Representations of $(\mathbb{C}^2)^{\otimes n}$ and Quantum Mechanics

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Today, I hope to introduce you to a lot of interesting quantum mechanics through the lens of a representation theoretic problem.

1. Quantum information

First: what is quantum information? Quantum information is the study of how we can use quantum systems to represent and process information. A major goal, for instance, is to build a quantum computer.

The basic model in quantum information is the 'simplest' quantum system – that of a **qubit**.

Definition 1.1. (**The qubit model**). A **qubit** is a vector in $\mathbb{C}^2$. We endow $\mathbb{C}^2$ with a preferred basis, $|+\rangle$, the **spin up** state and $|-\rangle$, the **spin down state**.

A **quantum operation** on a qubit will be a unitary transformation, i.e. a matrix in $U(2)$, the group of two-by-two matrices so that $M^\dagger M = 1$.

The qubit models, for instance, the spin of an electron. Recall the electron is a spin 1/2 particle, meaning its spin lives in the defining representation $\mathbb{C}^2$ of $SU(2)$.

An electron's spin can be controlled with a magnetic field. So here, quantum operations can be encoded as suitable changes in the magnetic field.

The qubit model is useful because it is simple and allows us to use some of the intution we might have for the 'bits' of classical computing, but also because it is practical. We can build lots of spin 1/2 particles, or things that are for all intents and purposes spin 1/2 particles, like the nucleons you discussed when investigating isospin. Electrons might be too hard to control to be the best practical building blocks for a quantum computer, but lots of these other spin 1/2 particles could very well be promising options.

2. Multi-qubit systems

Lots of problems in quantum information let you apply the tools of group and representation theory. I will focus on a really basic one which allows us to discuss interesting representation theory and physics as we solve it.

Our situation. Say that someone hands us n qubits clumped together. What can we say about them? For instance, helium has two electrons. What can we say about them?

(1) Well, n qubits live in $(\mathbb{C}^2)^{\otimes n}$.
(2) From the perspective of representation theory, the best we can do is write down all the natural group actions on this space, and study how they act.
(3) **The swap action**. We could swap the qubits, if we liked, and ask how that changed our description of them. So there is a natural action of S_n, the group of permutations of n elements, which swaps the order of our qubits.

    A basis of $(\mathbb{C}^2)^{\otimes n}$ is given by n-fold tensor products of qubits, $v_1 \otimes \cdots \otimes v_n$. Here,

$$SWAP(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$

    For instance, if (12) is the element which swaps one and two,

$$SWAP((12))(v_1 \otimes v_2) = v_2 \otimes v_1$$

(4) **Quantum operation action**. We could try to apply the same quantum operation to all of them. We might be able to do better, for instance by applying different unitary gates to different qubits, but if, for instance, our qubits are all electrons really close together, as in the case of helium, that could be hard – maybe the best we can do is apply the same magnetic field to all of them.

    But, at least, we can apply the same quantum operation to all our qubits. Given a unitary matrix U, this action reads

$$QO(U)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (Uv_1) \otimes (Uv_2) \otimes \cdots \otimes (Uv_n)$$

Now, notice that the **two actions commute**. That is, $QO(U)SWAP(\sigma) = SWAP(\sigma)QO(U)$. Hence, we can upgrade these two different actions to a **single** action of the product group $U(2) \times S_n$, defined by

$$QO \times SWAP(U \times \sigma)(v_1 \otimes \cdots \otimes v_n) = QO(U)SWAP(\sigma)(v_1 \otimes \dots) = SWAP(\sigma)QO(U)(v_1 \otimes \dots)$$

So,

Goal. We would like to understand the irreducible representations of $U(2) \times S_n$ on $(\mathbb{C}^d)^{\otimes n}$, and then understand what those irreducible representations tell us about the physics of n qubits.

To understand these irreps, we'll stop talking about physics for a while, and focus on the mathematics.

