The dihedral cathedral

1. Review of BSBim and one-color calculus

BSBim for $(W,S) = (S_2, \{s\})$:

$$S_2 \circlearrowright R\alpha_s \rightsquigarrow S_2 \circlearrowright R = \mathbb{R}[\alpha_s] \; (\deg \alpha = 2)$$
$$\alpha_s \mapsto -\alpha_s \qquad \alpha_s^k \mapsto (-\alpha_s)^k$$

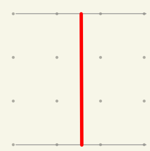
Objects: $BS(\underline{1}) = R$
(up to isom) $BS(\underline{s}) = R \otimes_{\mathbb{R}[\alpha_s^2]} R(1) = B_s$

$BS(\underline{s}^k) = B_s \otimes \underset{k)}{\cdots} \otimes B_s$

Morphisms: maps of $\mathbb{R}[\alpha_s]$-bimodules of any shift degree
Monoidal structure: $\otimes$.

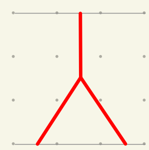
Now B_s is a Frobenius algebra object, meaning there are maps $\mu, \eta, \delta, \varepsilon$:

identity:
$$\begin{array}{ccc} B_s & = & R \otimes_s R \\ \uparrow & & \uparrow \\ B_s & = & R \otimes_s R \end{array} \qquad \begin{array}{c} f \otimes g \\ \updownarrow \\ f \otimes g \end{array}$$



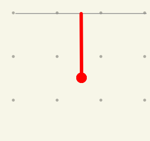
deg 0

multiplication: μ:
$$\begin{array}{ccc} B_s & = & R \otimes_s R \\ \uparrow & & \uparrow \\ B_s \otimes B_s & = & R \otimes_s R \otimes_s R(2) \end{array} \qquad \begin{array}{c} \partial_s(g) f \otimes h \\ \updownarrow \\ f \otimes g \otimes h \end{array}$$



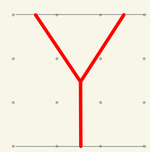
deg -1

unit: η:
$$\begin{array}{ccc} B_s & = & R \otimes_s R \\ \uparrow & & \uparrow \\ R & = & R \end{array} \qquad \begin{array}{c} \frac{1}{2}(1 \otimes \alpha_s + \alpha_s \otimes 1) \\ \updownarrow \\ 1 \end{array}$$



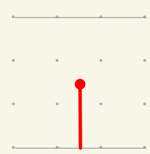
deg 1

comultiplication: δ:
$$\begin{array}{ccc} B_s \otimes B_s & = & R \otimes_s R \otimes_s R(2) \\ \uparrow & & \uparrow \\ B_s & = & R \otimes_s R \end{array} \qquad \begin{array}{c} f \otimes 1 \otimes g \\ \updownarrow \\ f \otimes g \end{array}$$



deg -1

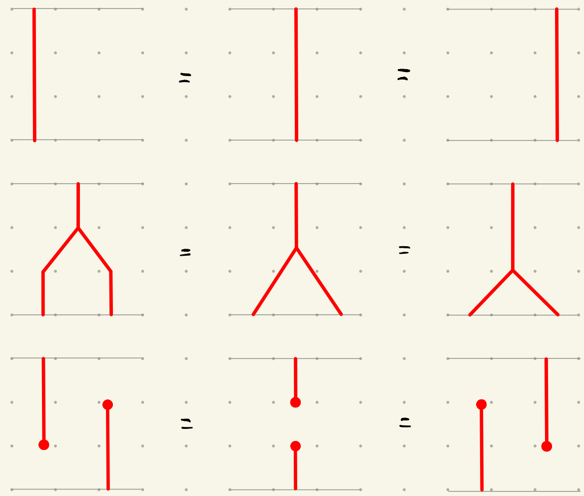
counit: η:
$$\begin{array}{ccc} R & = & R \\ \uparrow & & \uparrow \\ B_s & = & R \otimes_s R \end{array} \qquad \begin{array}{c} fg \\ \updownarrow \\ f \otimes g \end{array}$$



deg 1

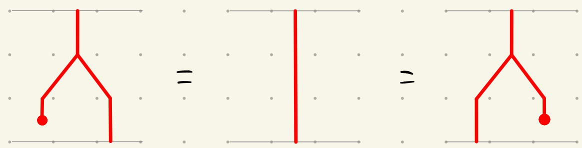
These satisfy the relations:

Axioms of strict monoidal category ⇝ Rectilinear isotopies:

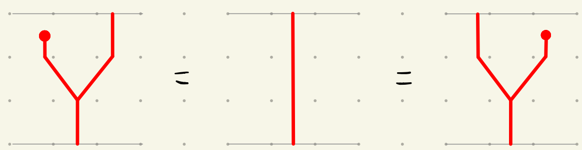


etc.

