Representation Theory and Physics
SHP Fall '19

Henry Liu

Last updated: February 1, 2020

Contents

0 Symmetries

Many mathematical objects and physical systems possess *symmetries*. A circle stays the same no matter how it is rotated; a rotation by θ for any angle θ is therefore a symmetry of the circle. On the other hand, a square stays the same only under rotation by multiples of π. From this simple example we see that, broadly, symmetries should be separated into two types.

1. The rotation symmetry of the circle is **continuous**: one can start with the unrotated circle and apply a given rotation by θ in a continuous fashion, without affecting the circle.

2. The rotation symmetry of the square is **discrete**: one *cannot* get from an unrotated square to a square rotated by (some multiple of) π in a continuous fashion.

In real life, common continuous symmetries include translations and rotations. Discrete symmetries are less obvious, but include time reversal (flipping the arrow of time), charge conjugation (swapping what we call positive vs negative charge), and translations in lattices (like for crystals/metals). It is important to study both continuous and discrete symmetries. The study of symmetry, in mathematics, is called **representation theory**.

Once we understand the symmetries of an object, the powerful machinery of representation theory kicks in and allows us to draw marvelous conclusions about the object itself. This is especially useful in physics, where often the symmetries are more obvious/intuitive than whatever conclusions we draw from them.

Example 0.1. The three-dimensional space we live in has translation and rotation symmetries. Then Noether's theorem, which we will see later, immediately implies the conservation of momentum and energy. Together with reflection symmetries, these symmetries form what is called the "Euclidean group" of symmetries of three-dimensional space.

Example 0.2. Three-dimensional space belongs to four-dimensional *spacetime*. In spacetime, it turns out there are additional symmetries which mix space and time called "Lorentz transformations". The statement that spacetime has these extra symmetries is the *only* postulate underlying the entire theory of special relativity. Putting the Lorentz transformations together with the usual Euclidean symmetries gives the "Poincaré group" of symmetries of four-dimensional spacetime.

Note that all these symmetries we just stated are continuous symmetries. Indeed, because many fundamental objects in physics are *continuous* objects, many of the interesting applications of representation theory to physics involve continuous symmetries. However continuous symmetries are more difficult to study than discrete symmetries. Hence we will begin with discrete symmetries, which are slightly less physically relevant, in order to familiarize ourselves with the basic objects of representation theory.

1 Groups

The first step in representation theory is to understand the structure of the set of symmetries of a given object. This set, which we'll call G, has some very special structure, which we'll discuss abstractly now. First, if g_1 and g_2 are two symmetries in G, then

> applying g_1, then applying g_2, is itself a symmetry of the object.

We'll denote this composite symmetry by $g_2 g_1$. (In this notation, we apply symmetries from right to left, e.g. g_1 is applied first. This is just a notational choice.) So the **composition** $g_2 g_1$ of two symmetries is still a symmetry, and therefore still belongs to the set G. Second,

> applying a symmetry *in reverse* is still a symmetry.

In other words, if there is a symmetry g which takes the object from state A to state B, then there is an inverse symmetry which takes the object from state B back to state A. We'll denote this inverse symmetry by g^{-1}. Finally, there is always a *trivial* symmetry, obtained by doing nothing to the object. The operation of doing nothing is always a symmetry, by definition.

Most sets do not have these two interesting structures, but we see that sets of symmetries always do. So, in order to study symmetries, we give a name to sets with such structures: they are called *groups*.

1.1 Definitions and first examples

Definition 1.1. A **group** G is a set that has a **group operation** $\star$. More precisely, this means that for any two elements a and b in G, we can apply the operation $\star$ to them to obtain an element $a \star b$. This operation must satisfy some axioms:

1. there must be an **identity element** e of G such that $e \star x = x$ for all x;
2. every element x must have an **inverse**, i.e. an element x^{-1} such that $x \star x^{-1} = e$;
3. (technical) the group operation must be **associative**, i.e. $(a \star b) \star c = a \star (b \star c)$.

