Cat of reps sl_2 $(\mathcal{U}_q(sl_2))$ $\longleftrightarrow$ Cat $\mathcal{O}$ for sl_n

f.d

add grading

max. sing. blocks

V_1 - fund rep

$$V_1^{\otimes n} \simeq K_0\Big(\bigoplus_{k=0}^{n} \mathcal{O}_{k,n-k}\Big)$$

$E \circlearrowleft F$ on v_1, v_0

$\cong$ Quadratic dual

$$V_1^{\otimes n} \simeq \bigoplus_{k=0}^{n} V_1^{\otimes n}(k) \qquad K_0\Big(\bigoplus_{k=0}^{n} \mathcal{O}^{k,n-k}\Big)$$

weight sp $\underline{\underline{2k-n}}$ max. parab. blocks.

big proj module P_k

$$V_1^{\otimes n}(k) \longleftrightarrow K_0(\mathcal{O}_{k,n-k})$$

$$v_{\underline{\mu}} \longleftrightarrow [M_{\underline{\mu}}]$$

$\mathcal{O}^{k,n-k}$ $\cup$ self-dual part $\mathcal{O}^{n,n}$

$\updownarrow$

proj. mod A^n $\cup$ $\cup$

$V_1^{\otimes n} \supset V_n$ $\mathrm{End}(P_k)$ $\cong$ $H_{k,n} = H^*(Gr(k,n))$

$\underline{v_{n-1}}$

$V_n \simeq S^n V_1$

$\underline{irr}$ $n \geq 0$ $sl(2)$.

$L \subset \mathbb{C}^n$, $\dim L = k$

$\mathrm{Inv}(V^{\otimes 2n}) \cap V^{\otimes 2n}$

$\underline{v}_{|\mu|} = \sum_{\mu} v_{\mu}$

k 0's, $n-k$ 1's, $|\mu| = n-k$

$\underline{v}_n$, $E \circlearrowleft F$, $\underline{v}_{n-1}$, $E \vdots F$, $\underline{v}_0$

basis of V_n

$\underline{\underline{V_n}} \subset V_1^{\otimes n}$ on cat. level.

$\underline{v}_k$ $P_k \in \mathcal{O}_{k,n-k}$

build a cat of V_n via $H_{k,n}$ $\dim H_{k,n} = \binom{n}{k}$

$H_{k,n}$-mod $\supset$ $H_{k,n}$-pmod

$$G_0(H_{k,n}) \longleftarrow K_0(H_{k,n}) \qquad [H_{k,n}] = \binom{n}{k}[L_k]$$

$$\mathbb{Z} \xleftarrow{\dim H_{k,n}} \mathbb{Z}$$

build from unique simple 1d module $L_k \subset \mathbb{C}$

V_n 2 integral lattices

$\underline{V}_n$ $\qquad$ $\underline{V}$

ind. lattice $\underline{V}_{n-1}$ $\qquad$ integral lattice $V'_{n,\mathbb{Z}}$ $\qquad$ $\underline{V}'_k$ $\qquad$ $\underline{V}'_k = \binom{n}{k}^{-1} \underline{V}_k$

$V_{n,\mathbb{Z}}$ $\quad$ $\underline{V}_0$

A

iso if A has fin. hom dims

$$K_0(A) \xrightarrow{\varphi} G_0(A)$$

$$[P] \longmapsto [P]$$

$$\mathbb{Z} \xrightarrow{\dim A} \mathbb{Z}$$

if A local/$\mathbb{k}$. $\operatorname{End}(L) = \mathbb{k}$

Categorify V_n

$$C_n = \hat{\bigoplus_{k=0}} H_{k,n}\text{-mod}$$

lift E, F to functors

$Gr(k,n)$ $\qquad$ L'/L

$$Fl(k,k+1,n) = \{0 \subset L \subset L' \subset \mathbb{C}^n\}$$

$\dim L = k$

$\dim L' = k+1$

$$Fl(k,k+1,n)$$

$$Gr(k,n) \swarrow \quad \searrow Gr(k+1,n)$$

$$H^*(Fl(k,k+1,n)) = H_{k,k+1,n}$$

$H^*(Gr(k,n)) \nearrow$ $\qquad$ $\nwarrow H^*(Gr(k+1,n))$

ring, view as bimodule only

$H_{k,k+1,n}$ - bimodule over $(H_{k,n}, H_{k+1,n})$

free $H_{k,n}$-mod, $H_{k+1,n}$-mod.