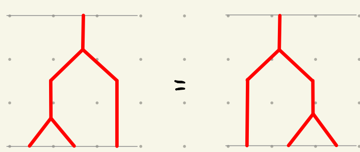
Unit ⇝



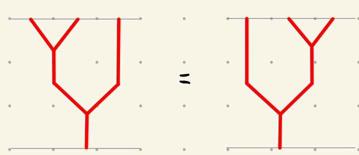
Counit ⇝



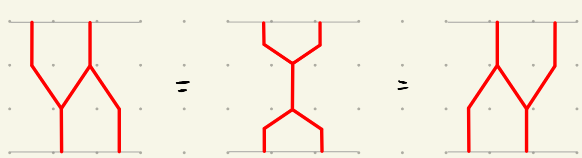
Associativity ⇝



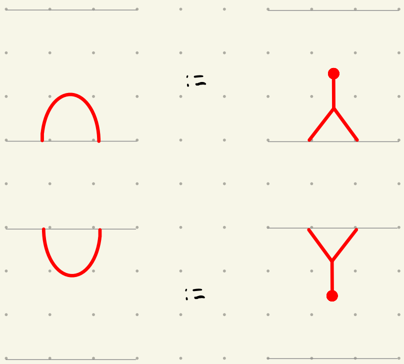
Coassociativity ⇝



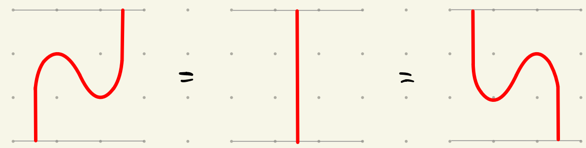
Frobenius associativity ⇝



We defined



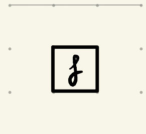
Frobenius



Relations imply: isotopic diagrams represent equal morphisms.

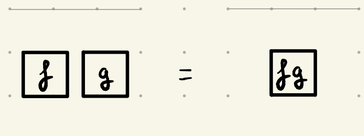
For $f \in R$, we also have

$$\begin{array}{ccc} R & & fh \\ \uparrow & & \uparrow \\ R & & h \end{array} \quad \rightsquigarrow$$

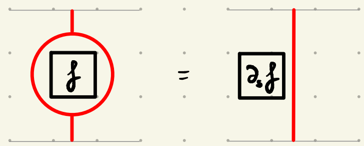


These satisfy:

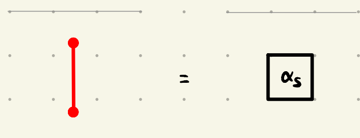
Multiplication $\rightsquigarrow$



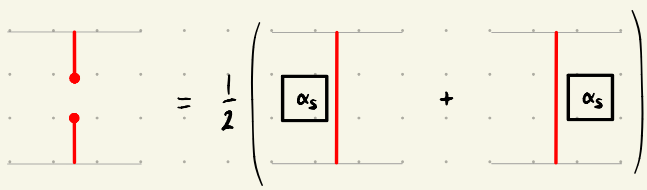
Keyhole $\rightsquigarrow$



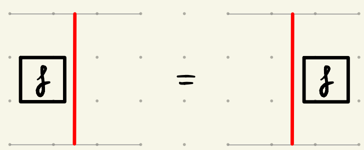
Barbell $\rightsquigarrow$



Fusion $\rightsquigarrow$



Polynomial slide $\rightsquigarrow$

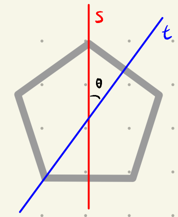


In fact, defining $\mathcal{H}_{BS}(s)$ as the R-linear category with objects $\{\underbrace{\bullet\ \bullet \cdots \bullet}_{k)} : k \geq 0\}$ and morphisms as above, we have

$\mathcal{H}_{BS}(s) \xrightarrow{\sim} \mathrm{BSBim}(S_2, \{s\})$ is an equivalence of categories.

