Cat $\mathcal{O}$ $\to$ categorified quantum groups $\underline{II}$

$\mathcal{O}$ for sl_n or gl_n reg block $\mathcal{O}_{reg}$ categorifies free rep of $\mathbb{Z}[S_n]$

if add grading $\mathbb{Z}(S_n)$

$1 \in S_i$ q

$T_i : \mathcal{O}_{reg} \to \mathcal{O}_{s_i} \to \mathcal{O}_{reg}$

$\uparrow$ i-th wall

integral

Singular blocks Parametrized by decompositions

$\uparrow$ $\underline{k} = (k_1, \dots k_e)$ $\sum k_i = n$

don't contain any $\lambda_{\underline{k}}$ $\cdot\lambda_{\underline{k}}^{W} = S_{k_1} \times \dots \times S_{k_e}$

f.d. modules / $sl(n)$,

sing block = cat. of f.d rep certain f.d alg $A_{\underline{k}}$

gl_n weight lattice $= \mathbb{Z}^n$ $\varepsilon_1 \dots \varepsilon_{\underline{n}}$

weight of fund. rep $V = \mathbb{C}^n$ $\varepsilon_i - \varepsilon_{i+1}$ simple roots of $sl(n)$.

$\lambda_k = (\underbrace{1, 1 \dots 1}_{k}, \underbrace{0 \dots 0}_{n-k}) - \rho$ $\quad$ $Stab(\lambda_k) = S_k \times S_{n-k}$

$\varepsilon_1 + \dots + \varepsilon_k - \rho$. $\quad$ dot. action $\quad \overset{\cap}{S_n}$

"max. sing block"

weights of Verma modules in this block $S_n \cdot \lambda_k$

$\lambda \quad (1, 0, \dots 0 1, 0 \dots 1) - \rho$ $\quad$ k 1's, n-k 0's.

$\hat{}$ possible weights of $M_\lambda \in \mathcal{O}_{k, n-k}$.

$\mathcal{O}_{k,n-k}$ - block for this orbit

$K_0(\mathcal{O}_{k,n-k}) \simeq$ free. ab rank $\binom{n}{k}$.

basis $\{[M_\lambda]\}_\lambda$ $\quad \lambda = (\alpha_1 \ldots \alpha_n) - \rho \quad \alpha_i \in \{0,1\}$

in $K_0(\mathcal{O}_{k,n-k})$ $\quad \sum \alpha_i = k$

bases $\{[L_\lambda]\}_\lambda$ - simple mod, $\{[P_\lambda]\}$ indec. proj. module

fin. hom. dim $\quad$ BGG reciprocity.

for a f.d. alg A $\nearrow K_0(A)$ - Groth group of f.g. proj.
$\searrow G_0(A)$ — Groth group of finite length modules

$\underline{K_0(A)}$

$[P]$ $\quad [P] = [P_1] + [P_2]$

$P \simeq P_1 \oplus P_2$

$\underline{G_0(A)}$ $\quad 0 \to M_1 \to M_2 \to M_3 \to 0$

$[M]$ $\quad [M_2] = [M_1] + [M_3]$.

$K_0(A) \xrightarrow{\gamma} G_0(A)$ $\quad$ neither inj nor surj.

if A has finite hom dim, then γ is an isomorphism

$[P] \longrightarrow [P]$

For blocks of $\mathcal{O}$, γ is an isomorphism.

$$\mathcal{O}_{0,n} \rightleftarrows \quad \mathcal{O}_{k,n-k} \underset{\mathcal{F}}{\overset{\mathcal{E}}{\rightleftarrows}} \mathcal{O}_{k+1,n-k-1} \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \rightleftarrows \mathcal{O}_{n,0}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n = \dim(\underline{V^{\otimes n}})$$

V - fund rep of $\underline{sl(2)}$ fix $\underline{\underline{2}}$

$$\mathcal{O}_n = \bigoplus_{k=0}^{n} \mathcal{O}_{k,n-k}$$

Two sl's here, $sl(2)$ is getting categorified by cat $\mathcal{O}$ for $sl(n)$, over all n.

$sl(2) \to sl(k)$.

