Parabolic Category $\mathcal{O}$

Ref: Humphreys Ch. 9

Notation (Old)

- $\mathfrak{g}$: semisimple Lie alg. $/\mathbb{C}$
- $\mathfrak{b}$: Fixed Borel subalgebra
- $\mathfrak{h}$: Cartan subalgebra
- Δ : Simple roots of positive roots $\Phi^+ \subseteq \Phi$ = root system associated to $\mathfrak{b}$

Notation (New) $I \subseteq \Delta$ be some fixed subset. We associate:

- $\mathfrak{p}_I := \bigoplus_{\alpha \in \Phi^+ \cup \Phi_I^-} \mathfrak{g}_\alpha$ "parabolic", where $\Phi_I := \Phi \cap \mathbb{Z}I$.

- $\mathfrak{l}_I := \mathfrak{h} \oplus \left(\sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha \right)$ "Levi"

- $\mathfrak{u}_I := \bigoplus_{\alpha \in \Phi^+ \setminus \Phi_I^+} \mathfrak{g}_\alpha$, $\mathfrak{u}_I^- := \bigoplus_{\alpha \in \Phi^- \setminus \Phi_I^-} \mathfrak{g}_\alpha$ "unipotent"

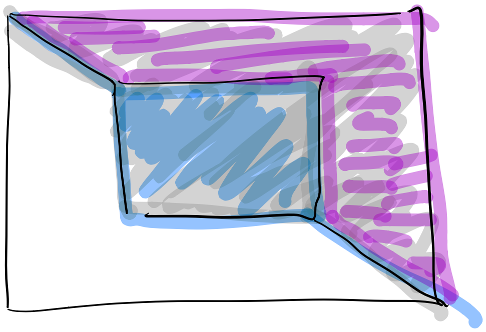
- $\mathfrak{g}_I := [\mathfrak{l}_I, \mathfrak{l}_I]$

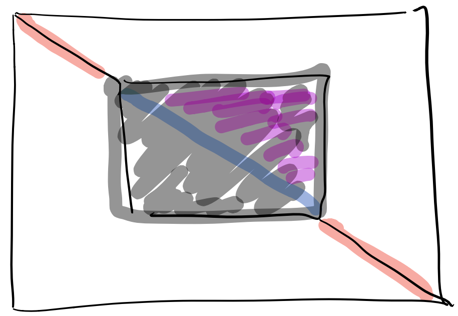
- $\mathfrak{h}_I := \bigoplus_{\alpha \in I} \mathbb{C} h_\alpha$

- $\mathfrak{n}_I := \bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_\alpha$, $\mathfrak{n}_I^- = \bigoplus_{\alpha \in \Phi_+^-} \mathfrak{g}_\alpha$

- $\mathfrak{z}_I = \bigcap_{\alpha \in I} \ker \alpha$ = center of $\mathfrak{l}_I$.

Observe: Choose $\mathfrak{p}_I \Rightarrow \mathfrak{l}_I, \mathfrak{n}_I, \mathfrak{g}_I, \mathfrak{h}_I, \mathfrak{u}_I$ fixed





• = $\mathfrak{p}_I$

• = $\mathfrak{l}_I$

• = $\mathfrak{u}_I$

• = $\mathfrak{g}_I$

• = $\mathfrak{h}_I$

• = $\mathfrak{n}_I$

• = $\mathfrak{z}_I$

$$\boxed{\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I}$$

$$\boxed{\mathfrak{g} = \mathfrak{u}_I^- \oplus \mathfrak{l}_I \oplus \mathfrak{u}_I}$$

$$\boxed{\mathfrak{g}_I = \mathfrak{n}_I^- \oplus \mathfrak{h}_I \oplus \mathfrak{n}_I}$$

$$\boxed{\mathfrak{l}_I = \mathfrak{g}_I \oplus \mathfrak{z}_I}$$

$$\boxed{\mathfrak{h} = \mathfrak{h}_I \oplus \mathfrak{z}_I}$$

<u>Rmk</u> There is bijection: Fix Borel $\mathfrak{b}$, simple roots Δ.