M $\qquad\qquad$ $\mathcal{E}(M)$

$$\circlearrowleft H_{k,n}\text{-mod} \underset{\mathcal{F}}{\overset{\mathcal{E}}{\rightleftarrows}} H_{k+1,n}\text{-mod} \circlearrowright$$

N

$$\mathcal{E}(M) = \underbrace{H_{k,k+1,n}} \otimes_{H_{k,n}} M$$

$$\mathcal{F}(N) = \underbrace{H_{k,k+1,n}} \otimes_{H_{k+1,n}} N$$

1991 BLM sl_k partial flags $\mathcal{E}, \mathcal{F}$

$V^{\otimes n} \curvearrowleft sl(n)$ $\qquad$ $\mathcal{U}_q(sl_2)$ / $\mathcal{U}_f(sl_2)$

Prop $\mathcal{E}, \mathcal{F}$ lift E, F in $sl(2)$. They are biadjoint, rich alg. of nat. trans. between arb. products of $\mathcal{E}, \mathcal{F}$.

$\mathcal{E}^m \to \mathcal{E}^m$ $\qquad$ mult. by 1st chern class of can. line bundle $\mathcal{E} \to \mathcal{E}$

$\mathcal{E}^2 \to \mathcal{E}^2$ $\qquad$ push-pull operator on cohomology

$$\begin{tikzcd} & H^*(Z) \arrow[dr] & \\ H^*(X) \arrow[ur] & & H^*(Y) \end{tikzcd}
\qquad
\begin{tikzcd} & Z \arrow[dl, "p'"'] \arrow[dr, "p''"] & \\ X & & Y \end{tikzcd}$$

$H^*(Gr(k,n)) = H_{k,n}$

$$\mathbb{k}[x_1 \ldots x_n]$$
$$\cup$$
$\mathrm{Sym}_{k,n-k}$ $\leftarrow$ symm under $S_k \times S_{n-k}$
$$\cong$$
$\mathrm{Sym}_k \otimes \mathrm{Sym}_{n-k}$ $\qquad$ $\mathrm{Sym}_k = \mathbb{k}[x_1 \ldots x_k]^{S_k}$

$\cup$ $\qquad$ $\uparrow$ free Sym_n-mod rank $\binom{n}{k}$.

Sym_n $\qquad$ $e_1^{(k)} \ldots e_k^{(k)}$ $\quad$ $e_1^{(n-k)} \ldots e_{n-k}^{(n-k)}$

$$R_{k,n-k} = \mathrm{Sym}_k \otimes \mathrm{Sym}_{n-k} \cong H_{GL(n)}^*(Gr(k,n)) \quad \uparrow \tbinom{n}{k}$$
$$\cup$$
$$R_n = \underline{\mathrm{Sym}_n} = H^*_{GL(n)}(\cdot)$$

mod out by augment. ideal in $\mathrm{Sym}_n \supset \underline{I_n}$ - pos. deg. el'ts Sym_n pos. deg $e_1 \ldots e_n = 0$

$R_n/I_n \simeq H^*(\cdot)$

$R_{k,n-k}/R_{k,n-k}I_n \simeq H^*(Gr(k,n))$] f.d.

$\mathcal{E}, \mathcal{F}$ Homs between bimodules & their ∂.

to look for gen, rels that exist

hold there

Aaron Lauda cat of $q.sl(2)$

$\mathcal{U}$ $\mathcal{E}, \mathcal{F}$

$\mathcal{U}^+(sl_2)$ $\mathcal{E}^n$

$\mathcal{U}^+(\mathfrak{g})$

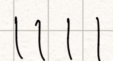
CR, Rouquier

Chuang-Rouquier

simply-laced case.

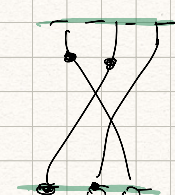
monoidal cat. via gen & relations.