V – fund. rep of $sl(2)$. Integral version of V, $sl(2)$.

$$E \uparrow \begin{matrix} v_1 \\ v_0 \end{matrix} \downarrow F \qquad V = \mathbb{Z}v_0 \oplus \mathbb{Z}v_1$$

$U_{\mathbb{Z}}(sl(2))$ – integral version (Humphreys, last chapter of H's book)

$V_{\mathbb{Z}}$ integral version

Kostant integral form of $sl(2)$
$$\left\{ \begin{array}{l} F,\ F^{(m)} \quad E,\ E^{(m)} = \dfrac{E^m}{m!} \quad \text{divided power} \\ \dbinom{H}{m} = \dfrac{H(H-1)\dots(H-m+1)}{m!} \end{array} \right.$$

$V_{\mathbb{Z}}^{\otimes n} \qquad v_{\alpha_1} \otimes v_{\alpha_2} \dots \otimes v_{\alpha_n} = v_\alpha \qquad \alpha_i \in \{0,1\}$

$\alpha = (\alpha_1 \dots \alpha_n)$

E, F

$$E(v_\alpha) = \sum_{\alpha_i = 0} v_{\alpha + \varepsilon_i} \qquad \varepsilon_i = (0 \dots 0, \overset{i}{1}, 0 \dots)$$

$$\alpha = (10110) \qquad E v_\alpha = v_{11110} + v_{10111}$$

$$F(v_\alpha) = \sum_{\alpha_i = 1} v_{\alpha - \varepsilon_i} \qquad F v_{10110} = v_{00110} + v_{10010} + v_{10100}.$$

$$\underset{M}{\mathcal{O}_{k,n-k}} \underset{\mathcal{F}}{\overset{\mathcal{E}}{\rightleftarrows}} \underline{\mathcal{O}_{k+1,n-k-1}} \ni N$$

$V = \mathbb{C}^n$ $\quad \varepsilon_i$ $\quad -\varepsilon_1 \dots -\varepsilon_n$

$$\mathcal{E} = \underbrace{pr_{k+1}(M \otimes V)}_{} \qquad \mathcal{F} = \underbrace{pr_k}_{}(N \otimes \underbrace{V^*}_{\text{weight}})$$

project onto $(k+1)$st sing block.

$\underline{M_\lambda \otimes V}$ has filtration with s. quotients

$M_{\lambda+\underline{\varepsilon_i}}$ $\quad 1 \le i \le n$

$\lambda = (1,\dots 0, 0, 1, 0 \dots)$ $\qquad M_\mu$

$\underline{M_{(1\ 0, 0, 2, 0, \dots 0)}}$ $\qquad \underline{M_{(1,1,0,1,0)}}$ $\qquad \mu = (1, \dots 0\ 0) \to k+1$ 1's

$0 \to 1$

M_μ $\qquad \mu - (0 \dots, 1, 0 \dots)$

ε_i

$$[\mathcal{E}] : K_0(\mathcal{O}_{k,n-k}) \to K_0(\underline{\mathcal{O}_{k+1, n-k-1}})$$

$$[\mathcal{E} M_\lambda] = \sum_{\substack{\mu = \lambda + \varepsilon_i \\ \lambda_i = 0}} [M_\mu]$$

$$[\mathcal{F} M_\mu] = \sum_{\substack{\lambda = \mu - \varepsilon_i \\ \mu_i = 1}} [M_\lambda]$$

$\mu = (\mu_1 . 1\ \mu_n)$

$\downarrow$

0

$\sum \lambda_i = k.$

$$K_0(\mathcal{O}_{k,n-k}) \xrightarrow{\simeq} V_2^{\otimes n}(k) \qquad \lambda_i \in \{0,1\}$$

$$[M_\lambda] \longrightarrow v_\lambda = v_{\lambda_1} \otimes \dots \otimes v_{\lambda_n}$$

