Category O for quantum affine algebras

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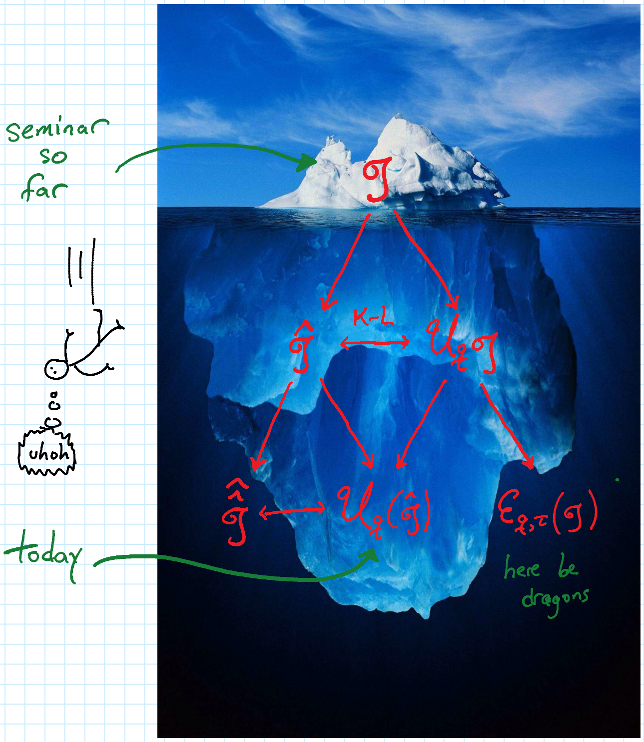


$\mathfrak{g}$ = ~~a symmetrizable Kac–Moody~~ finite Lie alg. (some "Cartan matrix" C)

$\cap$

$\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ (t: loop variable; c: central)

If $\mathfrak{g}$ is Lie alg (finite), some "affine Cartan" matrix $\hat{C}$

$$U_q\mathfrak{g} \subset U_q\hat{\mathfrak{g}}$$

($U_q\hat{\mathfrak{g}}$: quantum affinized algebra)

Use tools from $\mathfrak{g} \rightsquigarrow \hat{\mathfrak{g}}$ & $\mathfrak{g} \rightsquigarrow U_q(\mathfrak{g})$

e.g. evaluation maps $\quad$ e.g. braiding

$ev_a : \hat{\mathfrak{g}} \to \mathfrak{g}$
$\quad t \mapsto a \in \mathbb{C}^\times$

Simplest example: $U_q\hat{\mathfrak{sl}}_2$ and its reps

$\cup$

2-dim irrep V_1 of $U_q\mathfrak{sl}_2$:

$KeK^{-1} = q^2 e \leftarrow$ "$[h,e]$" (K: "q^h")
$KfK^{-1} = q^{-2} f \leftarrow$ "$[h,f]$"
$[e,f] = \frac{K-K^{-1}}{q-q^{-1}} \leftarrow$ "$[e,f]$"

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$C = (2)$

2-dim evaluation irrep $V_1(a)$ of $U_q\hat{\mathfrak{sl}}_2$: (a: evaluation parameter)

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad K_1 = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$\hat{C} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$e_0 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \qquad f_0 = \begin{pmatrix} 0 & a^{-1} \\ 0 & 0 \end{pmatrix} \qquad K_0 = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} = K_1^{-1}$$

E.g. what is $V_1(a)^\vee$? Fact: $= V_1(b)$ for some b.

To extract eval. parameter,

$$tr_{V_1(a)}(e_0 e_1) = a$$

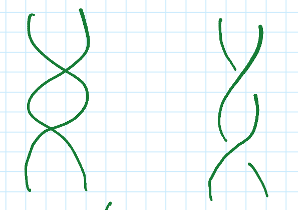
So it suffices to:

$$tr_{V_1(a)^\vee}(e_0 e_1) = tr_{V_1(a)}(\widetilde{Se_0} \cdot \widetilde{Se_1}) = q^2 a$$

where $\widetilde{Se_0} = -e_0 K_0^{-1}$, $\widetilde{Se_1} = -e_1 K_1^{-1}$

$$\Rightarrow \quad V_1(a)^\vee = V_1(q^2 a)$$

$$V_1(a)^{\vee\vee} = V_1(q^2 a)^\vee = V_1(q^4 a) \neq V_1(a)$$



A different presentation (Drinfeld):

$$[e(z), f(w)] = (\text{prefactor})(z,w)\,\frac{\psi^+(z)-\psi^-(z)}{z-w}$$

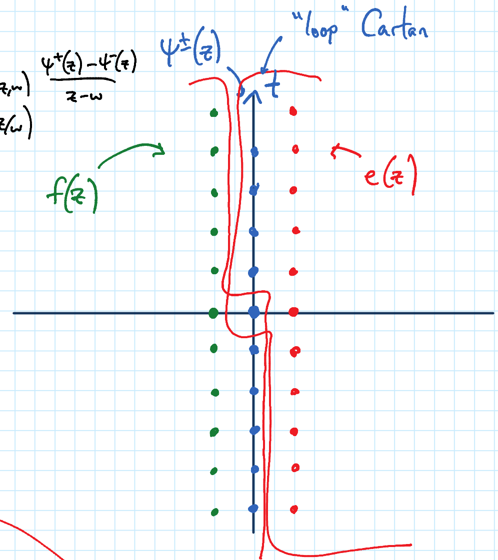
$\delta(z/w)$ ↑

"currents" (from physics)

$$e(z) = \sum_{n\in\mathbb{Z}} e_n z^{-n-1}$$

$$f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1}$$

$$\psi^\pm(z) = q^{\pm} \exp\left((q-q^{-1})\sum_{m\geq 1} h_{\pm m} z^{\mp m}\right)$$



Warning: $e_0, e_1, e_2, \ldots$ ← NOT the Chevalley generators!

the iso. w/ Chevalley gens is complicated.