$$\{\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{b}\} \longleftrightarrow \{I : I \subseteq \Delta\}$$

$$\mathfrak{p}_I \longleftarrow\!\shortmid I$$

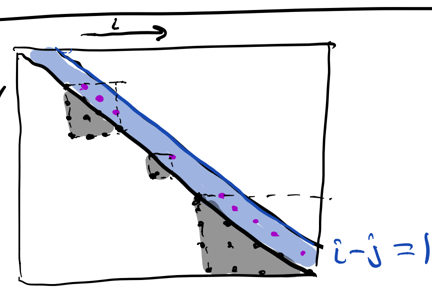
For $\mathfrak{sl}_n$, picture is clear: $i \downarrow$

$\mathfrak{b}$ = standard Borel

▮ $= \Delta = \{\varepsilon_i - \varepsilon_{i+1}\}$

▮ $= \mathfrak{p}_I$

⋮ $= I \subseteq \Delta$



§2 $\mathfrak{l}_I$ $(= \mathfrak{g}_I \oplus \mathfrak{z}_I)$ – modules

$$\mathfrak{h} = \mathfrak{h}_I \oplus \mathfrak{z}_I \Rightarrow \mathfrak{h}^* = \mathfrak{h}_I^* \oplus \mathfrak{z}_I^*$$

$$\lambda = \lambda|_{\mathfrak{h}_I^*} + \lambda|_{\mathfrak{z}_I^*}$$

"$\mathfrak{g}_I$ – dom int. weights"

<u>Def</u> $\Lambda_I^+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in \vec{\Phi}_I\}$

$\cup$ (equality if $I = \Delta$)

Λ^+

$\lambda \in \Lambda_I^+ \rightsquigarrow L_I(\lambda)$: Fin. dim irred $\mathfrak{g}_I$-module

$\Rightarrow$ is $\mathfrak{l}_I$-module by restricting λ action to $\mathfrak{z}_I$.

Conversely, $\mathfrak{z}_I$ acts on $L_I(\lambda)$ by scalars (Schur's lemma)

$\Rightarrow$ Every irred. $\mathfrak{l}_I$-module is of the form $L_I(\lambda)$: $\lambda \in \Lambda_I^+$.

<u>Def</u> "Verma for $\mathfrak{g}_I$" $\boxed{V_I(\lambda) := U(\mathfrak{l}_I) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_I)} \mathbb{C}_\lambda}$

$$\mathfrak{l}_I = \underline{\mathfrak{g}_I} + \cancel{\mathfrak{z}_I}$$

$$\mathfrak{h} + \mathfrak{n}_I = \underline{\mathfrak{h}_I + \mathfrak{n}_I} + \cancel{\mathfrak{z}_I}$$

BGG Resolution for $\mathfrak{g}_I$ $\Rightarrow$

$$\textcircled{+} \quad U_I(s_\varphi \cdot \lambda) \longrightarrow U_I(\lambda) \longrightarrow L_I(\lambda) \to 0$$

$\varphi \in I$

L.E.S of $\mathfrak{l}_I$-modules.

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§3: $\mathcal{O}^{\mathfrak{p}}$

For $\mathfrak{p} = \mathfrak{p}_I$

$(\mathcal{O}^{\mathfrak{p}}1)$ M fin. gen'd $U\mathfrak{g}$-module

$(\mathcal{O}^{\mathfrak{p}}2)$ M is $\mathfrak{l}_I$-semisimple

$(\mathcal{O}^{\mathfrak{p}}3)$ M is $\mathfrak{u}_I$-locally finite

For $I = \emptyset$, $\mathfrak{p} = \mathfrak{b} \Rightarrow \mathcal{O}^{\mathfrak{p}} = \mathcal{O}$

$I = \Delta$, $\mathfrak{p} = \mathfrak{g} \Rightarrow \mathcal{O}^{\mathfrak{p}} =$ semisimple $U\mathfrak{g}$-modules.

<u>lem</u> $M \in \mathcal{O}$, $\Pi(M) =$ weights of M. TFAE

(i) M locally $\mathfrak{n}_I^-$-finite

(ii) $\forall \alpha \in I$, $\mu \in \Pi(M)$, $\dim M_\mu = \dim M_{s_\alpha \mu}$

(iii) $\forall w \in W_I$ $(:= \langle s_\alpha \mid \alpha \in I \rangle)$, $\dim M_\mu = \dim M_{w\mu}$

(iv) $\Pi(M)$ is stable under W_I.