v_0

v_1

$$\begin{array}{ccccc} -n & & -n+2 & & n \\ V_{\mathbb{Z}}^{\otimes n}(0) & \underset{F}{\overset{E}{\rightleftarrows}} & V_{\mathbb{Z}}^{\otimes n}(1) & \cdots \underset{F}{\overset{E}{\rightleftarrows}} & V_{\mathbb{Z}}^{\otimes n}(n) \\ \wr\! \| & & \wr\! \| & & \wr\! \| \\ K_0(\mathcal{O}_{0,n}) & \underset{[\mathcal{F}]}{\overset{[\mathcal{E}]}{\rightleftarrows}} & K_0(\mathcal{O}_{1,n-1}) & \cdots \underset{[\mathcal{F}]}{\overset{[\mathcal{E}]}{\rightleftarrows}} & K_0(\mathcal{O}_{n,0}) \end{array}$$

$2k-n$ weight

$$\begin{array}{ccc} V_{\mathbb{Z}}^{\otimes n}(k) & v_\lambda & \text{product basis} \\ \wr\! \| & \updownarrow & \updownarrow \\ K_0(\mathcal{O}_{k,n-k}) & [M_\lambda] & \text{Verma module basis.} \end{array}$$

<u>Prop</u>

$$\begin{array}{ccc} 2k-n & & 2(k+1)-n \\ V_{\mathbb{Z}}^{\otimes n}(k) & \underset{F}{\overset{E}{\rightleftarrows}} & V_{\mathbb{Z}}^{\otimes n}(k+1) \\ \wr\! \| & & \wr\! \| \\ K_0(\mathcal{O}_{k,n-k}) & \underset{[\mathcal{F}]}{\overset{[\mathcal{E}]}{\rightleftarrows}} & K_0(\mathcal{O}_{k+1,n-k-1}) \end{array}$$

$\exists$ a comm. diag.

$\mathcal{E}, \mathcal{F}$ exact functors.

$\mathcal{E}$ $\quad M \longmapsto M \otimes V \xrightarrow{\mathrm{proj}_{k+1}} \mathcal{E}(M)$

$\quad N \longmapsto N \otimes V^* \xrightarrow{\mathrm{proj}_k} \mathcal{F}(N)$

$\underline{\otimes V}$, $\underline{\otimes V^*}$ biadjoint functors

matching summands are biadjoint

$\mathcal{E}, \mathcal{F}$ <u>biadjoint</u>

(Bernstein – S. Gelfand 1980 $f \otimes i$)

$\left\{ \begin{array}{l} \text{take } \underline{\text{projectives to projectives}} \\ \text{injectives to injectives.} \end{array} \right\}$

$\mathcal{E}, \mathcal{F}$ <u>exact</u> (

<u>Remark</u>

do hom. alg.

$T : C \to C$

T right exact

$T(M)$ - need proj. res. of M $\quad T_1, T_2$

$$P_\bullet \to M \xrightarrow{T_1} \underline{T_1(P_\bullet)} \longrightarrow \text{resolve } \widetilde{P}(T(P_\bullet))$$
$$\downarrow T_2$$

A - f.d. alg

$$\begin{array}{ccc} A\text{-pmod} & \underset{F}{\overset{\mathcal{E}}{\rightleftarrows}} & B\text{-pmod} \\ \cap & & \cap \\ A\text{-mod} & \rightleftarrows & B\text{-mod} \end{array}$$

"if biadjoint"

$$\begin{array}{ccc} K_0(A) & \rightleftarrows & K_0(B) \\ \downarrow & & \downarrow \\ G_0(A) & \rightleftarrows & G_0(B) \end{array}$$

$$V_{\mathbb{Z}}^{\otimes n} = \bigoplus_{k=0}^{n} V_{\mathbb{Z}}^{\otimes n}(k) \qquad \mathcal{D}_n = \bigoplus_{k=0}^{n} \mathcal{D}_{k,n-k}$$

$$\underline{V_{\mathbb{Z}}^{\otimes n}} \simeq K_0(\mathcal{D}_n). \qquad v_\lambda \leftrightarrow [M_\lambda].$$

$E \uparrow \downarrow F \qquad [\mathcal{E}] \downarrow \downarrow [\mathcal{F}]$

$$V_{\mathbb{Z}}^{\otimes n} \simeq K_0(\mathcal{D}_n)$$

$sl(2)$.

