Category $\mathcal{O}$: Verma's Thesis

Mrudul Thatte
August 28, 2020

1 Contravariant Forms

Recall that there is an anti-automorphism τ of $U(\mathfrak{g})$ given by $x_\alpha \mapsto y_\alpha, y_\alpha \mapsto x_\alpha, h_\alpha \mapsto h_\alpha$ for all simple roots α. Using this transpose map we define:

Definition 1.1. A symmetric bilinear form $(v, v')_M$ on a $U(\mathfrak{g})$-module M is called **contravariant** if

$$(u \cdot v, v')_M = (v, \tau(u) \cdot v') \; \forall u \in U(\mathfrak{g}), v, v' \in M.$$

1.1 Basic Properties

Proposition 1.2. *Suppose that $U(g)$-modules M, M_1, M_2 have contravariant forms. Then*

(a) Distinct weight spaces M_λ and M_μ of M are orthogonal.

(b) If $M = U(\mathfrak{g}) \cdot v^+$ is a highest weight module generated by a maximal vector v^+ of weight λ, then a nonzero contravariant form on M is uniquely determined up to a scalar multiple by the nonzero value $(v^+, v^+)_M$. The radical of this form is the unique maximal submodule N of M.

(c) The tensor product $M_1 \otimes M_2$ also has a contravariant form, given by $(v \otimes w, v' \otimes w') := (v, v')_{M_1}(w, w')_{M_2}$. If the forms on M_i are nondegenerate, so is the product form.

(d) For any submodule $N \subset M$, its orthogonal space $N^\perp := \{v \in M \mid (v, v_0)_M = 0 \forall v_0 \in N\}$ is also a submodule.

(e) If $M \in \mathcal{O}$, then the summands M^χ for distinct central characters χ are orthogonal.

We will prove part (b). Assuming (a), it's enough to look at the form on a weight space M_μ. Vectors $v, v' \in M_\mu$ can be written as $v = u \cdot v^+$ and $v' = u' \cdot v^+$ for some $u, u' \in U(\mathfrak{n}^-)$. Then

$$(v, v')_M = (u \cdot v^+, v')_M = (v^+, \tau(u)v')_M$$

Since u maps M_λ into M_μ, it's transpose $\tau(u)$ takes M_μ to M_λ, which is a one-dimensional space spanned by v^+. So, $\tau(u) \cdot v'$ is a scalar multiple of v^+ and $(v, v')_M$ is a scalar multiple of $(v^+, v^+)_M$ determined by the action of $U(\mathfrak{n}^-)$ on M.

Since N is a weight module which does not have λ as one its weights, $(v^+, N)_M = 0$. Then for any $u \in U(\mathfrak{g})$, we have $(u \cdot v^+, N)_M = (v^+, \tau(u)N) = 0$. This means that N is contained in the radical of the form. On the other hand, the radical of a nonzero form is a proper submodule of M and must be contained in N. $\square$

1.2 Example - Verma modules of $\mathfrak{sl}_2(\mathbb{C})$

Recall that $M(\lambda)$ has weights $\lambda, \lambda-2, \lambda-4, \dots$ and we may choose a basis of corresponding weight vectors $v_0, v_1, v_2, \dots$ such that

$$x \cdot v_i = (\lambda - i + 1)v_{i-1}$$
$$y \cdot v_i = (i+1)v_{i+1}$$

For a contravariant form on $M(\lambda)$, we must have $(v_i, v_j) = 0$ for $i \neq j$. Also, for every $i > 0$, we get

$$(v_i, v_i) = \left(\frac{1}{i} y \cdot v_{i-1}, v_i\right) = \frac{1}{i}(v_{i-1}, \tau(y) \cdot v_i) = \frac{1}{i}(v_{i-1}, x \cdot v_i) = \frac{\lambda - i + 1}{i}(v_{i-1}, v_{i-1})$$

Induction shows that

$$(v_i, v_i) = \frac{(\lambda - i + 1)(\lambda - i + 2)\dots(\lambda)}{i!}(v_0, v_0)$$

(It is easy to verify that a form defined using this formula is, in fact, a contravariant form.)

Note that since distinct v_i are orthogonal to each other, the form is non-degenerate iff (v_i, v_i) is nonzero for all $i \geq 0$ iff $\lambda \notin \mathbb{Z}^{>0}$ iff $M(\lambda)$ is simple. On the other hand, if $\lambda \in Z^{>0}$, then $(v_i, v_i) = 0$ for $i \geq \lambda + 1$ (vectors of weights $\leq -\lambda - 2$) and the radical of the form is $M(-\lambda - 2)$.