"one-color diagrammatic Hecke category"

Today: consider $(W,S) = (D_{2m}, \{s, t\})$ (possibly $m=\infty$)



$(\alpha_s ; \alpha_t^\vee) = -\frac{\cos \pi}{m_{st}}$

Dynkin diagram: 

$(m=3 \rightsquigarrow A_2)$

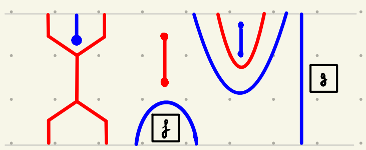
We define $\mathcal{H}_{BS} = \mathcal{H}_{BS}(W,S)$ so that $\mathcal{H}_{BS} \xrightarrow{\sim} \mathbb{BS}\mathrm{Bim}$ is an equivalence of categories.

Remark: technically, for D_∞ one should replace the geometric representation by some other realization $\{\mathfrak{h}, \{\alpha_s, \alpha_t\}, \{\alpha_s^\vee, \alpha_t^\vee\}\}$ where $\dim \mathfrak{h} \geq 3$ $(=3$ suffices$)$. Then $R = \mathrm{Sym}(\mathfrak{h}^*)$. When V_{geom} suffices, $\mathfrak{h} = V_{geom}^*$. The Demazure operator is defined as evaluation at $\alpha_s^\vee$.

2. Universal diagrams

Consider the diagrammatic category with objects $\{ \color{red}{\bullet}\ \color{red}{\bullet}\ \color{blue}{\bullet}\ \color{red}{\bullet}\ \color{blue}{\bullet} \cdots \color{red}{\bullet}\ \color{blue}{\bullet}\ \color{blue}{\bullet} \in \bigcup_{n \geq 0} \{\color{red}{\bullet}, \color{blue}{\bullet}\}^n \}$

and with morphisms coming from $\mathcal{H}_{BS}(s)$ and $\mathcal{H}_{BS}(t)$, that is, morphisms like



These are called universal diagrams, and the resulting category is the "universal 2-color diagrammatic Hecke category"
Denote this category by $\mathcal{H}_{BS}^\infty(s,t)$.

Theorem: The functor $\mathcal{H}_{BS}^\infty \to \mathbb{BS}\mathrm{Bim}$, for (W,S) infinite dihedral, is an equivalence of categories.

Idea (same for all (W,S))

Essentially surjective: obvious.

Full: it can be checked algebraically that every morphism of Bott-Samelson bimodules comes from a diagram
(general case: Libedinsky's light leaves)

Faithful: it suffices to show that Hom "dimensions" agree with sizes of (diagrammatic) basis of Hom spaces

How to find these dimensions?

3. Interlude on Soergel's Hom formula

Def. A standard bimodule is an R-bimodule of the form R_x for $x \in W$, where $R_x = R$ with twisted action $r \cdot m \cdot r' := rmx(r')$

Remarks:
- $R_x \otimes R_y \cong R_{xy}$
- $R_x = R \cdot 1 \cdot R$
- R_x is indecomposable
- $\mathrm{Hom}^\bullet(R_x, R_y) = \begin{cases} R & x=y \\ 0 & o/w \end{cases}$

Def. StdBim is the category of standard bimodules, their shifts and finite $\oplus$.

Rmk: $[\mathrm{Std\,Bim}]_{\oplus} \cong \mathbb{Z}[v^{\pm}]W$

Recall the elements $c_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$, $d_s = \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)$. Recall $\{1\otimes 1, c_s\}$, $\{1 \otimes 1, d_s\}$ are basis for B_s as a left or right R-module. In R-gbim we have

$$\begin{array}{ccccccccc} 0 & \to & R_s(-1) & \longrightarrow & B_s & \xrightarrow{\mu} & R(1) & \to & 0 \\ & & 1 & \longmapsto & d_s & & & & \\ & & & & f \otimes g & \longmapsto & fg & & \end{array} \qquad (\Delta)$$

$$\begin{array}{ccccccccc} 0 & \to & R(-1) & \longrightarrow & B_s & \xrightarrow{\mu} & R_s(1) & \to & 0 \\ & & 1 & \longmapsto & c_s & & & & \\ & & & & f \otimes g & \longmapsto & f\cdot s(g) & & \end{array} \qquad (\nabla)$$

Proof: For (Δ), we need to check that the first map is well-defined. Now $r \cdot m \cdot r' \mapsto f(rms(r')) = rms(r')d_s \underset{\substack{\text{computation in} \\ \text{the first talk}}}{=} rmd_s r'$. The kernel of μ is obviously spanned by $\alpha_s \otimes 1 - 1 \otimes \alpha_s$.