$$H v_\lambda = (2k-n) v_\lambda$$

$H \leftrightarrow$ "$\mathcal{H}$"

functor $\mathrm{Id}^{(2k-n)}$

$$E \rightsquigarrow E^{(m)} = \frac{E^m}{m!}$$

$$v_\lambda \xrightarrow{E^m} \Sigma$$

$(11..00100)$

$$\begin{array}{cccccccc} & & \uparrow & \uparrow & & & \uparrow & \\ (1 & 0 & 0 & 1 & 0 & 0 & 0) \end{array}$$

3rd 1st $\quad$ 2nd

$$\underline{\underline{E^m}}\, v_\lambda = \sum m!\, v_\mu$$

$\underline{\mu - \lambda = \Sigma}$ of m ε_i's in different places

$$\underline{\underline{E^{(m)}}}\quad \underline{v_\lambda = \sum v_\mu}$$
$\mu - \lambda = m$ ones, $n-m$ zeros

$$\mathcal{E}^{(m)}(M) = pr_{k+m}(M \otimes \Lambda^m V) \qquad [M_\lambda] \longrightarrow \sum_{\mu - \lambda} [M_\mu]$$

$M \in \mathcal{D}_{k,n-k}$

$\Lambda^m U$ - weights are m one's $n-m$ zeros in the sequence

$\varepsilon_1 \dots \varepsilon_n$ $\quad \varepsilon_{i_1} + \varepsilon_{i_2} \dots + \varepsilon_{i_m}$

$1 \le i_1 < \dots < i_m \le n$

$$[\mathcal{E}^{(m)} M_\lambda] = \sum_{\mu - \lambda = m \text{ ones}} [M_\mu]$$

$\mathcal{E}^{(m)}$ is $\frac{1}{m!}$ of $\mathcal{E}^m$ $\qquad$ able to divide m-th power of a functor by $m!$

$$\mathcal{F}^{(m)}(N) = pr_k\,(N \otimes \Lambda^m U^*)$$

$$\begin{array}{ccc} \mathcal{O}_{k,n-k} & \overset{\mathcal{E}^{(m)}}{\underset{\mathcal{F}^{(m)}}{\rightleftarrows}} & \mathcal{O}_{k+m,\, n-k-m} \\ \Psi & & \Psi \\ M & & N \end{array}$$

biadj functors $\mathcal{E}^{(m)}, \mathcal{F}^{(m)}$

Thm [BG 1980] Projective functors are isomorphic iff they induce the same map on Groth group.

$$\mathcal{E}^{(m_1)}\mathcal{E}^{(m_2)} = \binom{m_1+m_2}{m_1}\mathcal{E}^{(m_1+m_2)} \qquad E^{(m_1)}E^{(m_2)} = \binom{m_1+m_2}{m_1}E^{(m_1+m_2)}$$

$$\Downarrow$$

$$\bigoplus_{\binom{m_1+m_2}{m_1}} \mathcal{E}^{(m_1+m_2)}$$

$$\mathcal{F}^{(m_1)}\mathcal{F}^{(m_2)} = \binom{m_1+m_2}{m_1}\mathcal{F}^{(m_1+m_2)} \qquad F^{m_1}F^{m_2} = F^{m_1+m_2}$$

$[E,F] = H \qquad \underline{EF} - \underline{FE} = H \qquad H \quad (2k-n)\,Id$

$EF + (n-k)Id = FE + k\,Id \qquad V_{\mathbb{Z}}^{\otimes n}(k)$

$0 \le k \le n$

$$\Rightarrow \underline{\mathcal{E}\mathcal{F} \oplus Id^{n-k}} \cong \underline{\mathcal{E}\mathcal{F} \oplus Id^{k}}$$

isom due to same action on $K_0 \simeq V_{\mathbb{Z}}^{\otimes n}(\mathbb{Z})$.

$$\mathcal{E}^{(m_1)}\mathcal{F}^{(m_2)} \simeq \qquad E^{m_1}F^{m_2} = F^{m_2}E^{m_1} + \dots$$