$sl(2)$ $\mathcal{E}^n$

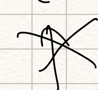


1 2 n 1 id: $\mathcal{E}^n \xrightarrow{id} \mathcal{E}^n$

| $\mathcal{E} \xrightarrow{id} \mathcal{E}$



$\mathcal{E}^n$ $\mathcal{E}^n$



$\mathcal{E}^m$ $\mathcal{E}^n$ $n \neq m$

x  $\mathcal{E} \xrightarrow{x} \mathcal{E}$

1 2 ∂  1 2 $\mathcal{E}^2 \xrightarrow{X} \mathcal{E}^2$

generators

relations

1) everything far away commutes

2) 

$\partial_i^2 = 0$

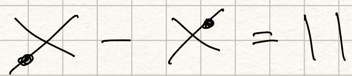
3) 

x_i

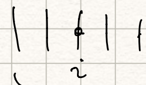
1 2 i

∂_i

1 i $i+1$

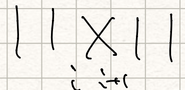


$Pol_n = \Bbbk[x_1, \dots x_n]$



mult by x_i endomorphism of Pol_n

$f \longmapsto x_i f$



∂_i Demazure – BGG divided diffs operator

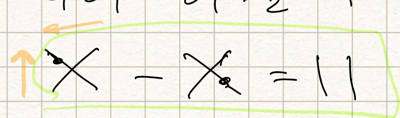
$$\partial_1 f(x_1, x_2) = \frac{f(x_1, x_2) - f(x_2, x_1)}{x_1 - x_2} \in Sym_2$$

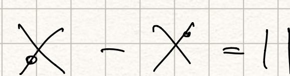
$$\partial_i f = \frac{f - {}^i f}{x_i - x_{i+1}}$$ ← transpose x_i, x_{i+1} in f.

$\partial_1^2 = 0$ $\quad \boxed{\partial_i^2 = 0}$

$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

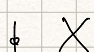
$x_1 \partial_1 - \partial_1 x_2 = 1$ $\qquad \partial_1 x_1 - x_2 \partial_1 = 1$





$x_i \partial_i - \partial_i x_{i+1} = 1$

pictorial



algebraic

x_i, ∂_i

End of Pol_n

as module over Sym_n

$\partial_i(gf) = g\partial_i(f)$

↑ symmetric in x_i, x_{i+1}

geometric

x_i – mult by 1st chern class of line bundle

$Pol_n = H^*_{GL_n}(Fl)$

$Sym_n = H^*_{GL_n}(\cdot)$

Fl

∂_i $\quad \mathbb{P}^1 \downarrow$ forget i-th subspace

Fl_i'

Def NH_n (nilHecke alg) alg gen by x_i, ∂_i acting on Pol_n.

Prop 1) NH_n has above defining relations.

2) $NH_n \longrightarrow \mathrm{End}_{Sym_n}(Pol_n)$ is an isom.

3) $\underline{NH_n} \simeq \underline{Mat(n!, Sym_n)}$. " $\frac{1}{n!}$ of 1 " idemp

$Pol_n \simeq Sym_n^{n!}$ works in graded case $n! \; [n]!$

Can think of $\bigcup_{n \geq 0} NH_n$ as describing a monoidal cat.

w/ a generating object $\mathcal{E}$. $\mathcal{E}^n$

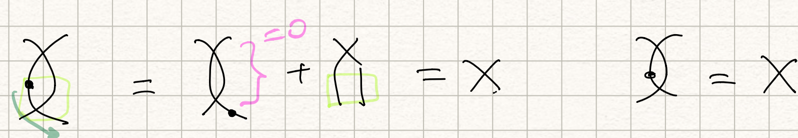
$$Hom(\mathcal{E}^n, \mathcal{E}^m) = \begin{cases} NH_n & n = m \\ 0 & \text{otherwise.} \end{cases}$$

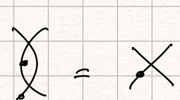
$n=0$ $NH_0 = \Bbbk$

$n=1$  $NH_1 = \Bbbk[x_1]$.

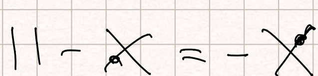
$n=2$  $NH_2 \simeq \underline{Mat(2, Sym_2)}$

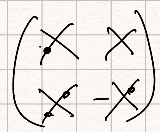
x_1, x_2, $\partial_1 = \partial$.