The computation for (∇) is analogous, using $f \cdot c_s = c_s \cdot f$. $\square$

We thus have $B_s = \dfrac{R(1)}{R_s(-1)}$, a filtration with subquotients in StdBim. This gives "standard filtrations" for every Soergel bimodule:

Application: Filtration for $B_s B_t$:

$$B_s B_t = \cfrac{R(2)}{\cfrac{R_s \otimes R_t}{R_s(-1) \otimes R_t(-1)}} = \begin{array}{c} R(2) \\ \hline R_s \\ \hline R_t \\ \hline R_{st}(-2) \end{array}$$

order respects Bruhat order $\rightsquigarrow$ "Δ-filtration"

Theorem (Soergel 2007): For a fixed enumeration of W respecting the Bruhat order, there exists a unique Δ-filtration, and the graded multiplicities of each standard bimodule are indep. of the enumeration.

Example (continued)

$$\begin{array}{c} R(2) \\ \hline R_s \\ \hline R_t \\ \hline R_{st}(-2) \end{array} \rightsquigarrow \begin{array}{l} h_1(B_sB_t) = 1 \cdot v^2 \\ h_s(B_sB_t) = 1 \\ h_t(B_sB_t) = 1 \\ h_{st}(B_sB_t) = 1 \cdot v^{-2} \end{array}$$

Now we can define $ch_\Delta(B) = \sum_{x \in W} v^{\ell(x)} h_x(B) \delta_x$

Examples:

$B_s = \dfrac{R(1)}{R_s(-1)} \rightsquigarrow \begin{array}{l} h_1 = v \\ h_s = v^{-1} \end{array} \qquad \rightsquigarrow ch_\Delta(B_s) = v^1 \cdot v^{-1} \cdot \delta_s + v \cdot \delta_1 = \delta_s + v$

$B_s B_t = \begin{array}{c} R(2) \\ \hline R_s \\ \hline R_t \\ \hline R_{st}(-2) \end{array} \rightsquigarrow \begin{array}{l} h_1 = v^2 \\ h_s = 1 \\ h_t = 1 \\ h_{st} = v^{-2} \end{array} \qquad \rightsquigarrow ch_\Delta(B_s B_t) = v^2 v^{-2} \delta_{st} + v\delta_s + v\delta_t + v^2 = \delta_{st} + v\delta_s + v\delta_t + v^2$

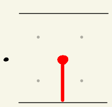
Remark: $ch_\Delta(B_x) = ch_\nabla(B_x) \ \forall x \in W$. Soergel's conjecture (now theorem) says $ch(B_x) = b_x$

Back to Hom spaces

Theorem (Soergel 2007) let B, B' be Soergel bimodules. Then the graded Hom $Hom^\bullet_{SBim}(B, B')$ is free as a left graded R-module and as a right graded R-module, of graded rank $(ch(B), ch(B'))$.

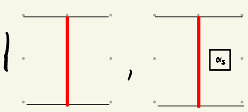
Examples:

$\text{rk } Hom^\bullet(B_s, R) = (b_s, 1) = \varepsilon(\bar{b}_s \cdot 1) = \varepsilon(\delta_s + v) = v \rightsquigarrow R \cdot$



deg 1

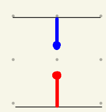
$\text{rk } Hom^\bullet(B_s, B_s) = (b_s, b_s) = (1, b_s^2) = (1, vb_s + v^{-1} b_s) = v^2 + 1 \rightsquigarrow R \cdot \{$



$\}$

deg 0 , deg 2

$\text{rk } Hom^\bullet(B_s, B_t) = (b_s, b_t) = (1, b_s b_t) = v^2 \rightsquigarrow R \cdot$