1.3 Universal Construction

Our goal is to construct contravariant forms on highest weight modules. We start by constructing a form on $U(\mathfrak{g})$. Let $\varepsilon^+ : U(\mathfrak{n}) \to \mathbb{C}$ and $\varepsilon^- : U(\mathfrak{n}^-) \to \mathbb{C}$ be the maps sending all nonconstant PBW basis elements to 0. Use the PBW theorem to define the linear map $\varphi := \varepsilon^- \otimes id \otimes \varepsilon^+ : U(\mathfrak{g}) \equiv U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}) \to U(\mathfrak{h})$. This gives us a symmetric bilinear form on $U(\mathfrak{g})$

$$C(u, u') := \varphi(\tau(u)u').$$

Since τ is an anti-automorphism, we have

$$C(u_0 u, u') = C(u, \tau(u_0)u')$$

for all $u_0, u, u' \in U(\mathfrak{g})$.

For a weight λ, let $\varphi_\lambda = \lambda \circ \varphi$ and define a form on $U(\mathfrak{g})$ by

$$C^\lambda(u, u') := \varphi_\lambda(\tau(u)u').$$

Now consider a highest weight module M generated by maximal vector v^+ of weight λ. Suppose that $u_1, u_2 \in U(\mathfrak{g})$ satisfy $u_1 \cdot v^+ = u_2 \cdot v^+$. By writing u_i in the PBW basis and comparing the the components of $u_i \cdot v^+$ of weight λ, we obtain $\varphi_\lambda(u_1) = \varphi_\lambda(u_2)$. For any $u \in U(\mathfrak{g})$, we get $uu_1 \cdot v^+ = uu_2 \cdot v^+$ and therefore $\varphi_\lambda(uu_1) = \varphi_\lambda(uu_2)$. Thus, $\varphi_\lambda(U(\mathfrak{g})(u_1 - u_2)$ is zero and $(u_1 - u_2)$ lies in the radical of C^λ. This allows us to define a form on M by

$$(v, v')_M := C^\lambda(u, u')$$

where $v = u \cdot v^+$ and $v' = u' \cdot v^+$ for $u, u' \in U(\mathfrak{n}^-)$. It is easy to check that this is a nonzero contravariant form. Thus we have

Theorem 1.3. *If M is a highest weight module of weight λ, there exists a (nonzero) contravariant form $(v, v')_M$ on M. The form is unique (up to scalar multiples) and completely determined by $(v^+, v^+)_M$. Its radical is the unique maximal submodule of M. In particular, the form is nondegenerate if and only if M is the simple module $L(\lambda)$.* □

2 Simple Submodules of Verma Modules

Proposition 2.1. *$M(\lambda)$ has a unique simple submodule.*

Proof. Recall that as $U(\mathfrak{n}^-)$-modules, $M(\lambda)$ and $U(\mathfrak{n}^-)$ are isomorphic. Under such an isomorphism, we may identify nonzero submodules of $M(\lambda)$ with nonzero left ideals of $U(\mathfrak{n}^-)$. Since $U(\mathfrak{n}^-)$ is left noetherian and does not have any zero divisors, any two nonzero left ideals of intersect non-trivially. Thus, any two nonzero submodules of $M(\lambda)$ must intersect non-trivially. This is impossible for distinct simple submodules. □

Example 2.2. In case of $\mathfrak{sl}_2(\mathbb{C})$, if $\lambda \in \mathbb{Z}^{>0}$, then the unique maximal submodule $M(-\lambda - 2) \subset M(\lambda)$ is simple. Otherwise, $M(\lambda)$ itself is simple.

3 Homomorphisms between Verma Modules

Theorem 3.1. *Let $\lambda, \mu \in \mathfrak{h}^*$. Then*

(a) Any nonzero homomorphism $\varphi : M(\mu) \to M(\lambda)$ is injective.

(b) In all cases, $\dim \operatorname{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)) \leq 1$.

(c) The unique simple submodule of $M(\lambda)$ is a Verma module.

Proof. (a) Let v_μ^+ and v_λ^+ be maximal vectors in $M(\mu)$ and $M(\lambda)$, respectively. Let $u \in U(\mathfrak{n}^-)$ be such that $\varphi(v_\mu^+) = u \cdot v_\lambda^+$. As left $U(\mathfrak{n}^-)$-modules, $M(\mu) = U(\mathfrak{n}^-)v_\mu^+ \equiv U(\mathfrak{n}^-) \equiv U(\mathfrak{n}^-)v_\lambda^+ = M(\lambda)$ so that φ corresponds to the map on $U(\mathfrak{n}^-)$ given by $u' \mapsto u'u$. Since $U(\mathfrak{n}^-)$ does not have zero divisors, φ must be injective.

(b) Note that any nonzero homomorphism $M(\mu) \to M(\lambda)$ must descend to an isomorphism between the unique simple submodules of $M(\mu)$ and $M(\lambda)$. Thus, if φ_1, φ_2 are two such homomorphisms, there exists a scalar $c \in \mathbb{C}$ such that $\varphi_1 - c\varphi_2$ kills L. By part (a), we conclude $\varphi_1 - c\varphi_2 = 0$.

(c) Suppose that $L(\mu)$ is the unique simple submodule of $M(\lambda)$. Then the composition $M(\mu) \twoheadrightarrow L(\mu) \hookrightarrow M(\lambda)$ gives a nonzero homomorphism between Verma modules. By part (a), this is injective and $M(\mu) = L(\mu)$.

