MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

PRACTICE FINAL SOLUTIONS

The exam is **170 minutes** (1:10pm – 4:00pm). No additional material or calculators are allowed. There are 100 points in total,

- Write your **name and UNI** clearly on your exam booklet.
- **Show your work** and reasoning, not just the final answer. Partial credit will be given for correct reasoning, even if the final answer is completely wrong.
- **Don't cheat**!
- Don't panic!

(1) (10 points) State whether the following are true/false. No explanation necessary.

(a) An anti-derivative of a polynomial is always a polynomial.

Solution. $\boxed{\text{True}}$. For $n \geq 0$, the antiderivative of x^n is $x^{n+1}/(n+1)$, which is a polynomial.

(b) Given a function f, there is exactly one function F such that $F'(x) = f(x)$.

Solution. $\boxed{\text{False}}$. You can add any constant C to the antiderivative F without affecting its derivative.

(c) If f and g are continuous functions,

$$\int f(x)g(x)\,dx = \left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right).$$

Solution. $\boxed{\text{False}}$. Just like how the product rule tells us $(fg)' \neq f'g'$, the integral of a product is not the product of the integrals. For example, take $f(x) = g(x) = 1$.

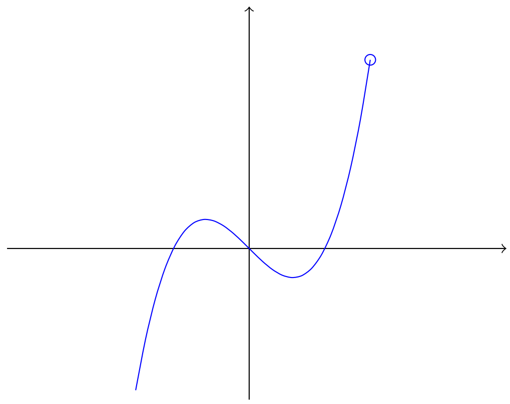
(d) An alternate but equivalent way to define the derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

Solution. $\boxed{\text{True}}$. The usual notion of a secant line at x involves fixing the point $(x, f(x))$ and looking at a point $(x+h, f(x+h))$ which approaches it. We can modify the first point to move as well, e.g. $(x-h, f(x-h))$. Then the slope of the resulting secant line is the expression above, and in the limit becomes the slope of the tangent line at x, which is $f'(x)$.

(e) It is possible for a continuous function on (a, b) to have a local maximum but not a global maximum.

Solution. $\boxed{\text{True}}$. For example, take a function like



which has no global max as drawn but has one local max.

(2) Compute the limit.

(a) (5 points)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \cos(1 + k/n).$$

Solution. The summation along with $\lim_{n\to\infty}$ should suggest that this limit is a definite integral. Since there is an overall factor of $1/n$, we can guess that the rectangle widths are $1/n$. In the sum, we are plugging k/n into the function $f(x) = \cos(1 + x)$. So it makes sense to guess the interval is $[0, 1]$, and that we are plugging in right endpoints. The limit becomes

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \cos(1 + k/n) = \int_0^1 \cos(1 + x)\, dx = \boxed{\sin(2) - \sin(1)}.$$

(b) (5 points) $\lim_{z\to 0} z^3 \cos^{99}(2/z)$

Solution. Use the squeeze theorem. Since $-1 \leq \cos \leq 1$, we have

$$-z^3 \leq z^3 \cos^{99}(2/z) \leq z^3.$$

Since $\lim_{z\to 0} \pm z^3 = 0$, the middle limit must also be $\boxed{0}$.

(c) (5 points)

$$\lim_{h\to 0} \frac{(x + h)^{10} - x^{10}}{h}$$

Solution. This limit is exactly the definition of a derivative:

$$\lim_{h\to 0} \frac{(x + h)^{10} - x^{10}}{h} = \frac{d}{dx} x^{10} = \boxed{10x^9}.$$

(d) (5 points) $\lim_{u\to 3}\left(f(u)g(-u)\right)^{10}$ where

$$f(u) = \begin{cases} 1/(u-3) & u < 4 \\ u^2 & u \geq 4 \end{cases}, \quad g(u) = \begin{cases} -2u-6 & u < 0 \\ \sin(u) & u \geq 0 \end{cases}.$$

Solution. When u is close to 3, we have

$$f(u) = \frac{1}{u-3}, \quad g(-u) = -2(-u) - 6 = 2u - 6.$$

The other pieces don't matter when we just want to take the limit. So the limit becomes

$$\lim_{u\to 3}\left(f(u)g(-u)\right)^{10} = \lim_{u\to 3}\left(\frac{2u-6}{u-3}\right)^{10} = \lim_{u\to 3} 2^{10} = \boxed{2^{10}}.$$

(e) (5 points) $\lim_{t\to\infty} t^{3/2}\cos(1/t)$

Solution. This is *not* an indefinite form. As $t \to \infty$, we have $1/t \to 0$. So $\cos(1/t) \to \cos(0) = 1$. Hence

$$\lim_{t\to\infty} t^{3/2}\cos(1/t) = \lim_{t\to\infty} t^{3/2} = \boxed{\infty}.$$

(3) Compute the derivative dy/dx.

(a) (5 points)

$$y = \int_{-x}^{x^2} \arctan(t^3)\, dt$$

(Hint: $\int_{-x}^{x^2} = \int_{-x}^{0} + \int_0^{x^2}$.)