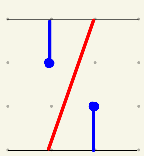


deg 2

$\text{rk } Hom^\bullet(B_s B_t, B_t B_s) = (1, b_s b_t b_s b_t) = v^4 + 2v^2$

$stst + vsts + vst^2 + v^2 st + vs^2 t + v^2 s^2 + v^2 st + v^3 s + vtst + v^2 ts + v^2 t^2 + v^3 t + v^2 st + v^3 t + v^3 s + v^4$

$\rightsquigarrow R \cdot \{$



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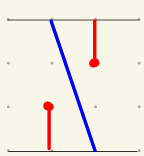

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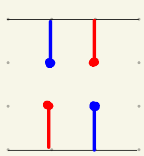


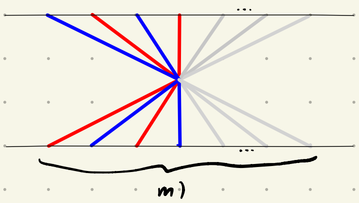
deg 4 $\}$

If $m = 2$ (i.e. type A_1), we have $b_s b_t b_s b_t = (\delta_s + v)^2 (\delta_t + v)^2 = (v + v^{-1})^2 b_s b_t = (v + v^{-1})^2 \cdot (v^2 + v\delta_s + v\delta_t + \delta_{st})$

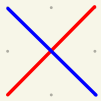
$\rightsquigarrow v^4 + 2v^2 + 1$

This suggests that we need a new morphism.

Def: Define the 2m-valent morphism as:



deg 0

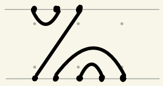
E.g.: for $m=2$, we have 

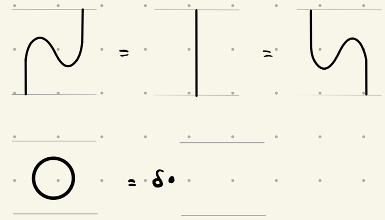
Theorem: Defining $\mathcal{H}_{BS}$ as above, plus the 2m-valent morphism, and imposing three relations that we will see later, we have an equivalence of categories $\mathcal{H}_{BS} \xrightarrow{\sim} \mathbb{B}SBim$ for (W, S) finite dihedral.

Next, we motivate the 2m-valent vertex and the new relations.

4. The two-color Temperley-Lieb category

Def: Temperley-Lieb monoidal category $\mathcal{L}_\delta$ is given by:

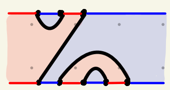
- Objects: $\{\underbrace{\bullet\bullet\cdots\bullet}_{n} \mid n \geq 0\}$
- Morphisms: $\mathbb{Z}[\delta]\cdot\{$  $\}$ (crossingless matchings), subject to

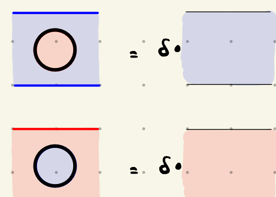


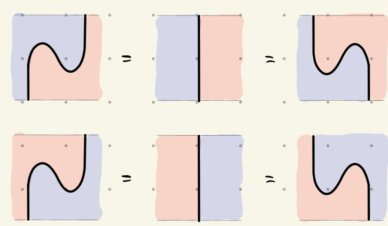
- Monoidal structure: concatenation.

Rmk: specializing to $\delta = -(q+q^{-1})$, $\mathcal{L}_\delta \cong \mathrm{Fund}(U_q(\mathfrak{sl}_2))$
$\delta = -2$, $\mathcal{L}_\delta \cong \mathrm{Fund}(\mathfrak{sl}_2\mathbb{C})$

Def: 2-colored Temperley-Lieb 2-category $2\mathcal{L}_\delta$ is given by:

- Objects: $\{ -, - \}$
- 1-morphisms: $\{$  $\mid n \geq 0\}$
- 2-morphisms: $\mathbb{Z}[\delta]\cdot\{$  $\}$ (crossingless matchings), subject to





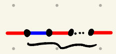
Vertical composition and horizontal composition as usual