<u>Pf</u> | (i) $\Rightarrow$ (ii)

Look at submodule N of M gen'd by the $\mathfrak{sl}_2$ action on M_μ.

Assumption M is $\mathfrak{n}_I^-$-finite & $M \in \mathcal{O}$

$\Longrightarrow N$ is fin. dim.

$\overset{\mathfrak{sl}_2}{\Longrightarrow}$ All weight spaces which are conjugate have same dimension.

$\Rightarrow$ (ii), (iii), (iv).

(iv) $\Rightarrow$ (i) $\quad v \in M_\mu \overset{M \in \mathcal{O}}{\Longrightarrow} \mathcal{U}(\mathfrak{n}_I) \cdot v$ fin. dim

$\Rightarrow$ weights in $\Pi(M)$ are $\mu + \nu : \nu \in \mathbb{Z}^+ \Phi_I^+$

for finitely many ν.

$\overset{(d)}{\Longrightarrow}$ $w_I^0 :=$ longest elem of $W_I : \Phi_I^+ \to \Phi_I^-$

weights in ΠM are $\underbrace{w_I^0 \cdot \mu}_{\in \Pi M} + \nu$, $\nu \in \mathbb{Z}^+ \Phi_I^-$

For fin. many ν

$\Rightarrow$ M is locally $\mathfrak{n}_I^-$-finite $\blacksquare$

__Cor__ (a) $M \in \mathcal{O}$ lies in $\mathcal{O}^{\mathfrak{p}} \iff M$ satisfies any (i)–(iv) of lemma.

(b) $\mathcal{O}^{\mathfrak{p}}$ is closed under $M \to M^\vee$

(c) $\mathcal{O}^{\mathfrak{p}}$ is closed under $\oplus$, $\subseteq$, $/$, $\overline{}$ (extension),

$- \otimes$ (fin-dim).

(d) If $M \in \mathcal{O}^{\mathfrak{p}} \leq \mathcal{O}$ decomposes as

$$M = \bigoplus_{\xi : Z_{\mathfrak{g}} \to \mathbb{C}} M^{\xi} \quad , \text{ then } M^{\xi} \in \mathcal{O}^{\mathfrak{p}}$$

(e) If $L(\lambda) \in \mathcal{O}^{\mathfrak{p}} \Rightarrow \lambda \in \Lambda_I^+$.

Pf (a) $M \in \mathcal{O}^{\mathfrak{p}} \Rightarrow (\mathcal{O}^{\mathfrak{p}}2) \Rightarrow$ (i) of lemma

$M \in \mathcal{O} \Rightarrow (\mathcal{O}^{\mathfrak{p}}1)$ & $(\mathcal{O}^{\mathfrak{p}}3)$

By lem(i), $\forall v \in M_{\mu}$, $U(\mathfrak{g}_I).v$ is fin. dim

$\Rightarrow$ Every element of M spans a fin. dim $U(\mathfrak{l}_I)$-module

$\xRightarrow[\text{Reducibility}]{\text{Complete}}$ M is direct sum of $U(\mathfrak{l}_I)$-modules

$\Rightarrow (\mathcal{O}^{\mathfrak{p}}1)$ ✓

(b) $\operatorname{ch} M = \operatorname{ch} M^{\vee} \Rightarrow$ (iv) of lemma holds for $M \Leftrightarrow M^{\vee}$.

(c) & (d) Statements hold for $\mathcal{O}$ + cor (a) + lemma $\Rightarrow$ hold for $\mathcal{O}^{\mathfrak{p}}$.

(e) $L(\lambda) \in \mathcal{O}^{\mathfrak{p}}$, v^+ = max'l vector, then for any

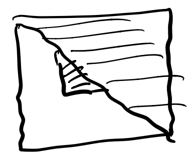
$\alpha \in I$, $y_{\alpha}^{n} \cdot v^+ = 0$ for $n \gg 0$.

study of $\mathfrak{sl}_2$-modules $\Rightarrow \lambda \in \Lambda_I^+$ ∎

Def $\boxed{\text{Truncation functor}}$: $\overline{(-)} : \mathcal{O} \longrightarrow \mathcal{O}^{\mathfrak{p}}$

$M \in \mathcal{O} \longmapsto \underline{M} := \{ \mathfrak{n}_I^- \text{-finite vectors in } M \}$

$=$ unique max'l submodule of M inside $\mathcal{O}^{\mathfrak{p}}$.