It is common to call the inverse x^{-1} because we often pretend the group operation is "multiplication" and refer to the group operation as a "product".

Many familiar objects that do not necessarily arise from the study of symmetries have group structures, with various group operations. It is important to note that, although the notation we use for abstract groups is "multiplicative", sometimes the group operation may be addition, or some other operation. So we often write $(G, \star)$ to mean a group G with the group operation $\star$, to make it clear what the group operation is. When it is clear from context, we sometimes just refer to the group as G.

Example 1.2. The set of integers, called $\mathbb{Z}$, forms a group using addition as the group operation. (To be precise, we should write $(\mathbb{Z}, +)$.) Clearly, given two integers x and y, their sum $x + y$ is still an integer.

1. The identity element is 0, because $0 + x = x$ for any integer x.
2. The operation of addition is associative, because $(x + y) + z = x + (y + z)$ for all integers x, y, z.
3. The inverse of an integer x is the integer $-x$ (which always exists), because $x + (-x) = 0$.

Exercise. Show that $\mathbb{Z}$ with multiplication as the group operation is *not* a group. Is it possible to "fix" $\mathbb{Z}$ so that it is?

Exercise. Let $\mathbb{Z}/n$ denote the group of **integers modulo** n, using addition modulo n as the group operation. In other words, it is the set $\{0, 1, 2, \ldots, n-2, n-1\}$ where the result of the group operation on a and b is the remainder of $a + b$ upon dividing by n. Check that $\mathbb{Z}/n$ is a group.

Example 1.3. Given an object, its **symmetry group** is the group of all symmetries of the object, using composition as the group operation. The identity element e for this operation is always the symmetry which takes the object and does nothing to it; every object clearly has such a symmetry. The inverse of a symmetry is the symmetry "in reverse".

There are many structural properties which are already illustrated by these examples. For example, groups whose elements are numbers usually have the following very special property. It is important to emphasize that most groups, particularly symmetry groups, do *not* have this property!

Definition 1.4. A group G is **abelian** if

$$x \star y = y \star x$$

for every x and y in G. We say the group operation is **commutative**.

We also want to speak about the size of groups, namely how many elements they contain. It is possible of course for a group to contain infinitely many elements, like $\mathbb{Z}$, so we usually only talk about the size of *finite* groups.

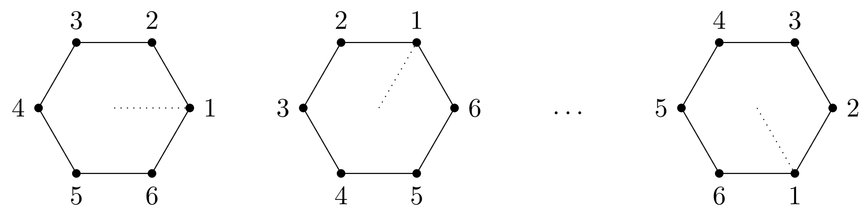
Definition 1.5. The number of elements, or **cardinality** or **order**, of a group G is written $|G|$. We say G is **finite** or **infinite** depending on its cardinality.

1.2 The dihedral group

One simple yet very interesting example of a symmetry group is the symmetry group of a regular polygon with n (equal) sides. Its symmetry group is called the **dihedral group**, and written D_n. To reduce confusion when studying D_n, it is best to label each corner of the polygon with a number, to keep track of what each symmetry does.

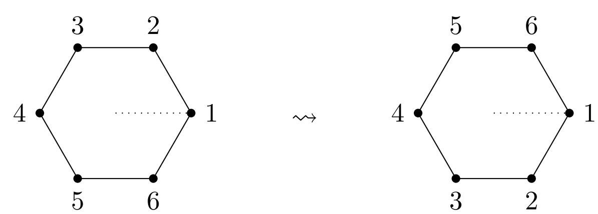
The first step in understanding D_n is to identify some of its elements, and to give names to them.