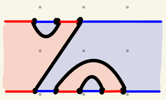
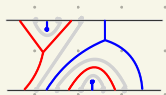
Now specialize $2\mathcal{L}_\delta$ to $\delta = a_{st} := -2\cos\left(\frac{\pi}{m_{st}}\right)$ $\left(= -(q+q^{-1}) \text{ for } q = \sqrt[2m]{1}\right)$
$= \partial_t(\alpha_s) = \partial_s(\alpha_t)$

Then we have a functor

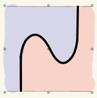
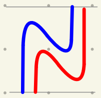
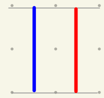
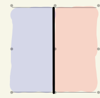
$$\Sigma : 2\mathcal{L}_{a_{st}} \longrightarrow \mathbb{BS}\mathrm{Bim}$$

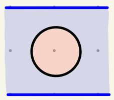
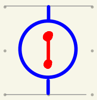
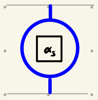
sending $-, - \mapsto$ "unique object"

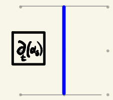
 $\mapsto B_s \otimes B_t \otimes \cdots \otimes B_s$

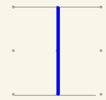
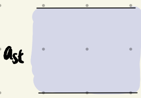
 $\mapsto$  $\mapsto \varphi$ (this functor factors through $\mathcal{H}_{BS}^{\infty}$)

Proof that this is well-defined:

 $\mapsto$  $=$  $\mapsfrom$ 

 $\mapsto$  $=$ 

$=$ 

$= a_{st}$  $\longmapsfrom a_{st}$ 

Two similar observations:

- In Fund, we have an idempotent map $V^{\otimes n} \twoheadrightarrow L(n) \hookrightarrow V^{\otimes n}$
- In BSBim, $w_0 = \underbrace{st\ldots s}_{m}$ is the longest element $\Rightarrow B_s B_t \ldots B_s \twoheadrightarrow B_{w_0} \hookrightarrow B_s B_t \ldots B_s$

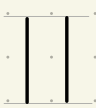
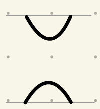
Both of these phenomena can be explained using Temperley-Lieb:

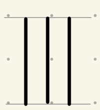
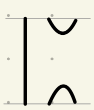
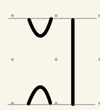
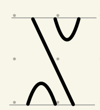
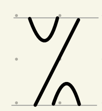
Denote $TL_n = \mathrm{End}_{\mathcal{D}_S}(\underbrace{\bullet\bullet\cdots\bullet}_{n})$

Prop (Jones-Wenzl elements) There is a unique element $JW_n \in TL_n$ satisfying:

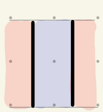
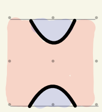
- Capping or cupping any two strands (when possible) sends the element to 0.
- The coefficient of $id_n \in TL_n$ in JW_n is 1

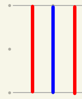
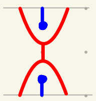
Furthermore, JW_n is idempotent.

Examples $JW_2 =$  $- \frac{1}{\delta}$ 

$JW_3 =$  $+ \frac{\delta}{\delta^2-1}$  $+ \frac{\delta}{\delta^2-1}$  $+ \frac{1}{\delta^2-1}$  $+ \frac{1}{\delta^2-1}$ 

There are entirely analogous 2-colored JW_n's:

$JW_2 =$  $- \frac{1}{\delta}$ 

Now its image in $\mathcal{H}^{\infty}_{BS}$ is  $-\frac{1}{a_{st}}$  . This is still an idempotent!

In fact mapping this to a morphism in $\mathbb{BSB}$im, we have found $B_sB_tB_s = B_s \oplus \mathrm{Im}(JW)$
$= B_s \oplus B_{sts}$

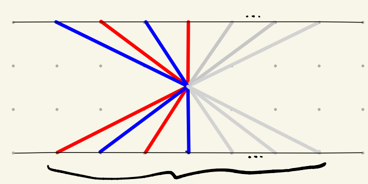
Upshot: diagrammatics have provided us with the explicit map realizing $B_{sts} \subseteq_{\oplus} B_sB_tB_s$!

In fact, we have the following.