3. Tensor powers

A very general and powerful duality, called **Schur-Weyl duality**, gives a beautiful and general answer to the problem. For the sake of simplicity, we'll work it out in examples.

Example 3.1. Let $n = 2$, so we're investigating the decomposition of $(\mathbb{C}^2) \otimes (\mathbb{C}^2)$ into $S_2 \times U(2)$ representations.

S_2 is abelian, so has only one-dimensional irreducible representations. There are two of them.

- The **symmetric** representation, sending $x \to x$ (This corresponds to the the diagonal subspace $x \otimes y + y \otimes x \to x \otimes y + y \otimes x$.
- The **antisymmetric** representation, sending $x \to -x$. (This corresponds to the antidiagonal subspace, $v \otimes w - w \otimes v \to w \otimes v - v \otimes w$).

The subspaces of $\mathbb{C}^2 \otimes \mathbb{C}^2$ irreducible under the action of S_2 are so important that they have names.

Definition 3.2. $S^2\mathbb{C}^2$, the **second symmetric power of** $\mathbb{C}^2$, is the subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ generated by symbols like $x \otimes y + y \otimes x$. (The subspace on which S_2 acts trivially.)

$\Lambda^2\mathbb{C}^2$, the **second exterior power of** $\mathbb{C}^2$, is the subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ generated by symbols like $x \otimes y - y \otimes x$. (The subspace on which S_2 acts as -1.)

Now, we'd like to split $\Lambda^2\mathbb{C}^2$ and $S^2\mathbb{C}^2$ into further irreducible representations of $U(2)$. But it turns out $\Lambda^2\mathbb{C}^2$ and $S^2\mathbb{C}^2$ are **already irreducible representations of** $U(2)$, namely the spin 1 one representation and the spin zero representation.

So we found a correspondence:

$$\text{antisymmetric rep of } S_2 \iff \Lambda^2\mathbb{C}^2 \text{ rep of } U(2)$$
$$\text{symmetric rep of } S_2 \iff S^2\mathbb{C}^2 \text{ rep of } U(2)$$

Example 3.3. Let n arbitrary, so we're investigating the decomposition of $(\mathbb{C}^2)^{\otimes n}$ into $S_n \times U(2)$ representations.

Recall 3.3.1. Recall that you know a lot about $SU(2)$ representations, which are effectively the same as $\mathfrak{su}_2$ representations: they are labelled by **spin**, a half-integer s. For each such s, there is a $2s+1$-dimensional representation generated by raising and lowering operators.

Clearly, $SU(2) \subset U(2)$. So for the moment we will study irreducible representations of the $SU(2)$ action on $(\mathbb{C}^2)^{\otimes n}$. How do these decompose into irreducible representations?

Example 3.4. Consider $\mathbb{C}^2 \otimes \mathbb{C}^2$. It has a natural tensor product basis $|+\rangle \otimes |+\rangle, |+\rangle \otimes |-\rangle, |-\rangle \otimes |+\rangle, |-\rangle \otimes |-\rangle$.

If $SU(2)$ acts by $U(v_1 \otimes v_2) = (Uv_1) \otimes (Uv_2)$, the Lie algebra acts by $L(v_1 \otimes v_2) = (Lv_1) \otimes v_2 + v_1 \otimes (Lv_2)$ (because of the product rule for differentiation).

So the raising operator E is no longer diagonal in the normal tensor product basis, for instance

$$E(|-\rangle \otimes |-\rangle) = (E\,|-\rangle) \otimes |-\rangle + |-\rangle \otimes (E\,|-\rangle) = |+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle$$

In fact, if we apply

$$E(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle) = 2\,|+\rangle\,|+\rangle$$

Which suggests we define a new basis on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\begin{aligned} |+\rangle\,|+\rangle \\ |+\rangle\,|-\rangle + |-\rangle\,|+\rangle \\ |-\rangle\,|-\rangle \\ |+\rangle\,|-\rangle - |-\rangle\,|+\rangle \end{aligned}$$

The first three basis elements, as just shown, form a spin 1 subspace conventionally called by physicists the **triplet**. It corresponds to $S^2\mathbb{C}^2$ in our previous example.