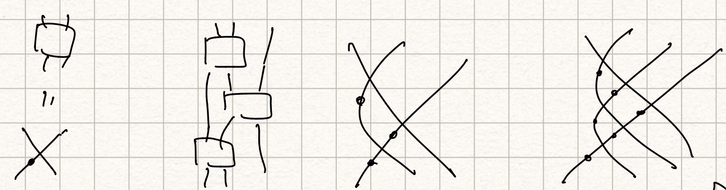
$e_1 =$  $e_1^2 = e_1$

$1 - e_1 =$ 

    $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$ $e_{ij}e_{jk} = e_{ik}$

check signs

$Sym_2 = \Bbbk[x_1+x_2,\ x_1x_2]$ 

$e_1 =$ 

min. idempotent

deg $x_i = 2$, deg $\partial_i = -2$

diff in deg is 2

2 $q+q^{-1}$ $Pol_2 \cong Sym_2 \cdot 1 \oplus Sym_2 \cdot x_1$, -1 1

$\mathcal{E}^2$ End$(\mathcal{E}^m)$ add idemp. to our category

(idempotent completion)

Karoubi envelope

$\mathcal{C}$ $1, \mathcal{E}, \mathcal{E}^2, \mathcal{E}^3 \ldots$ $\longrightarrow$ Kar$(\mathcal{C})$

$\mathcal{E}^{(n)} = (\mathcal{E}^n, e_{(n)})$ "$\frac{1}{n!}\mathcal{E}^n$" $\mathcal{E}^{(n)}$

min. idemp.

$\mathcal{E}^n = \bigoplus_{n!} \mathcal{E}^{(n)}$ $\mathcal{E}^{(n)} = \frac{1}{n!}\mathcal{E}^n$ $E^{(n)} = \frac{1}{n!}E^n$

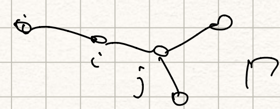
$\mathcal{E}^n = \bigoplus(\mathcal{E}^{(n)})[n]!$

$\mathcal{E}^2 = \mathcal{E}^{(2)}\{1\} \oplus \mathcal{E}^{(2)}\{-1\}$.

Categorify divided powers of an operator.

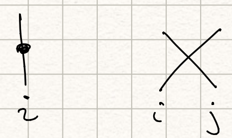
$E, X \longrightarrow \mathcal{E}$.

$\mathcal{U}^+(\mathfrak{g})$ simply-laced   Serre relations of $\mathcal{U}^+$



generalize NH_n

algebras   strands colored by vertices (simple roots)

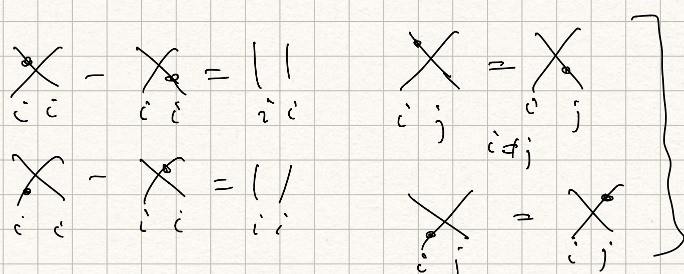


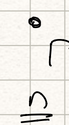
$R(\nu)$ $\nu = \sum_i \nu_i \cdot i$

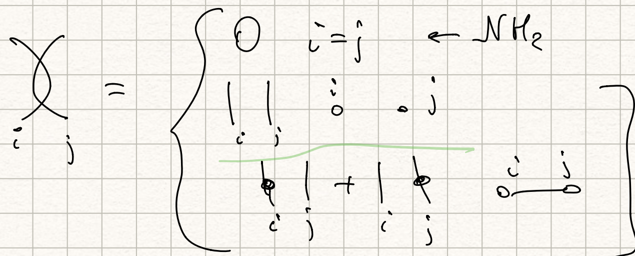
↑ # of i-strands.