1. There are n different symmetries obtained by rotation.



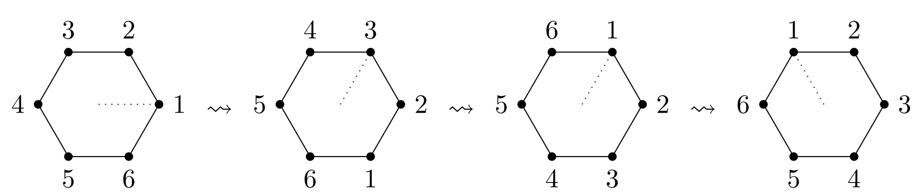
The first one is the identity element e. If we call the second one r, note that the other rotations are just compositions of r with itself. So the rotation symmetries are
$$e, r, r^2, r^3, \ldots, r^{n-1}.$$
Note that $r^n = e$, which is the statement that rotating a full 360° is the same as not doing anything. From this we can tell that $r^{-1} = r^{n-1}$.

2. There is a symmetry given by flipping the polygon across a fixed axis, which we'll take to be the x-axis for simplicity.



Call this symmetry s. Note that $s^2 = e$, since flipping twice is the same as not doing anything.

What about flips across other lines? In the same way that all rotations are obtained by compositions of r, those other flips may be obtained by an appropriate composition of r and s. For example, for the hexagon, flipping across the line between 2 and 5 is the same as the composition rsr^{-1}.



Exercise. Check that rs is *not* the same symmetry as sr, and therefore conclude that the dihedral group is *not* abelian.

Exercise. Check that $rs = sr^{-1}$. Conclude that $r^k s = sr^{-k}$ for any integer k.

We can use this last exercise to obtain a full description of the dihedral group as follows. Suppose we are given a complicated composition of r and s, like

$$r^{27} srsr^8 s^3.$$

Any such expression can be simplified into the form r^k or sr^k for some integer k using the following two steps.

1. Use that $r^n = e$ and $s^2 = e$ to simplify the exponents.

2. "Move" all the occurrences of s to the front using $r^k s = sr^{-k}$.

Example 1.6. Let's simplify $r^{27} srsr^8 s^3$ for the hexagon. Since $r^6 = e$ and $s^2 = e$, we get

$$r^{27} srsr^8 s^3 = r^3 srsr^2 s.$$

Then we move the first s to the front:

$$(r^3 s) rsr^2 s = (sr^{-3}) rsr^2 s = sr^{-2} sr^2 s.$$

Moving the second s now gives

$$s(r^{-2} s) r^2 s = s(sr^2) r^2 s = r^4 s.$$

Finally, moving the last s gives

$$r^4 s = sr^{-4} = sr^2.$$

So even though we can write down very complicated compositions of rotations and flips, after simplifying we see that D_n actually only contains $2n$ elements:

1. n rotations $e, r, r^2, \ldots, r^{n-1}$;

2. n rotations-with-a-flip $s, rs, r^2 s, \ldots, r^{n-1} s$.

This makes a lot of sense, because any symmetry of the n-gon must take the corner labeled 1 to some position. We can use rotations to place the 1 there. Then we are left with only two possibilities: either the numbers of corners adjacent to the 1 are already correct, in which case we have identified the symmetry as r^k for some k, or the numbers are flipped, in which case we apply an extra flip to get sr^k.

