–Thimm)

For $k \geq 0$ and $m_1, \ldots, m_k, m_{k+1} \geq 1$,

$$\frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{w \in R(m_1, \ldots, m_{k+1}) \\ o_{\sigma(1)}(w) > \cdots > o_{\sigma(k)}(w)}} \prod_{i=1}^{k} [m_{\sigma(i)} - \sum_{j=i+1}^{k+1} c(\boldsymbol{e}_{\sigma(i)}, \boldsymbol{e}_{\sigma(j)})]_\kappa = \frac{[m_1 + \cdots + m_{k+1}]_\kappa!}{[m_{k+1}]_\kappa! \prod_{i=1}^{k-1} [m_i - 1]_\kappa!}$$

where $o_i(w)$ is the index of the first occurrence of i in w, and $c(\boldsymbol{e}_i, \boldsymbol{e}_j) \approx$ the number of inversions in w for i and j.

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where $o_i(w)$ is the index of the first occurrence of i in w, and $c(\boldsymbol{e}_i, \boldsymbol{e}_j) \approx$ the number of inversions in w for i and j.

Alternatively, can sidestep this by a trick using the freedom to choose $p \geq 1$ in the framing functor $F_{k,p} = \cdots \oplus L^{\oplus p}$.

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Mochizuki-style (cf. [Nakajima–Yoshioka], [Kuhn–Tanaka]):

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However, $\widetilde{\tau}^0$-stable loci on auxiliary stacks are interesting in their own right, e.g. they include $\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^3}^{\oplus 2})$;

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Finally, we may try to obtain formulas for DT/PT descendent transformations.

Thank you!