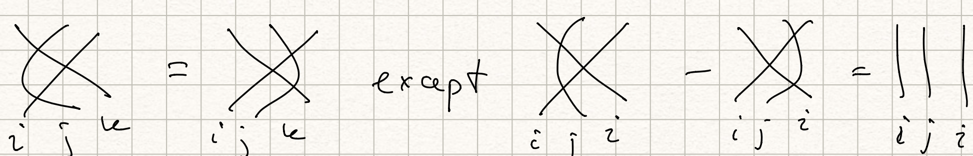
Each i-rel's in NH_n

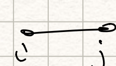
NH_n



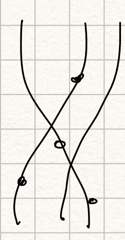








$R(\nu)$



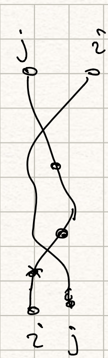
$\mathcal{E}^n \to \mathcal{E}^n$ $NH_n = \mathrm{End}(\mathcal{E}^n)$

$\mathcal{E}_i$ $i \in \Gamma$

$\mathrm{Hom}(\mathcal{E}_{i_1} \ldots \mathcal{E}_{i_n}, \mathcal{E}_{j_1} \ldots \mathcal{E}_{j_n})$

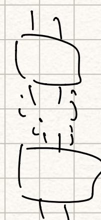
pictures / mod. rel's   prescribed colors on boundary

$\mathrm{Hom}(\mathcal{E}_i\mathcal{E}_j, \mathcal{E}_j\mathcal{E}_i)$



mod rel's.

get a basis of $R(\nu)$



$R(\nu)$ has idempotents

$\underline{i} \in \mathrm{Seq}(\nu)$

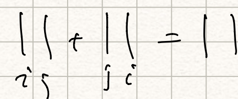
i appears ν_i times.

$\mathrm{Seq}(i+j) = \{ij, ji\}$

$1_{\underline{i}}$ - idemp for seq $\underline{i}$.

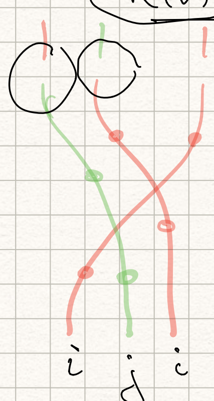
$1 = \sum_{\underline{i} \in \mathrm{Seq}(\nu)} 1_{\underline{i}}$

$1_{ij} + 1_{ji}$ $\qquad \nu = i+j$



$R(\nu)$ categorifies ν-weight space of $\mathcal{U}^+(\mathfrak{g})$

$\cup$



1_{iji}

$\mathcal{E}_j\mathcal{E}_i\mathcal{E}_i$

$= \alpha$

$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i$

$E_{\underline{i_1}} E_{\underline{i_2}} \ldots E_{\underline{i_n}}$

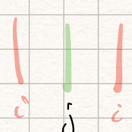
$E_i E_j, E_j E_i$

$\mathcal{E}^2 \rightsquigarrow \mathcal{E}^{(2)}$

$e_1 \in \mathrm{End}(\mathcal{E}^2)$

$\alpha 1_{iji} = \alpha$

$1_{iji}\alpha = 0$ $\qquad 1_{jii}\alpha = \alpha$



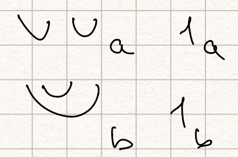
1_{iji}

$\mathcal{E}_1, \mathcal{E}_2$

$\mathcal{E}_\nu = \oplus \mathcal{E}_{\underline{i}}$ $\qquad \mathcal{E}_{\underline{i}} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\ldots\mathcal{E}_{i_n}$

$\mathcal{E}_{\text{"}2i+j\text{"}} = \mathcal{E}_i\mathcal{E}_i\mathcal{E}_j \oplus \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \oplus \mathcal{E}_j\mathcal{E}_i\mathcal{E}_i$ $\qquad R(\nu) = \mathrm{End}(\mathcal{E}_{\text{"}2i+j\text{"}})$

$1_{\underline{i}}$ $\quad R(\nu) = \bigoplus_{\underline{i},\underline{j}\in Seq(\nu)} 1_{\underline{j}} R(\nu) 1_{\underline{i}}$