($\mathfrak{u}_P =$  - finite)

$\overline{M} := \left(\underline{M^\vee}\right)^\vee =$ largest quotient of M inside $\mathcal{O}^{\mathfrak{p}}$.

Aside $\Gamma_{\mathfrak{p}} : \mathcal{O} \longrightarrow \mathcal{O}^{\mathfrak{p}}$

$M \longrightarrow$ max'l locally $\mathfrak{u}_{\mathfrak{p}}$-finite submodule

called Zuckerman functor. Is left adjoint to $i : \mathcal{O}^{\mathfrak{p}} \hookrightarrow \mathcal{O}$.

(Right adjoint is $\Gamma_{\mathfrak{p}}^*(M) =$ " —— " quotient)

Key Lemma (Enright–Wallach)

Let $d = \dim \mathfrak{l}_I - \dim \mathfrak{h}$.

1. For $i > d$, $R^i\Gamma_{\mathfrak{p}} : \mathcal{O} \longrightarrow \mathcal{O}^{\mathfrak{p}}$, $R^i\Gamma_{\mathfrak{p}}(M) = 0$.
2. Projective functors ($\supseteq$ Translation functors) commute with $\Gamma_{\mathfrak{p}}$.
3. $\left(M \overset{\in \mathcal{O}}{\longmapsto} R^i\Gamma_{\mathfrak{p}} M\right) \underset{\text{can.}}{\cong} \left(M \longmapsto R^{d-i} \underbrace{\Gamma_{\mathfrak{p}}(M^\vee)^\vee}_{\text{"}\Gamma_{\mathfrak{p}}^*(M)\text{"}}\right)$
4. $R^d\Gamma_{\mathfrak{p}} =$ largest quotient lying in $\mathcal{O}^{\mathfrak{p}}$.

§4 Let $L_I(\lambda)$ = fin. dim l_I-module, $\lambda \in \Lambda_I^+$

λ is $\mathfrak{g}_I$-dom.

not. nec. 1-dim.

Def Parabolic Verma $\boxed{M_I(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L_I(\lambda)}$

- $M_I(\lambda)$ is gen'd as $U\mathfrak{g}$-mod $1 \otimes v_+$ weight λ

  (Universal property) $\exists$ $M(\lambda) \twoheadrightarrow M_I(\lambda)$.

  and $L(\lambda)$ is unique quotient of $M_I(\lambda)$.

- PBW basis of $M_I(\lambda)$: $Y_{i_1}^{c_1} \cdots Y_{i_r}^{c_r} u \cdot v^+$,

  where $Y_i \in \Phi^+ \setminus \Phi_I$ (use $U(\mathfrak{n}^-) = U(\mathfrak{n}_I^-) \otimes U(\mathfrak{u}_I^-)$)

Thm Let $\lambda \in \Lambda_I^+$,

(a) $M_I(\lambda) \in \mathcal{O}^{\mathfrak{p}}$ ($\Rightarrow L(\lambda) \in \mathcal{O}^{\mathfrak{p}}$)

(b) $\exists$ exact sequence

$$\bigoplus_{\alpha \in I} M(s_\alpha \cdot \lambda) \to M(\lambda) \to M_I(\lambda) \to 0$$

(c) $M_I(\lambda) = \overline{M(\lambda)}$ ($\stackrel{\text{def}}{=} \Gamma_{\mathfrak{p}} M(\lambda)$)

(Rmk : (b) extends to BGG Resolution where i^{th} term is $\bigoplus_{l(w)=i,\, w \in W_I} M(w \cdot \lambda)$)

Pf (a) By prop, STS (iv). We know weights of

- $M_I(\lambda)$ are $\mu - \nu$: $\mu \in \Pi(L_I(\lambda))$, $\nu \in \mathbb{Z}_{\geq 0}$ - linear comb. of $\Phi^+ \setminus \Phi_I$.
- W_I permutes $\Pi L_I(\lambda)$
- $\forall s_\alpha : \alpha \in I$, s_α permutes $\Phi^+ \setminus \alpha$, $\Phi_I^+ \setminus \alpha$.
- so $\forall w \in W_I$, $w\mu - w\nu \in \Pi(M_I(\lambda))$ ✓