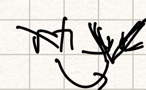
$$E^{(a)}F^{(b)} = \sum_{j=0}^{\min(a,b)} F^{(b-j)} \binom{H-a-b-2j}{j} E^{(a-j)}$$

$$\binom{H}{a} = \frac{H(H-1)\cdots(H-a+1)}{a!}$$

$V_{\mathbb{Z}}^{\otimes n} \otimes_{\mathbb{C}} V^{\otimes n} \qquad \underline{V = \mathbb{C}^2}$

$sl(3) \to$ take blocks $\qquad \underbrace{2,1,0}$ $(2,1,0 \dots 0)$

more complicated pieces of $\mathcal{O}$



Sussan, Stroppel-Mazorchuk

$\nwarrow$ sing. blocks

$D^b\ \mathcal{O}_{k,n-k} \qquad \mathcal{O}_n = \bigoplus \mathcal{O}_{k,n-k}$

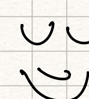
$\wr\!\wr \qquad \updownarrow$ Koszul duality

$D^b\ \mathcal{O}^{k,n-k}$ – parabolic blocks inside an regular block

$\cup \qquad$ ex. $n \to 2n \quad k \to n$

<u>take all self-dual</u> projectives P_a in $\mathcal{O}^{n,n}$ $\qquad P = \bigoplus_a P_a$

$\mathrm{End}(P) \simeq$ arc algebra <u>2n endpoints</u>



$\mathcal{O}^{n,n}$

$[\mathcal{E}]$
$[\mathcal{F}] \circlearrowright K_0(\mathcal{O}_n)$
$\wr\!\wr$

<u>lift</u> $\quad \mathcal{E}, \mathcal{F} \circlearrowright \mathcal{O}_n$

Schur-Weyl duality.

$E, F \circlearrowright V_{\mathbb{Z}}^{\otimes n} \circlearrowleft TL_{-2} \qquad sl(2)$

$sl(2) \circlearrowright V^{\otimes n} \circlearrowleft \mathbb{C}[S_n]$ $\quad$ $sl(2)$ $\quad$ $V = \mathbb{C}^2$

E, F $\quad$ TL_d $\quad$ $V^{\otimes n}$

$\underline{d = -2}$ $\quad$ $V \cong V^*$ $\quad$ $w \otimes v$ / $v \otimes w$

$\bigcirc =$ $\quad$ $\bigcirc = -2$

$d = -q - q^{-1}$

$q = 1$

V $\quad$ $V \otimes V$ $\quad$ $\mathbb{C}$ $\quad$ V

$1 \mapsto v_0 \otimes v_1 - v_1 \otimes v_0 = v_{01} - v_{10}$

$v_{01} \mapsto -1$ $\quad$ $v_{00} \to 0$

$v_{10} \mapsto 1$ $\quad$ $v_{11} \to 0$

$V^{\otimes 3}$ $\quad$ $= |$ $\quad$ $\bigcirc = -\ominus = -2$ $\quad$ $\mathbb{C} \to V \otimes V \to \mathbb{C}$

TL_{-2} acts on $V^{\otimes n}$, preserves $V_{\mathbb{Z}}^{\otimes n}$

u_i $\quad$ $u_i^2 = -2u_i$ $\quad$ $\bigcirc = -2$

1 2 i

$u_i u_{i\pm1} u_i = u_i$

$U_{\mathbb{Z}}(sl_2) \circlearrowright \underline{V_{\mathbb{Z}}^{\otimes n}} \circlearrowleft TL_{\underline{-2}}$

E, F

after cat

$\mathcal{C}^{ab}$

$\mathcal{E}, \mathcal{F} \circlearrowright \mathcal{O}_n \circlearrowleft$?