Definition 1.7. Any element in the dihedral group can be written as a composition of r and s, so we say D_n is **generated by** r and s. The rules we impose on how multiple r and s interact are called **relations**, and we identified three:

$$r^n = e, \quad s^2 = e, \quad rs = sr^{-1}.$$

A full description of D_n is given by a **presentation** using generators and relations, written

$$D_n = \langle r, s \mid r^n = e,\ s^2 = e,\ rs = sr^{-1} \rangle.$$

1.3 The symmetric group

A more complicated example of a symmetry group is the symmetry group of n indistinguishable objects, e.g. point particles. Such objects may be permuted in any order, and all permutations are symmetries. We label the objects from 1 to n, in which case permutations look like

$$\underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \underset{3}{\bullet} \quad \underset{4}{\bullet} \quad \underset{5}{\bullet} \quad \underset{6}{\bullet} \qquad \rightsquigarrow \qquad \underset{3}{\bullet} \quad \underset{2}{\bullet} \quad \underset{6}{\bullet} \quad \underset{4}{\bullet} \quad \underset{1}{\bullet} \quad \underset{5}{\bullet}$$

This symmetry group is called the **symmetric group**, and written S_n. We can immediately note that it consists of $n!$ elements. One way to write elements is to just list the permuted labels under the original labels, like

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 4 & 1 & 5 \end{pmatrix}$$

for the above example. (We will see however that writing elements like this isn't the best way to uncover the hidden structures in S_n.)

We can ask for a generators-and-relations presentation of S_n like we did for D_n, and the first step is to identify some special kinds of elements and give names to them.

1. For any two labels i and j, we can consider the permutation which swaps i and j and leaves everything else alone. Such permutations are called **transpositions**, and are written (i, j).

2. More generally, given a sequence of labels $i_1, i_2, \ldots, i_m$, we can consider the permutation which sends i_1 to i_2, and i_2 to i_3, and so on, and i_m back to i_1. Such permutations are called **cycles**, and are written $(i_1, i_2 \ldots, i_m)$. The **length** of a cycle is the number of items involved in it.

Theorem 1.8. *S_n is generated by transpositions.*

Proof. Given any permutation σ in S_n, if we can sort out its items in increasing order using just transpositions (to get to the identity element e), then the inverse sequence of transpositions is equal to σ. But sorting is easy: the first transposition should swap the first element in σ with 1, the second should then swap the second element with 2, etc. $\square$

Exercise (Hard)**.** Show that S_n is actually generated by *adjacent transpositions* $\sigma_i = (i, i+1)$ for $1 \leq i < n$, and that their compositions are governed by the relations

- $\sigma_i^2 = e$ for all i;
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ when $|i - j| > 1$;
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all i.

Because of the theorem, it is useful to write elements of S_n as compositions of transpositions. But this can often become cumbersome to write. Instead, we write them as compositions of *cycles*, due to the following exercise.

Exercise. Show that cycles are just shorthand for compositions of transpositions, because

$$(i_1, i_2, \ldots, i_m) = (i_1, i_2)(i_2, i_3) \cdots (i_{m-1}, i_m).$$

To decompose a given permutation σ into a product of cycles, it is easiest to start with the label 1 and write down the sequence $1, \sigma(1), \sigma(\sigma(1)), \ldots$ until we return to 1; this forms a cycle. Then take the next smallest label not included in this cycle, and form a new cycle starting with it, and so on. Note that sometimes there will be cycles of length 1, which we *omit* writing.

Example 1.9. Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 1 & 6 & 2 & 5 & 7 \end{pmatrix}$ in S_7.

1. There is a cycle $1 \to 3 \to 1$. This is written $(1, 3)$.
2. The next smallest number not involved in a cycle so far is 2. There is a cycle $2 \to 4 \to 6 \to 5 \to 2$. This is written $(2, 4, 6, 5)$.
3. The next smallest number not involved in a cycle so far is 7. There is a cycle $7 \to 7$. This is a length-1 cycle and we do not write it.
4. There are no more labels not involved in a cycle, so we are done.

Hence $\sigma = (1, 3)(2, 4, 6, 5)$.

Note that it does not matter which order we compose *disjoint* cycles, i.e. cycles that involve no common labels. Disjoint cycles commute with each other.

1.4 Homomorphisms

Now we return to discussing groups more abstractly. Given a group G, it is conceptually helpful to consider its "multiplication" table, where we write down all products of elements in the group. The convention we will use is to multiply the row element by the column element, not vice versa.