<u>Theorem</u>: Take a (reduced) expression $\underline{w} = \underbrace{st\ldots s}_{n}$ with $n < m_{st}$. Then the image of the colored JW_n is an idempotent map $BS(\underline{w}) \twoheadrightarrow B_w$

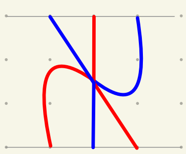
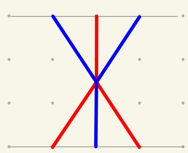
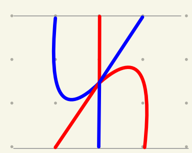
<u>Back to the 2m-valent morphism.</u>

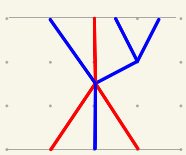
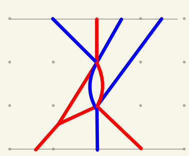
Notice that in $\mathbb{BSB}$im, $\underbrace{sts\ldots s}_{m} = \underbrace{tst\ldots t}_{m}$ $\Rightarrow$ this equals the longest element
$\Rightarrow B_{w_0} \subseteq_{\oplus} B_sB_t\cdots B_s$ and $B_{w_0} \in B_tB_sB_t\cdots B_t$,

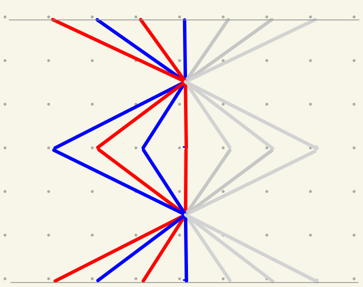
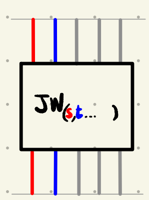
so we have $B_{sts\ldots s} \twoheadrightarrow B_{w_0} \hookrightarrow B_{tst\ldots t}$. This is the map that  corresponds to.

m)

What about the three relations?

Cyclicity:  =  =  (for all m, both parities, color swaps)

2-color associativity:  =  (for all m, both parities, color swaps)

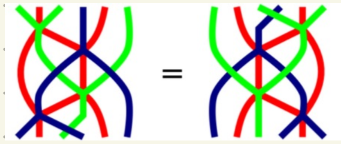
Elias-Jones-Wenzl:  = 

<u>Theorem</u>: these are enough.

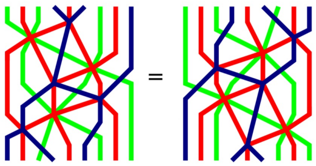
6. A word on more colors.

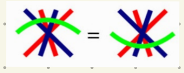
One can play the same game for Coxeter groups with 3 generators. There are 4 possible Coxeter groups:

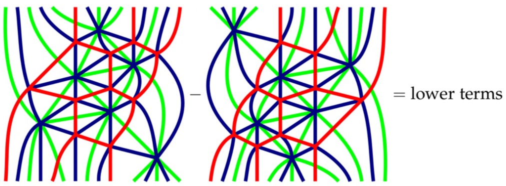
A_3, B_3, $A_1 \times I_2(m)$, H_3.

Type A_3:  ("Zamolodchikov relation")

analogously one has the following Zamolodchikov relation for $B3$:

Type B_3: 

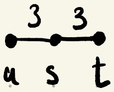
Type $A_1 \times I_2(m)$: 

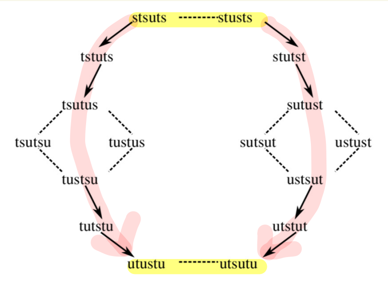
Type H_3: 

however, despite considerable effort, we have not been able to compute the lower terms which appear. The question of what these lower terms are could in principle be decided by computer, however the computation is impossible with our current algorithms and technology. This is the caveat mentioned earlier: we do not have a completely explicit presentation of the category $\mathbb{BS}$Bim when W contains a parabolic subgroup of type H_3, knowing this Zamolodchikov relation only in the rough form above.

These relations would suffice. In fact 4-color relations are not needed!

Type A_3:





$\rightsquigarrow$ Maps in SBim (explaining the Zamolodchikov relation)

Such cycles in Coxeter groups are made up of disjoint braid relations and Z relations from rank 3 parabolic

This doesn't mean morphisms with a fixed source and target are equal in $\mathcal{H}_{BS}$, but making the right choices they will be equal "mod lower terms".

Next: light leaves?