(b) Recall $\bigoplus U_I(s_\alpha \cdot \lambda) \longrightarrow V_I(\lambda) \to L_I(\lambda) \to 0$

$\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{p}_I} -$ $\hookrightarrow$ $\bigoplus M_I(s_\alpha \cdot \lambda) \longrightarrow M(\lambda) \to M_I(\lambda) \to 0$ ✓

(c) $\lambda \in \Lambda_I^+ \implies s_\alpha \cdot \lambda \notin \Lambda_I^+$

$\implies L(s_\alpha \cdot \lambda) \notin \mathcal{O}_{\mathfrak{p}} \implies \underline{M(s_\alpha \cdot \lambda)} \notin \mathcal{O}^{\mathfrak{p}}$

$\implies M(s_\alpha \cdot \lambda) \subseteq \ker(M(\lambda) \longrightarrow M_I(\lambda))$

$\overset{(b)}{\implies} \exists\, \overline{\varphi} : M_I(\lambda) \longrightarrow \overline{M(\lambda)}$

$\overset{(a)}{\implies} M_I(\lambda) \in \mathcal{O}^{\mathfrak{p}}$, since $\overline{M(\lambda)}$ universal h.w object in $\mathcal{O}^{\mathfrak{p}} \implies \overline{\varphi}$ is isom. ■

Cor $M \in \mathcal{O}$ lives in $\mathcal{O}^{\mathfrak{p}} \iff$ all composition factors $L(\lambda)$ satisfy $\lambda \in \Lambda_I^+$

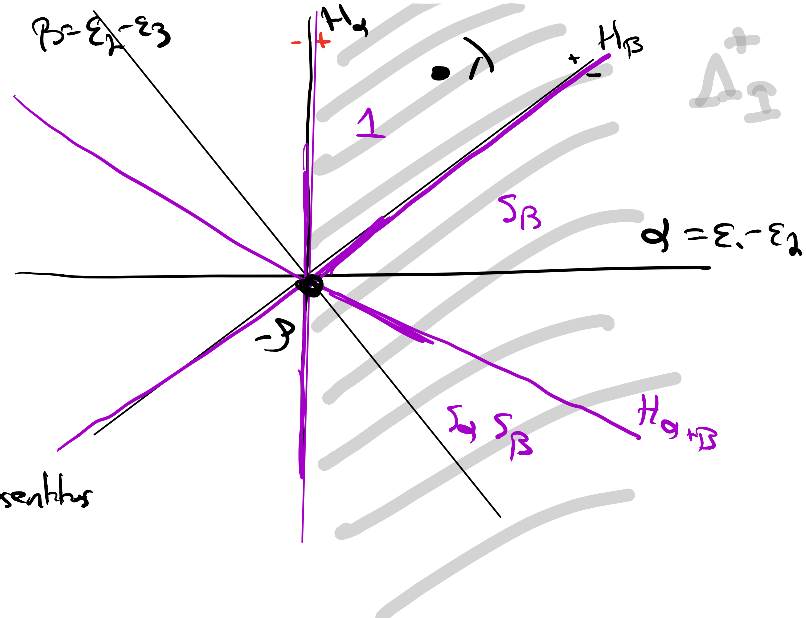
§5 $\mathfrak{sl}_3$

Let $I = \{\alpha\}$

$\Delta = \{\alpha, \beta\}$

λ = dom. reg weight

$1, s_\beta, s_\alpha s_\beta$ = min'l length representatives of $W_I \backslash W$



Character of $M_I(\lambda)$ known (BGG resolution) $\Rightarrow$ ($\forall \lambda \in \Lambda_I^+$) ⊛

$$\boxed{0 \longrightarrow M(s_\alpha \cdot \lambda) \longrightarrow M(\lambda) \longrightarrow M_I(\lambda) \to 0}$$

1) Let $\mu = s_\beta s_\alpha \cdot \lambda$ is min'l among linked weights in Λ_I^+.

$$\Rightarrow \boxed{M_I(s_\beta s_\alpha \cdot \lambda) = L(\mu)} \quad (= \text{simple})$$