The state $|+\rangle\,|-\rangle - |-\rangle\,|+\rangle$ is a spin 0 subspace, called the **singlet**. It corresponds to $\Lambda^2\mathbb{C}^2$ in our previous example.

So we have:

$$\text{Spin } 1/2 \otimes \text{Spin } 1/2 = \text{Spin } 1 \oplus \text{Spin } 0$$

In general, it suffices to compute $\text{Spin } 1/2 \otimes \text{Spin k}$. Index the states of *Spin k* from $|0\rangle$, the lowest state, to $2k$, the highest state. Now, applying raising operators starting from the lowest state $|0\rangle \otimes |-\rangle$ will give you a chain of length $2k+1$, because

$$\begin{aligned} E(|0\rangle \otimes |-\rangle) &= |1\rangle \otimes |-\rangle + |0\rangle \otimes |+\rangle \\ E(|1\rangle \otimes |-\rangle + |0\rangle \otimes |+\rangle) &= |2\rangle \otimes |-\rangle + 2\,|1\rangle \otimes |+\rangle \\ E^j(|0\rangle \otimes |-\rangle) &= |j\rangle \otimes |-\rangle + j\,|j-1\rangle \otimes |+\rangle \end{aligned}$$

So inside $\text{Spin } 1/2 \otimes \text{ Spin k } \supset \text{Spin k} + 1/2$, a $2k+2$-dimensional subspace.

The lowest state not included in this chain is $|1\rangle \otimes |-\rangle - |0\rangle \otimes |+\rangle$.

We can compute

$$\begin{aligned} E(|1\rangle \otimes |-\rangle - |0\rangle \otimes |+\rangle) &= |2\rangle \otimes |-\rangle \\ E^j(|1\rangle \otimes |-\rangle - |0\rangle \otimes |+\rangle) &= |j+1\rangle \otimes |-\rangle + (j-1)\,|j\rangle \otimes |+\rangle \end{aligned}$$

So this is a chain of length $2k$. We can count dimensions:

$$dim(Spin(1/2) \otimes Spin(k)) = 2(2k+1) = 4k+2 = 2k+(2k+2) = dim(Spin(k-1/2)) + dim(Spin(k+1/2))$$

so this is everything. So

Proposition 3.5.

$$\text{Spin } 1/2 \otimes \text{Spin k} = \text{Spin k-1/2} \oplus \text{Spin k+1/2}$$

Now we can use this to compute inductively the irreducible representations of $(\mathbb{C}^2)^{\otimes n}$:

$$\begin{aligned} (\text{ Spin } 1/2)^n &= (\text{Spin } 1/2)^{n-2}(\text{Spin } 1 \oplus \text{Spin } 0) \\ &= (\text{ Spin } 1/2)^{n-3}(\text{Spin } 3/2 \oplus 2 \cdot \text{Spin } 1/2) \\ &= (\text{ Spin } 1/2)^{n-4}(\text{Spin } 2 \oplus 3 \cdot \text{Spin } 1 \oplus 2 \cdot \text{Spin } 0) \\ &= (\text{ Spin } 1/2)^{n-5}(\text{Spin } 5/2 \oplus 4 \cdot \text{Spin } 3/2 \oplus 5 \cdot \text{Spin } 1/2) \end{aligned}$$

The numbers appearing are interesting and complicated. They are given by the **Catalan triangle**, $C(n,k) = \frac{(n+k)!(n-k+1)!}{k!(n+1)!}$.

How do these representations transform under the symmetric group? We gave the simplest example earlier. Here is the next one.

Example 3.6. $Spin(1/2)^3 = Spin(3/2) \oplus 2Spin(1/2)$. How does it decompose under S^3?