$\mathcal{E}^{(m)}, \mathcal{F}^{(m)}$

projective functors

Zuckerman functors / Bernstein functors

pass to hom on derived cat

u_i $\quad$ $\mathcal{U}_i$

$\bigcirc = -2$

$\mathfrak{p} \supset \mathfrak{b}^+$ cat of $\mathfrak{p}$ loc. fin $U(\mathfrak{g})$-mod

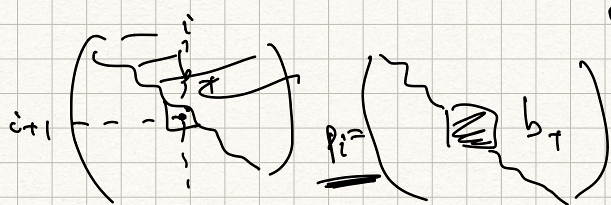
 $\mathfrak{p}$

$\mathcal{O} \supset \mathcal{O}^{\mathfrak{p}}$ - parabolic subcategory. modules M in $\mathcal{O}$ as $U(\mathfrak{p})$-mod loc. fin. dim.

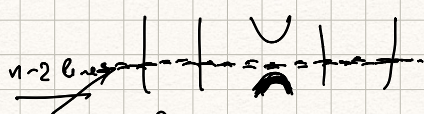
$$\dim(U(\mathfrak{p}) \cdot v) < \infty. \qquad v \in M$$

parabolic V. modules $(M_\lambda \to M_\lambda^{\mathfrak{p}})$.
max. q. $U(\mathfrak{p})$-loc. finite

$\mathfrak{p}_i$



$\Gamma_i(M) =$ max locally $U(\mathfrak{p}_i)$-finite submodule of M. Zuckerman



n-2 lines

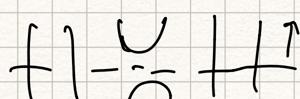
n lines

$u_i \in TL_{-2}$ [BFK] [ES] Enright-Shelton

functor U_i - derived $V \simeq \mathbb{C}^n$

$\mathrm{inc} \circ \Gamma_i$ $\Gamma_i(M) \subset \mathcal{O}_{k,n-k}$ $V^{\otimes n} \to V^{\otimes(n-2)} \to V^{\otimes n}$

$\mathcal{O}_{k,n-k}$ $\mathfrak{p}_i$-loc fin. $\simeq \mathcal{O}_{k-1,n-k-1}$



$\mathcal{O}_{k,n-k} \xleftarrow{i} \mathcal{O}_{k,n-k}$ $\mathfrak{p}_i$ - loc. fin $\leftarrow \mathcal{O}_{k,n-k}$

$\mathfrak{g}^0$

(L_0) - f.d. mod

$\uparrow$

$n=2$ $\mathfrak{sl}(2)$ reg. block 0 0 h.w. M_0

$\cup$

$M_0 = P_0$ P_1 - big proj 0 1 l.w. $M_1 = L_1$

$\mathcal{O}(0) = \mathcal{O}$ 0

$\mathrm{End}_{sl(2)}(P_1) \simeq \underline{\Bbbk[x]}/(x^2) \qquad x = (1\,0\,1)$.

$\mathcal{O}_{reg}(sl_2) \simeq \mathrm{End}_{\Bbbk[x]/(x^2)}\left(\Bbbk \oplus \Bbbk[x]/(x^2)\right)$.

$\circlearrowright_x$

$p = sl(2)$

max quotient $\underline{\underline{sl(2)\text{ loc finite}}}$

right exact

	$\mathcal{O}_{reg} \to$	$sl(2)$-f.d. $\to$	$\mathcal{O}_{reg}$
	$M_0 \to$		$\circlearrowright$
	$L_0 \to$	$\underline{\underline{L_0}}$ $\to$	L_0
	$M_0 \to$	$L_0 \to$	L_0
	$M_1 \to$	$0 \to$	0
dual verma	$M_0^* \to$	0	
$0 \to M_0 \to P_1 \to M_1 \to 0$	$P_1 \to$	0.	