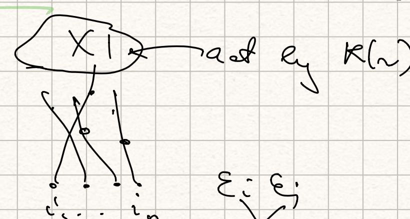


$, E_iF_j = F_jE_i$ $\quad \overset{i}{\circ} \quad \overset{j}{\circ}$

$1 = \sum_{a\in B^n} 1_a$

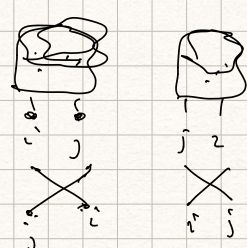
$R(\nu)$ $\quad 1_{\underline{i}}$ $\quad R(\nu)1_{\underline{i}} = P_{\underline{i}}$ - left proj $R(\nu)$-module

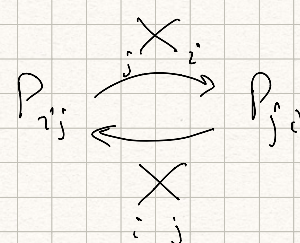
$R(\nu) = \bigoplus_{\underline{i}\in Seq(\nu)} P_{\underline{i}}$

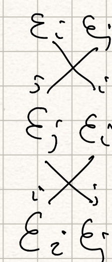


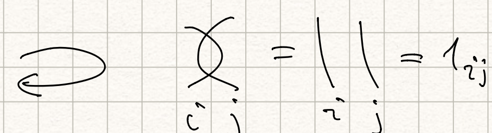
$\mathcal{E}_i\mathcal{E}_j \simeq \mathcal{E}_j\mathcal{E}_i$ $\quad i\circ \quad \circ j$

$P_{ij} \simeq P_{ji}$

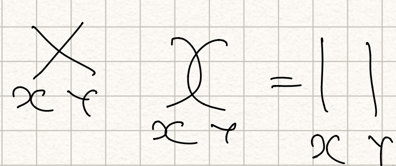


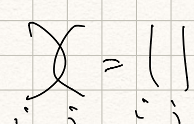


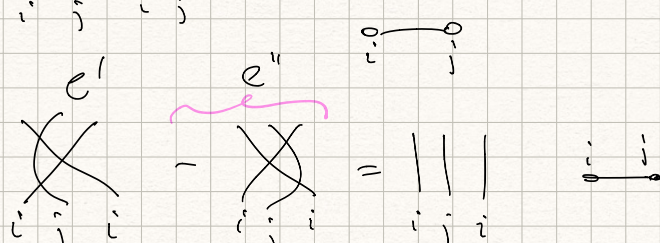




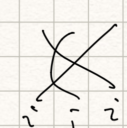
$XY = YX$ $\quad x, Y$



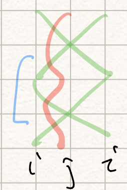
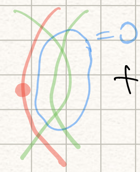




$e' =$



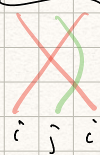
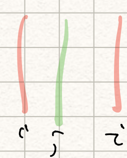
$(e')^2$

$(e')^2 =$  $=$  $+$  $=$  $= $ 

 $= e'$ idempotent $\quad e'' = -$ 

$e' + e'' = 1_{iji} \qquad e'e'' = 0 = e''e'$

$e' \qquad e''$

 $-$  $=$ 

$P_{iji} \cong R(2i+j)e' \oplus R(2i+j)e''$

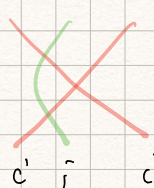
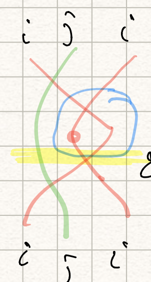
$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i$