- Well, S_3 has a natural 3D representation on $\mathbb{C}^3$, permuting basis vectors e_1, e_2, e_3. Is this irreducible? Notice that it preserves the sum of the coefficients: if $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ is a state, then $\lambda_1 + \lambda_2 + \lambda_3$ is left invariant. So it's reducible: we can split it as a 2D representation spanned by the subspace $\lambda_1 + \lambda_2 + \lambda_3 = 0$, which has basis $x_1 - x_2, x_2 - x_3$, and a remaining 1D trivial subspace spanned by $x_1 + x_2 + x_3$. We call the 2D representation defined here the **defining representation**.
- This splitting, it turns out, is exactly like what we have here. $Spin(3/2)$ is constructed starting with the symmetric state $|-\rangle|-\rangle|-\rangle$ and symmetric raising operators, so it **transforms trivially under the symmetric group**.
- The two $Spin(1/2)$s are formed by a preferred choice of basis: each is a spin 0 in two of the vectors and a spin 1/2 in the other. The spin 0s are antisymmetric in $e_1 - e_2, e_2 - e_3$, so it turns out they transform into each other under the defining representation of S_3! So we have

$$Spin(1/2)^3 = Spin(3/2) \oplus \text{Defining}(S_3) \otimes Spin(1/2)$$

In general, it turns out that for *any* $(\mathbb{C}^d)^{\otimes n}$, a representation of the unitary group uniquely labels a representation of the symmetric group, and there is a map between the two, called the **Schur transform**. This is called **Schur-Weyl duality**, and is used in quantum information theory to study the statistical properties of random qubits. But this story is a little too complicated for us to delve into.

4. Composite particles

However, there are lots of down-to-earth physical problems we can now solve.

Example 4.1. What is the total spin of the nucleus of molecular hydrogen? Note, the nucleus is composed of two protons, and protons are spin 1/2.

The answer is that the spin can be 1 or 0. **Parahydrogen** is hydrogen with a spin 0 nucleus. **Orthohydrogen** is hydrogen with a spin 1 nucleus.

For **small** molecules and atoms, the energy coming from spin is roughly

$$E_S \propto S(S+1)$$

where S denotes the total spin. Warning, this is just an approximation which gets poor very quickly.

So it turns out that parahydrogen has slightly lower energy than orthohydrogen, by about $1.06kJ/mol$.

Example 4.2. Covalent bonding means when elements bind together by sharing electrons. For instance, the hydrogen molecule is formed by two electrons shared between two nuclei.

Because, by assumption, the two nuclei 'share' the electrons, we shouldn't be able to distinguish between them. So they should live in a spatially trivial representation of S_2.

Remember that due to the **spin statistics**-theorem, when I swap two fermions – particles of noninteger spin– in space, I pick up a $-$ sign. So we actually want the hydrogen electron to live in an *antisymmetric* representation of S_2. The two minus signs will then cancel each other. That means the electrons are in a spin 0 representation.

Example 4.3. Why does helium not like to form covalent bonds with hydrogen? Here's a super over-simplified explanation. It has one proton and two electrons. The lowest energy state for the electrons of helium is for them to live in a spin 0 state.

If I add an electron,

$$Spin(1/2) \otimes Spin(0) = Spin(1/2)$$

So the electron can't pair up simply to form a spin 0 state. One of them would have to be excited to live in a higher spin state, which is energetically costly.

5. Entanglement

Remark 5.1. Physicists say that the states $|+\rangle|-\rangle + |-\rangle|+\rangle$, $|+\rangle|-\rangle - |-\rangle|+\rangle$ are **entangled states** because they cannot be decomposed into a tensor product $u \otimes v$, where u is a state of the first qubit and v is a state of the second qubit.

On the other hand, $|+\rangle|+\rangle$ and $|-\rangle|-\rangle$ are **unentangled states**.

The existence of entangled states is weird. Two electrons can be distant from each other, but still 'interact' with each other through entanglement. This is sometimes colourfully called 'spooky action at a distance'.