2) Let $\mu = s_\beta \cdot \lambda$, Then ⊛ $\Rightarrow$

$$\operatorname{Ker}(M(\mu) \twoheadrightarrow M_I(\mu)) = M(s_\alpha \cdot \mu)$$

$$= M(s_\alpha s_\beta \cdot \lambda)$$

$$= \frac{L(s_\alpha s_\beta \cdot \lambda)}{L(w_0 \cdot \lambda)}$$

$$\Rightarrow \boxed{M_I(\mu) = \frac{L(s_\beta \cdot \lambda)}{L(s_\beta s_\alpha \cdot \lambda)}}$$

3) Let $\mu = \lambda$, Then (*) $\Rightarrow$

$$M_I(\lambda) = \underbrace{M(\lambda)}_{6\text{-terms}} \Big/ \underbrace{M(s_\alpha\cdot\lambda)}_{4\text{ terms}}$$

$$\boxed{M_I(\lambda) = \frac{L(\lambda)}{L(s_\beta\cdot\lambda)}}$$

$$\dim \operatorname{Hom}(M_I(\lambda), M_I(s_\beta\cdot\lambda)) = 0$$
$$\dim \operatorname{Hom}(M_I(s_\beta\cdot\lambda), M_I(\lambda)) = 1 \quad (\&\ \ker \cong L(s_\beta s_\alpha\cdot\lambda))$$
$$\dim \operatorname{Hom}(M_I(\lambda), M_I(s_\beta s_\alpha\cdot\lambda)) = 0$$
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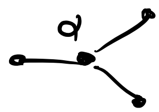
$$\dim \operatorname{Hom}(M_I(s_\beta s_\alpha\cdot\lambda), M_I(s_\beta\cdot\lambda)) = 1$$

Rmk Type D_4,  , $I = \alpha$,

We have example where $\dim \operatorname{Hom}(\text{Verma}, \text{Verma}) = 2$ (Irving 9.6 "Proj. modules in $\mathcal{O}_S$")

Recall for $\mathcal{O}$,

① $\boxed{\operatorname{Hom}(M(\lambda), M(\mu)) \neq 0 \iff \mu \uparrow \lambda}$

$\iff \mu = s_{\alpha_1}\cdots s_{\alpha_r}\cdot\lambda < \cdots < s_{\alpha_r}\cdot\lambda < \lambda$

for some $\alpha_i \in \Delta$

(2) $\dim \operatorname{Hom}(M(\lambda), M(\mu)) \le 1$

(3) Hom always injective.

For sl_3, $\nexists\ M_I(s_\beta s_\alpha \cdot \lambda) \to M_I(\lambda) \Rightarrow$ (1) Fails

$\ker(M_I(s_\beta \cdot \lambda) \to M_I(\lambda)) = L(s_\beta s_\alpha \cdot \lambda) \Rightarrow$ (3) fails

Let $\lambda, \mu \in \Lambda_I^+ : \mu \uparrow \lambda$.

$$\begin{tikzcd}
M(\mu) \arrow[r, hook, "\varphi"] \arrow[d] \arrow[dr, "\pi\circ\varphi"] & M(\lambda) \arrow[d, "\pi"] \\
M_I(\mu) \arrow[r, dashed, "\varphi_I"'] & M_I(\lambda)
\end{tikzcd}$$

$\exists$ by UP of $M_I(\mu)$ in $\mathcal{O}^p$

φ_I = Standard map

<u>Thm</u> (Lepowsky - Boe) Let $\lambda, \mu \in \Lambda_I^+$ & $\varphi_I : M_I(\mu) \to M_I(\lambda)$.

(a) φ_I may be 0 or may fail to be injective.

(b) If $\varphi_I = 0$ & $[M(\lambda) : L(\mu)] \ge 2$, there could be nonzero morphism $M_I(\mu) \to M_I(\lambda)$.

(c) $\varphi_I = 0 \iff \mu \uparrow \lambda$ by chain of weights with at least one not in Λ_I^+.

<u>Cor(c)</u> $\lambda \in \Lambda$, $w' < w$, $\ell(w) - \ell(w') = 1$,

then $\varphi_I : M_I(w'\cdot\lambda) \longrightarrow M_I(w\cdot\lambda)$ is <u>non zero</u>.

dim $L_I(\lambda)$ = dim $L_I(\mu) = 1$

Case when $M_I(\lambda), M_I(\mu)$ = "scalar type" (§ 9.11 Humphreys)

then $\operatorname{Hom}(M_I(\lambda), M_I(\mu))$ behaves as in $\mathcal{O}$. ∎