$E_iE_jE_i$

$2E_iE_jE_i = E_i^2E_j + E_jE_i^2$

$[2] = (q+q^{-1})$

$\boxed{E_iE_jE_i = E_i^{(2)}E_j + E_jE_i^{(2)}}$

$e' =$  $=$ 

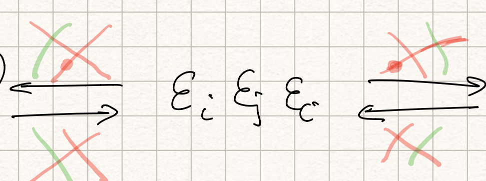
idemp for $\mathcal{E}_i^{(2)}$

$iji \qquad jii \qquad jii$

$\mathcal{E}_j\mathcal{E}_i^{(2)}$

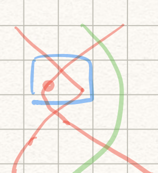
$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \cong \mathcal{E}_i^{(2)}\mathcal{E}_j \oplus \mathcal{E}_j\mathcal{E}_i^{(2)}$

$1_{iji} = e'' + e'$



$\mathcal{E}_j\mathcal{E}_i^{(2)} \rightleftarrows \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \rightleftarrows \mathcal{E}_i^{(2)}\mathcal{E}_j$

$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \cong \mathcal{E}_j\mathcal{E}_i^{(2)} \oplus \mathcal{E}_i^{(2)}\mathcal{E}_j$

$-$  $= -$ 

Thm $\bigoplus_{\nu} K_0(R(\nu)-\mathrm{pmod}) \simeq \mathcal{U}^{+}_{\mathbb{Z}}(\mathfrak{g})$

$$[P_{\underline{i}}] \longrightarrow E_{\underline{i}} = E_{i_1} \cdots E_{i_n}$$

$$[P_{i^n, e_n}] \longleftrightarrow E_i^{(n)}$$

$$R(\nu) \otimes R(\nu') \hookrightarrow R(\nu + \nu')$$

Ind, Res $\longrightarrow$ Groth. group

give mult, comult on $\mathcal{U}^{+}_{\mathbb{Z}}(\mathfrak{g})$

invert powers of q into mult. rules for $\mathcal{U}^{+}_{\mathbb{Z}}(\mathfrak{g})$.

don't have cartan

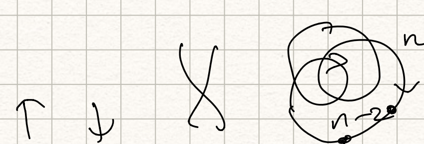
$$(x \otimes x')(y \otimes y') = q^{d(x', y)}\, xy \otimes x'y'$$

$(\mathcal{U}^{+})^{\otimes 2} \xleftarrow{\Delta} E_i$

weight spaces

$E_i, \ F_i$

$\mathcal{E}_i, \ \mathcal{F}_i$ – biadjoint



Brundan–Kleshchev

Varagnolo–Vasserot.

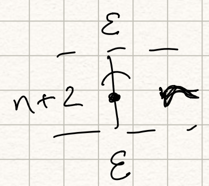
Lauda, Introducing $\mathfrak{g}$ & cat. quantum $sl(2)$

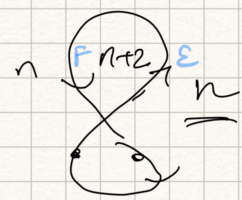
$NH_n \circlearrowright Pol_n$

x_i, ∂_i

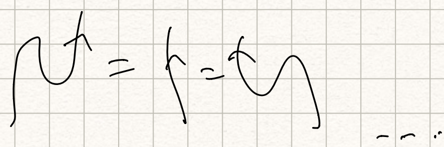
$$R(\nu) \circlearrowright \left(\bigoplus_{\underline{i} \in Seq(\nu)} Pol_{\underline{i}} \right)$$

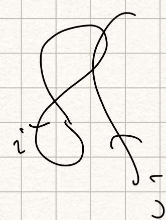
K_i $\quad q^{\cdot\cdot} \to$ grading shift





$$V(n) \underset{F}{\overset{E}{\rightleftarrows}} V(n+2)$$





isotop

$\Updownarrow$

biadjointness

$\mathcal{E}_i, \mathcal{F}_i$

$\binom{K_i}{n}$ $\quad \frac{K_i(K_i-1)}{2}$

$\sim \underline{K_i} \sim \underline{H_i}$

Categorify $\quad \frac{K(K-1)}{2} \quad \frac{K^2}{2} \quad \underline{NH_2}$