

Things I should know about Category \mathcal{O} , but I don't

Summer/Fall 2020

1 Part I: Fundamentals

All sections, theorems, propositions in this section refer to [1].

(1) Category \mathcal{O} : Properties

- (a) Introduce category \mathcal{O} and basic properties such as the theorem on page 14 and prove part (d). Introduce highest weight modules and show they are in category \mathcal{O} , are indecomposable and prove the corollary on page 17. [1.1-1.3]
- (b) Introduce central character and the dot action. Define linkage classes, regular and singular weights and the Harish-Chandra homomorphism. Explain the theorem on page 26. (Prove) Category \mathcal{O} is Artinian and thus finite length. [1.7-1.1.11]
- (c) Introduce \mathcal{O}_χ and show \mathcal{O} decomposes into a direct sum

$$\mathcal{O} = \bigoplus_{\lambda} \mathcal{O}_{\chi_\lambda} = \bigoplus_{\lambda \in \mathfrak{h}^*/(W \cdot)} \mathcal{O}_{\chi_\lambda}$$

Define blocks of abelian categories and show that each block of category \mathcal{O} is contained inside a \mathcal{O}_χ . Show $\mathcal{O}_{\chi_\lambda}$ is a block of category \mathcal{O} for λ an integral weight. [1.12-1.13]

(2) Category \mathcal{O} : Methods

- (a) Prove the proposition on page 48 about Ext groups. Introduce the duality functor and show it is exact and induces a self-equivalence on category \mathcal{O} . Then compute some examples. [3.1-3.3]
- (b) Sketch a proof the theorem on page 56 and introduce standard filtrations. Prove that the number of times $M(\lambda)$ shows up in the standard filtration of a module M is equal to $\dim \text{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee)$. [3.6-3.7]
- (c) Define ρ -dominant and ρ -antidominant weights. Show category \mathcal{O} has enough projectives (and thus enough injectives by duality) [3.5, 3.8]

(3) Projectives in Category \mathcal{O}

- (a) Introduce projective covers and prove part (a) of the theorem on page 62. Show all projective modules in category \mathcal{O} has a standard filtration and then prove BGG Reciprocity. [3.9-3.11]
- (b) Introduce contravariant forms and the universal construction. Use this to give a construction of the Shapovalov form on any highest weight module and prove the theorem on page 71. [3.14-3.15]
- (c) Show that $\mathcal{O}_{\chi_\lambda}$ is equivalent to the category of representations over a finite-dimensional algebra (Projective Generator Trick). [3.13]

(d) (Show that for \mathfrak{sl}_2 every block only has 5 indecomposable modules.) [3.12]

(4) **Verma's Thesis**

(a) Prove the theorem on page 75 about morphisms between Verma modules and prove the Simplicity Criterion for Verma modules in the integral case. [4.1-4.2, 4.4]

(b) State Embedding theorems for Verma modules and state enough facts so that you can prove the theorem on page 83 giving the block decomposition of $\mathcal{O} = \bigoplus_{\lambda \text{ } \rho\text{-antidominant}} \mathcal{O}_\lambda$. [4.5-4.9]

(c) Prove that the standard filtration of the projective cover of $L(\lambda)$ for λ integral and ρ -antidominant has a very special form. [4.10]

(d) (Explain how Verma's crucial mistake in his thesis implies $[M(\lambda) : L(\mu)] = 1$) [4.14]

(5) **Jantzen's Thesis**

(a) Define what it means for two weights to be strongly linked and state the BGG theorem on which simples can appear in $M(\lambda)$. Define Bruhat Ordering and then rephrase the BGG theorem for λ antidominant and regular in terms of the Bruhat order. [5.1-5.2]

(b) State what the Jantzen filtration of a Verma module $M(\lambda)$ is and give an example in \mathfrak{sl}_3 . Use the Jantzen filtration to prove the BGG theorem. [5.3-5.5]

(c) (Sketch a proof of Jantzen's Theorem) [5.7]

(d) (Sketch a proof of Shapovalov's Determinantal Formula for the Shapovalov matrix or just do some examples for fun) [5.8-5.9]

(6) **BGG Resolution and Extensions**

(a) Define what it means for a module to have BGG resolution and state the Weak BGG Resolution for $L(\lambda)$ where λ is integral dominant. Prove the theorem on extensions between Verma modules on page 113 and show Weak BGG \implies BGG [6.1-6.2, 6.5]

(b) Sketch proofs of the Ext criterion for having a standard filtration. [6.10-6.13]

(c) (Prove Bott's theorem as a corollary of the BGG resolution, giving an explicit description of the \mathfrak{g} -module structure of certain lie algebra cohomology groups. Use this to prove Borel-Weil-Bott and the Weyl Character Formula.) [6.6]

(d) (Prove that the character of any module in \mathcal{O} can be expressed as the Euler characteristic of certain Ext groups.) [6.14]

(7) **Translation Functors**

(a) Define what it means for two weights to be compatible and define translation functors $T_\lambda^\mu : \mathcal{O}_\lambda \rightarrow \mathcal{O}_\mu$. Prove that T_λ^μ is exact and sends projectives to projectives and that T_λ^μ is left and right adjoint to T_μ^λ . [7.1-7.2]

(b) Assume λ, μ are integral weights from now on so $\lambda^\natural = \lambda$ and $W_{[\lambda]} = W$. Define facets of a root system and give examples and state the Key lemma on page 135. Use the Key lemma to show that under suitable conditions, translation functors send Verma modules to Verma modules. [7.3, 7.5-7.6]

- (c) Show that under suitable conditions, translation functors send simples to simples or 0 and give an example in \mathfrak{sl}_2 . Show that under suitable conditions T_λ^μ induces an isomorphism between Grothendieck groups $K_0(\mathcal{O}_\lambda) \rightarrow K_0(\mathcal{O}_\mu)$ and in fact gives equivalences of categories $\mathcal{O}_\lambda \rightarrow \mathcal{O}_\mu$. Conclude with the application to character formulas. [7.7-7.9]

2 Part II: Advanced Topics

You're on your own here. Good luck!

(8) Kazhdan Lusztig Conjectures

- (a) Expand $C_s^2, C_s C_t C_s$ in the Kazhdan Lusztig basis in the Hecke algebra.

(9) Parabolic Versions of Category \mathcal{O}

(10) Projective Functors and Harish-Chandra Modules

(11) Tilting Modules

- (a) For algebraic groups in any characteristic, knowing the character formula for tiltings is equivalent to knowing the character formula for irreducibles.

(12) Categorification in Category \mathcal{O}

- (a) Essentially blocks of regular Category \mathcal{O} /parabolic versions of \mathcal{O} can be used to categorify representations of \mathfrak{g} and also the Weyl group W . Zuckerman and projective functors categorify the action.
- (b) Can be used to construct knot, link, and tangle invariants.

(13) Koszul Duality

(14) Symplectic Duality

- (a) Hypertoric Category \mathcal{O} .

References

- [1] James E. Humphreys. Representations of semisimple Lie algebras in the BGG category \mathcal{O} , volume 94 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008.