Think: $x_n = $ "$x \cdot t^n$"

Characters of $U_q\mathfrak{g}$

$$\chi^{(i)}(V) = tr_V(K_i)$$

Characters of $U_q\hat{\mathfrak{g}}$

$$\chi_q^{(i)}(V)(z) = tr_V(\psi_i^\pm(z))$$

↑ q-character (Frenkel-Reshetikhin)

↖ kind of hard, e.g.

$$\chi_q(V\otimes W) = \chi_q(V)\chi_q(W)$$

Def: (some types of $U_q\hat{\mathfrak{g}}$ modules)

diagonalizable: $U_q\mathfrak{h}$ acts semisimply diagonalizable : $U_q \mathfrak{h}$ acts semisimply (finite) Cartan (i.e. $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$)

kind of hard, e.g. $\chi_q(V \otimes W) = \chi_q(V)\chi_q(W)$ — would need to check $\Delta \psi_i^\pm = \psi_i \otimes \psi_i$ + off-diag

integrable : diagonalizable, $\dim V_\lambda < \infty$ $\forall \lambda \in \mathfrak{h}^*$

$V_{\mu \pm r\alpha_i} = 0 \quad \forall r > R(\mu, i)$

can raise with $e_i(z) = \sum_{n \in \mathbb{Z}} e_{i,n} z^{-n-1}$ (independent of n.)

discrepancy !

in category $\mathcal{O}$: diagonalizable, $\dim V_\lambda < \infty$ $\forall \lambda \in \mathfrak{h}^*$

$\{\lambda \mid \dim V_\lambda > 0\} \subset$ descendents of a finite collection $\lambda_1, \ldots, \lambda_n$

wts for finite Cartan

μ s.t. $\mu < \lambda_i$

Source of a lot of problems for $U_q \hat{\mathfrak{g}}$:

$\exists$ ∞-ly many raising/lowering ops for each root α.

$V_\lambda \longrightarrow V_{\lambda \pm \alpha_i}$ $\quad (e_{i,n}, f_{i,n} \quad \forall n \in \mathbb{Z})$

E.g. general highest wt module is not integrable.

"smallest"

$$\text{highest wt} \quad v_0 \longmapsto \left.\begin{matrix} \vdots \\ f_{i,1} v_0 \\ f_{i,0} v_0 \\ f_{i,-1} v_0 \\ \vdots \end{matrix}\right\} \in V_{\lambda - \alpha_i}$$

is ∞-dim !!

in general, linearly independent

Thm: [Frenkel–Reshetikhin, Hernandez]

Highest wt module with $\psi_i^\pm(z) v_0 = \sum_{n \in \mathbb{Z}} \phi^\pm_{i, \pm n} v_0 \, z^{\mp n}$ (some scalar $\phi^\pm_{i,\pm n}$)

is :

integrable $\iff$ exists polynomials $P_i(z)$ s.t. ("Drinfeld polys.")

$$\sum_{n \geq 0} \phi^\pm_{i, \pm n} z^{\mp n} = q_i^{\deg P_i} \frac{P_i(z q_i)}{P_i(z/q_i)}$$

do the appropriate series expansion

in category $\mathcal{O}$ $\Rightarrow$ exists polynomials $P_i(z)$ s.t in category $\mathcal{O}$ $\Rightarrow$ exists polynomials $P_i(z)$ s.t

how you define $\delta(z)$

$$P_i(z)\left[\sum_{n\geq 0}\phi^+_{i,n}z^{-n} - \sum_{n\leq 0}\phi^-_{i,-n}z^n\right] = 0$$

Think: are expansions of $\frac{1}{1-z}$

e.g. $(1-z)\left[\cdots + z^{-2} + z^{-1} + 1 + z + z^2 + \cdots\right] = 0$

minus $-$ expansion; $+$ expansion

More generally, all "loop weights" of integrable modules are of the form

$$\psi_i^{\pm}(z) = q_i^{\deg P_i - \deg Q_i}\,\frac{P_i(zq_i)}{P_i(z/q_i)}\,\frac{Q_i(z/q_i)}{Q_i(zq_i)}$$

$\Rightarrow$ data of a q-character χ_q is (roots of P) & (roots of Q) for each loop weight space.

$\prod_{\text{roots }\alpha} Y_\alpha$ $\qquad$ $\prod_{\text{roots }\beta} Y_\beta^{-1}$

formal variables indexed by $\gamma \in \mathbb{C}$

E.g. $\chi_q\big(V_1(a)\big) = Y_a + Y_{q^2a}^{-1}$

$P = (z-a)$

Saw earlier that $V_1(a)^\vee = V_1(q^2a)$

$$0 \to ?? \to V_1(a)\otimes V_1(q^2a) \to \mathbb{C} \to 0$$

evaluation pairing

$$\chi_q\big(V_1(a)\otimes V_1(q^2a)\big) = \left[Y_a + Y_{q^2a}^{-1}\right]\left[Y_{q^2a} + Y_{q^4a}^{-1}\right]$$

$$= Y_aY_{q^2a} + Y_aY_{q^4a}^{-1} + 1 + Y_{q^2a}^{-1}Y_{q^4a}^{-1}$$

$\chi_q(\,??\,)$ $\leftarrow$ know now.

Fact: this is q-char of $V_2(qa)$ $\leftarrow$ 3-dim irrep of $\mathfrak{sl}_2$

Can ask general questions about

$$\bigotimes_{i=1}^N V_i(a_i)$$

1. when is it reducible? & into what factors?

2. do all f.d. irreps arise from these ?