□

Remark 3.2. Whenever there is a nonzero homomorphism $M(\mu) \to M(\lambda)$, we may write $M(\mu) \subset M(\lambda)$.

4 Simplicity Criterion and Embeddings

Theorem 4.1. *Let $\lambda \in \mathfrak{h}^*$.. Then $M(\lambda) = L(\lambda)$ if and only if λ is ρ-antidominant, i.e., $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ for all positive roots α.*

Proof. We begin with integral weights.

Part (1) Suppose that $M(\lambda)$ is simple. Since λ is integral, it is ρ-antidominant iff $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ for all *simple* roots α. If this fails for some simple root, then we have $s_\alpha \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha < \lambda$. This means that there is a nonzero homomorphism $M(s_\alpha \cdot \lambda) \to M(\lambda)$. However, such a morphism must be an embedding, which is impossible if $M(\lambda)$ is simple.

Part (2) We know that the highest weights of composition factors of $M(\lambda)$ must be of the form $w \cdot \lambda \leq \lambda$ with $w \in W$. If λ is ρ-antidominant, then the only weight satisfying this constraint is λ and only $L(\lambda)$ can occur as a composition factor. Since $\dim M_\lambda = 1$, we see that it occurs only once and therefore $M(\lambda) = L(\lambda)$.

To extend the first part of the proof to the general case, we need embeddings of the form $M(s_\alpha \cdot \lambda) \to M(\lambda)$ for arbitrary positive roots. It turns out that such embeddings exist as long as $s_\alpha \cdot \lambda \leq \mu$:

Theorem 4.2. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha > 0$. If $\mu = s_\alpha \cdot \lambda \leq \lambda$, then there exists an embedding $M(\mu) \subset M(\lambda)$.*

The second part of the proof can be generalized by replacing W by the reflection subgroup $W_{[\lambda]}$. □

Example 4.3. For $\mathfrak{sl}_2(\mathbb{C})$, there is a unique positive root 2 and $\rho = 1$. So, λ is ρ-antidominant iff $\langle \lambda + 1, 1 \rangle = \lambda + 1 \notin \mathbb{Z}^{>0}$ iff $\lambda \notin \mathbb{Z}^{\geq 0}$. We already know that these are precisely the weights for which $M(\lambda)$ is simple.

Corollary 4.4. *If λ is ρ-antidominant, then $L(\lambda)$ is the unique simple submodule and therefore a composition factor of $M(w \cdot \lambda)$ for all $w \in W_{[\lambda]}$.*

Proof. The unique simple submodule is a Verma module whose highest weight is in the orbit $W_{[\lambda]} \cdot \lambda$. We know that λ is the only ρ-antidominant weight in this orbit. □

5 Block Decomposition of Category O

Theorem 5.1. *For a ρ-antidominant λ, let $\mathcal{O}_\lambda$ be the subcategory of modules whose composition factors all have highest weights linked to λ by $W_{[\lambda]}$. Such $\mathcal{O}_\lambda$ are precisely the blocks of $\mathcal{O}$.*

Proof. Consider a Verma module $M(\mu)$ and let $L(\lambda) = M(\lambda)$ be its unique simple submodule. By the simplicity criterion, λ is ρ-antidominant. Then the composition factors of $M(\mu)$, including $L(\lambda)$ and $L(\mu)$ lie in the same block. The highest weights of these factors all must be in the orbit $W_{[\lambda]} \cdot \lambda$. On the other hand, we have already shown that any Verma module with highest weight in the orbit $W_{[\lambda]} \cdot \lambda$ has $L(\lambda)$ as its unique submodule. □

6 Error in Verma's Thesis

Warning: Section contains false results.

Verma believed that he had proved the following Lemma:

Lemma 6.1. *Let M be the submodule generated by a weight vector v_μ of $M(\lambda)$. Then the submodule M' of M generated by vectors $x_\alpha \cdot v_\mu$ is either 0 or M.*

However, there is gap in the proof of this Lemma and it leads to some interesting results -

Theorem 6.2. *Every submodule M of $M(\lambda)$ are generated by the maximal vectors in M.*

Proof. Since M is a weight module, it's enough to prove the theorem when M is generated by a single weight vector v_μ. If $\mu = \lambda$, then $M = M(\lambda)$ and we are done. Assume that the result holds for all weights $\mu' > \mu$. Then the module M' in the above Lemma is generated by maximal weights. Either $M = M'$ or $M' = 0$, which means that v_μ is itself a maximal vector. $\square$

Consider a composition factor M_i/M_{i-1} of $M(\lambda)$. By the above Theorem, we may assume that M_i is generated by M_{i-1} along with one maximal vector of weight μ. This weight vector generates a copy of $M(\mu)$ in $M(\lambda)$ and we have $M_i/M_{i-1} = L(\mu)$. Thus, every composition factor of $M(\lambda)$ comes from an embedding of a Verma module. This means that every composition factor appears with multiplicity 1.