$0 \to \overset{P_0}{\overset{\shortparallel}{M_0}} \to P_1 \to \overset{P_0}{\overset{\shortparallel}{M_0}} \to L_0 \to 0$ $\qquad$ sub m. p Zuckerman

$M_0 \supset M_1$

$0 \to \overset{\text{verma}}{P_0} \to P_1 \to \overset{\text{verma}}{P_0} \to L_0 \to 0$ $\qquad$ q. mod p Bernstein

his proj, apply

$0 \to \underset{-2}{\underline{\underline{L_0}}} \to 0 \to \underset{0}{\underline{\underline{L_0}}} \to 0$

$\underline{\underline{G}} = \cup \cap$ $\qquad \underline{L_0 \oplus L_0[2]}$ $\qquad \bigcirc = \underline{\underline{-2}}$

$\uparrow$

$\underline{L_0}$

$G(L_0) = \underline{L_0} \oplus \underline{L_0}[2]$

$G[-1]$ to correspond to $u_i \in TL_{-2}$

$\mathcal{O}_{k-1, n-1-k}$

ES.

$\mathcal{E}, \mathcal{F} \circlearrowright \mathcal{O}_n \circlearrowleft$ TL acts by functors (derived)

Zuckerman / Bernstein.

$\downarrow \quad \downarrow K_0 \quad \swarrow$

$E, F \circlearrowright V_{\mathbb{Z}}^{\otimes n} \circlearrowleft \underline{TL_{-2}}$ $\quad u_i$

$\underline{U_{\mathbb{Z}}(sl_2)}$

graded case (graded modules, g. functors, g. bimodules)

q - version

$E, F \circlearrowright \overline{V_{\mathbb{Z}[q,q^{-1}]}^{\otimes n}} \circlearrowleft TL_{-q-q^{-1}}$ $\quad q \rightsquigarrow$ grading shift

$U_{\mathbb{Z}[q,q^{-1}]}(sl_2)$

$$\underline{\mathcal{O}_n} \xleftrightarrow{\text{K. dual}} \mathcal{O}^n$$

$\uparrow \quad \uparrow$

$\oplus$ sing. blocks $\quad \oplus$ parabolic blocks

$\mathcal{E}, \mathcal{F} \circlearrowright \mathcal{O}^n \circlearrowleft$ proj. functors $\longleftrightarrow$ perverse sheaf

Zuckerman functors

$D^b(\mathcal{O}_n) \longleftrightarrow \hat{\bigoplus}_{k=0} D_{Sh}(Gr(k,n))$

$L \subset \mathbb{C}^n$

$\{ \dim L = k$

Get to categorified
q-group explicitly.

Get to combinatorics

Homs are $\underline{\underline{\mathcal{E}^n}}, \ldots$ $\qquad \mathcal{E}^{(n)} = \dfrac{``\mathcal{E}^n"}{n!}$

2D diagrammatics of Mon. cat that categorify
q-groups

cross-sect of
a foam in 3D

$\underline{3D \text{ foam}}$

$\underline{Q^2 = Q}$ vs $\underline{Q^2 = n\,Q}$

more options.

$S_n = \{1, \ldots n\}$

$S \subset S_n \quad |S| = k \qquad S \mapsto \sum T$

$T \supset S$

$|T| = |S|+1$

$0.0,\ 1..1,\ 2.2$

$S \mapsto \sum T$

$T \supset S$

$|T| = |S| + m$

$\underline{\underline{sl(3)}}$ $\qquad V^{\otimes n}$

$V \simeq \mathbb{C}^3 \qquad v_0, v_1, v_2$

$Gr(k, n)$

sets $\longleftarrow$ vect. spaces $\qquad L \subset \mathbb{C}^n \qquad \mathbb{F}_q$

$\dim L = k$

$\ldots$ sets $\xleftrightarrow[q=1]{}$ $L \subset \underline{\mathbb{F}_q^n}$

$q=1$

$\mathbb{F}_1$ Connes,
geometry Manin, Kapranov-Smirnov

$\mathcal{E}, \mathcal{F} \circlearrowright \mathcal{O}_n^{gr} \circlearrowleft$ Zuckerman f's in graded case $\qquad q \leftarrow$ grading shift $\{1\}$

$$\mathcal{E}^{(2)} = \text{"}\frac{\mathcal{E}^2}{q+q^{-1}}\text{"}$$

$$\mathcal{E}^{(2)}\{1\} \oplus \mathcal{E}^{(2)}\{-1\} \simeq \mathcal{E}^2 \qquad G = SL(n)$$

$$\mathcal{O}_n^{gr} \simeq D^b_{perv}($$

$$\cup\!\!\shortparallel \qquad \bigoplus \mathcal{O}_{k,n-k}$$

$\dim L = k$

$L \subset \mathbb{C}^n$

$$\mathcal{O}^n_{gr} \simeq D^b_{perv}\Big(\coprod^{\binom{n}{k}} \underset{\mathcal{S}_k}{Gr(k,n)}\Big)$$