Example 1.10. The symmetric group S_2 (of two objects) has two elements, with the following multiplication table.

	e	$(1,2)$
e	e	$(1,2)$
$(1,2)$	$(1,2)$	e

Example 1.11. The group $\mathbb{Z}/2$ (of integers mod 2) also has two elements, with the following multiplication table.

	0	1
0	0	1
1	1	0

Note that, in some sense, we've written the same multiplication table twice but with elements renamed. The way to translate between S_2 and $\mathbb{Z}/2$ while preserving the multiplication table is

$$e \leftrightarrow 0, \quad (1,2) \leftrightarrow 1.$$

Using this dictionary, the two groups are actually *equivalent*. This notion of equivalence is expressed mathematically as follows.

Definition 1.12. Let G and H be two groups, with group operations $\star_G$ and $\star_H$. We say G and H are **isomorphic**, written

$$G \cong H,$$

if there exists a function $f\colon G \to H$ which:

1. is a **bijection**, i.e. a one-to-one correspondence between the elements of the two sets;
2. is a **homomorphism**, meaning that

$$f(a \star_G b) = f(a) \star_H f(b).$$

If we view f as a "dictionary" between elements of G and H, being a homomorphism means that the dictionary is compatible with the group operations in G and H, and being an isomorphism means the dictionary covers all elements of G and H.

Exercise. Show that D_3 is isomorphic to S_3.

Exercise. Show that D_n cannot be isomorphic to S_n for $n > 3$, using cardinality.

Importantly, it is possible for $f\colon G \to H$ to be a homomorphism without being an isomorphism. One trivial way is to send everything in G to the identity element e_H in H. Then clearly

$$f(a \star_G b) = e_H = f(a) \star_H f(b).$$

Example 1.13. Consider the map $f\colon \mathbb{Z}/2 \to D_3$ given by

$$0 \mapsto e, \quad 1 \mapsto s.$$

This is not an isomorphism because $\mathbb{Z}/2$ is much smaller than D_3. But it *is* a homomorphism. The most important check is

$$f(1+1) = e = s^2 = f(1)f(1).$$

The existence of this homomorphism means that there is a copy of $\mathbb{Z}/2$ hiding inside D_3.

Definition 1.14. A subset $H \subset G$ which itself is a group is called a **subgroup** of G. This is written $H \leq G$.

Exercise. Show that, in D_3, the elements e, r, r^2 form a subgroup isomorphic to $\mathbb{Z}/3$. On the other hand, show that e, r, sr^2 does not form a subgroup. Are there any other subgroups of D_3 that we haven't found yet?

Suppose we want to specify a homomorphism $f\colon G \to H$, and G has generators a, b, and c. Then it is actually enough to specify what $f(a)$, $f(b)$, and $f(c)$ are. This is because any element in G can be written as some product of a, b, and c, and therefore e.g.

$$f(a^7 b^{11} c^{-3}) = f(a)^7 f(b)^{11} f(c)^{-3}.$$

So a homomorphism is fully specified by what it does to generators.

Example 1.15. A homomorphism $\phi\colon \mathbb{Z} \to \mathbb{Z}$ is completely determined by the integer $\phi(1)$. This is because

$$\phi(n) = \phi(\underbrace{1 + \cdots + 1}_{n \text{ times}} = \underbrace{\phi(1) + \cdots + \phi(1)}_{n \text{ times}} = n\phi(1).$$

We also speak about generators of a *subgroup*. For example, the set of even integers forms a subgroup of $\mathbb{Z}$. It is often written $2\mathbb{Z}$, because it is generated by the element 2.

1.5 Operations on groups

Whenever we define a type of mathematical object (e.g. a group) along with some notion of equivalence (e.g. isomorphism of groups), we can start asking about *classification*. Namely,

> can we classify all the different objects of this type?

If the answer turns out to be yes, then usually the result is that every such object is built from a small collection of basic building blocks. In our case, this means we require a way to build a bigger group using two smaller ones.

Definition 1.16. Given two groups G and H, their **product** $G \times H$ is the group whose elements are pairs (g, h) with $g \in G$ and $h \in H$, and group operation given by

$$(g_1, h_1) \star (g_2, h_2) = (g_1 \star_G g_2, h_1 \star_H h_2).$$

Example 1.17. The group $\mathbb{Z}/2 \times \mathbb{Z}/2$ has elements

$$\{(0,0), (0,1), (1,0), (1,1)\}$$

and multiplication table

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

Note that even though it has order 4, it is *not* isomorphic to $\mathbb{Z}/4$. One way to see this is that every element in $\mathbb{Z}/2 \times \mathbb{Z}/2$ becomes zero when added to itself, but this is not true for every element of $\mathbb{Z}/4$.

Exercise. Show that $\mathbb{Z}/2 \times \mathbb{Z}/3$ is isomorphic to $\mathbb{Z}/6$.

Exercise. Show that $\mathbb{Z}/n \times \mathbb{Z}/m$ is isomorphic to $\mathbb{Z}/nm$ whenever $\gcd(n,m) = 1$. Hint: construct an isomorphism

$$\phi\colon \mathbb{Z}/nm \to \mathbb{Z}/n \times \mathbb{Z}/m$$

by picking wisely what $\phi(1)$ should be.

1.6 Classification

Now we can return to the problem of classifying different types of groups. The simplest type we can start thinking about are the *finite* and *abelian* ones. It is clear that $\mathbb{Z}/n$ and products of $\mathbb{Z}/n$'s are finite abelian groups, while $\mathbb{Z}$ (infinite) and S_3 (non-abelian) are not. If we start listing the non-isomorphic finite abelian groups of small order, it turns out we get

cardinality	non-isomorphic groups
1	1
2	$\mathbb{Z}/2$
3	$\mathbb{Z}/3$
4	$\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$
5	$\mathbb{Z}/5$
6	$\mathbb{Z}/2 \times \mathbb{Z}/3$
7	$\mathbb{Z}/7$
8	$(\mathbb{Z}/2)^3$ and $\mathbb{Z}/2 \times \mathbb{Z}/4$ and $\mathbb{Z}/8$
$\vdots$	$\vdots$

It is not obvious why they are all products of $\mathbb{Z}/n$'s. In fact this empirical observation is true in general.

Theorem 1.18 (Classification of finite abelian groups)**.** *Any finite abelian group is isomorphic to*

$$\mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k$$

for some integers $n_1, \ldots, n_k \geq 2$ which are all prime powers.

Even though this only classifies a very special type of group, the proof of this theorem is already somewhat intricate, and we will skip it. The complexity of group theory is evident even in this special case.

We can go further and now ask for a classification of *all* finite groups, regardless of whether they are abelian. In the non-abelian case it turns out there are different ways to "take the product" of two groups, called *semidirect products*. So trying to decompose a group G into a product is not the best approach. Instead, we can write a **composition series** for G. This is a sequence

$$1 = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_n = G$$

of normal subgroups such that H_i is a *largest possible* normal subgroup of H_{i+1}. Equivalently, H_{i+1}/H_i is a simple group.

Definition 1.19. A group G with no normal subgroups aside from 1 and itself is called **simple**.

Finite simple groups are the building blocks for finite (non-abelian) groups. Unlike the abelian case, where the building blocks have a nice classification, the classification of finite simple groups involves 18 infinite families and 26 sporadic groups. The classification was a major mathematical milestone, "completed" in February 1981 (with some minor holes that were patched by 2004). The complete proof of the classification spans over 10,000 pages and is spread out across 500 or so papers. There is a current ongoing project to simplify and coalesce the proof into a 12-volume series, expected in 2023.