$\mathcal{S}_k$ ← schubert str.

$$\cup\!\!\shortparallel \quad \bigoplus_{k=0}^{n} \mathcal{O}^{k,n-k}_{gr}$$

$$\underline{H^{\oplus}(Gr(k,n))} = H^*(\mathbb{C}_{Gr(k,n)})$$

grading $\qquad$ bimodules

$$\mathcal{O}_n = \bigoplus_{k=0}^{n} \mathcal{O}_{k,n-k}$$

↑ max. sing. block

$S_k \times S_{n-k}$

$$\bigoplus_{k_1,k_2} S_{k_1} \times S_{k_2} \times S_{n-k_1-k_2}$$

k_1	k_2	$n-k_1-k_2$
0	1	2

$$\underline{\mathcal{O}^3_n} = \bigoplus_{k_1,k_2} \mathcal{O}_{k_1,k_2,n-k_1-k_2}$$

$sl(3)$

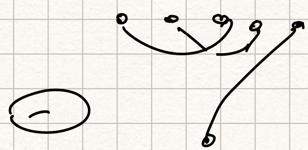
$$\mathcal{E}_i, \mathcal{F}_i \circlearrowright \underline{\mathcal{O}^3_n} \circlearrowleft \mathbb{Z}[S_n]/\underline{J_{3,n}}$$

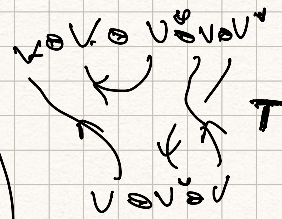
$\text{sing} = \underline{\underline{V^{\otimes n}}} \qquad V \simeq \underline{\underline{\mathbb{C}^3}}$

sing + parabolic cond $\underline{\underline{V^{\otimes}}}$.

next case: Kuperberg's webs

$\underline{V \cong V^*}$ $sl(2)$ $\underline{\underline{sl(3)}}$





$V \otimes \underline{V^*} \otimes V \otimes \underline{V^*}$

$\cap \quad \cap \qquad \cap$

$\underline{V \otimes \underbrace{V \otimes V} \otimes V \quad \otimes \underbrace{V \otimes V}} \longrightarrow \mathcal{O}_n^3$

$V^* \subset V \otimes V$ $sl(3)$

$sl(n)$.

J. Sussan, C. Stroppel

Mazorchuk

$V \cong \mathbb{C}^3$

$\mathcal{U}(sl_3) \curvearrowright V^{\otimes n} \curvearrowleft \mathbb{Z}[S_n] / J_{3,n}$

$\underline{\Lambda^4 V = 0}$

$$\sum_{g \in S_4} \operatorname{sgn}(g)\, g = 0$$

$(n+1)$-dim irr. h.w.

v_0

v_1 $\quad V^{\otimes n} \supset \underline{V_n}$ $\qquad v_0^{\otimes n}$

$\mathcal{O}_n \quad \mathcal{O}^n$

$\frac{1}{n} \sum v_0 \ldots v_1$

$\dfrac{v_0^{\otimes n}}{\downarrow F}$

$F v_0^{\otimes n}$

$\downarrow$

$\vdots$

$v_1^{\otimes n}$

$\frac{1}{k!} \sum_{|\lambda| = k} v_\lambda$

$H^*(Gr(k,n))$-mod

Π

local alg

$\binom{n}{k}$

simple

proj.

$\binom{n}{k}$

$A = \mathbb{k}[x]/(x^n)$-mod

A-mod $\qquad A$-proj.

$$G_0(A) \Longleftarrow K_0(A)$$

$$\underline{\mathbb{Z}[L]} \longleftarrow \mathbb{Z}[A]$$

$$\underline{n[L]} \longleftarrow [A]$$