Solution. Use the hint so that we can apply the fundamental theorem of calculus. Since $\int_{-x}^{0} = -\int_0^{-x}$, we have

$$y = -\int_0^{-x} \arctan(t^3)\, dt + \int_0^{x^2} \arctan(t^3)\, dt.$$

Then apply FToC, being mindful of the chain rule:

$$\begin{aligned} y' &= -\arctan((-x)^3)\cdot(-1) + \arctan((x^2)^3)\cdot 2x \\ &= \boxed{\arctan(-x^3) + 2x\arctan(x^6)}. \end{aligned}$$

(b) (5 points) $y = (\tan x)^{1/x}$

Solution. This has both base and exponent depending on x, so use logarithmic differentiation. First take logs:

$$\ln y = \frac{1}{x}\ln\tan x.$$

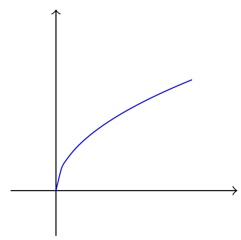
Then implicitly differentiate:

$$\frac{1}{y}y' = -\frac{1}{x^2}\ln\tan x + \frac{1}{x}\frac{1}{\tan x}\sec^2 x.$$

Move y to the other side (and simplify a little) to get

$$y' = \boxed{(\tan x)^{1/x}\left(-\frac{\ln\tan x}{x^2} + \frac{1}{x\sin x\cos x}\right)}.$$

(4) You are making a bowl by rotating the curve $y = x^{1/4}$ around the x-axis.



(a) (5 points) If your bowl has height h, what is the total volume of water it can hold?

Solution. The water in the bowl is the solid of revolution of $y = x^{1/4}$ on $[0, h]$. It has volume

$$V = \int_0^h \pi(x^{1/4})^2\,dx = \pi\int_0^h x^{1/2}\,dx = \boxed{\pi\frac{2}{3}h^{3/2}}.$$

(b) (5 points) You fill the bowl with water at a rate of 4 cm^3/s. Using (a), how fast is the water level increasing when the water level is $h = 1$ cm?

Solution. Differentiate the formula $V = (2\pi/3)h^{3/2}$ with respect to time t to get:

$$\frac{dV}{dt} = \pi h^{1/2}\frac{dh}{dt}.$$

So if $dV/dt = 4$ and $h = 1$, then $dh/dt = \boxed{4/\pi}$ cm/s.

(5) (5 points) Find the equation of the tangent line to $4x^2 + y^2 = 8$ at $(x, y) = (1, 2)$.

Solution. First find the slope y'. Use implicit differentiation to get

$$8x + 2y \cdot y' = 0$$

So $y' = -4x/y = -2$ is the slope of the tangent line. So the tangent line has the form $y = -2x + C$. Since it passes through the point $(1, 2)$, we get $2 = -2 \cdot 1 + C$, i.e. $C = 4$. The equation of the line is therefore $\boxed{y = -2x + 4}$.

(6) (10 points) Use linear approximation to explain why for x close to zero, $\sin x \approx x$ and $\cos x \approx 1$. Use this to estimate

$$\int_{-0.1}^{0.1} e^{(\sin x)^3}(\sin x)^2(1 + \ln(\cos x))\,dx.$$

Solution. Using linear approximation for $x \approx 0$,

$$\sin(x) \approx \sin(0) + \cos(0) \cdot x = x$$
$$\cos(x) \approx \cos(0) + (-\sin(0)) \cdot x = 1.$$

In the integral, x is close to 0. So everywhere we see $\sin x$ we can replace it with x, and everywhere we see $\cos x$ we can replace it with 1. This gives

$$\int_{-0.1}^{0.1} e^{(\sin x)^3} (\sin x)^2 (1 + \ln(\cos x))\, dx \approx \int_{-0.1}^{0.1} e^{x^3} x^2\, dx.$$

Use the substitution $u = x^3$, so that $(1/3)du = x^2\, dx$:

$$\int_{-0.1}^{0.1} e^{x^3} x^2\, dx = \frac{1}{3}\int_{-0.001}^{0.001} e^u\, du = \boxed{\frac{1}{3}(e^{0.001} - e^{-0.001})}.$$

(7) Compute the integral.

(a) (5 points)

$$\int_{-2}^{2} e^{x^2} \sin(x)^3\, dx$$

Solution. The entire integrand is odd. This is because x^2 is even, and $\sin(x)$ is odd and therefore $\sin(x)^3$ is as well. Just to check:

$$e^{(-x)^2} \sin(-x)^3 = e^{x^2}(-\sin(x))^3 = -e^{x^2} \sin(x)^3.$$

So the integral is $\boxed{0}$.

(b) (5 points)

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx$$

Solution. This is the area under a semicircle of radius 2. So the answer is $(1/2)\pi \cdot 2^2 = \boxed{2\pi}$.

(8) (10 points) Sketch the graph of $y = x^4 - 8x^2 + 8$ by finding its critical points, inflection points, and regions of convexity/concavity.

Solution. We will need the first and second derivatives

$$y' = 4x^3 - 16x = 4x(x-2)(x+2)$$
$$y'' = 12x^2 - 16 = 4(3x^2 - 4).$$

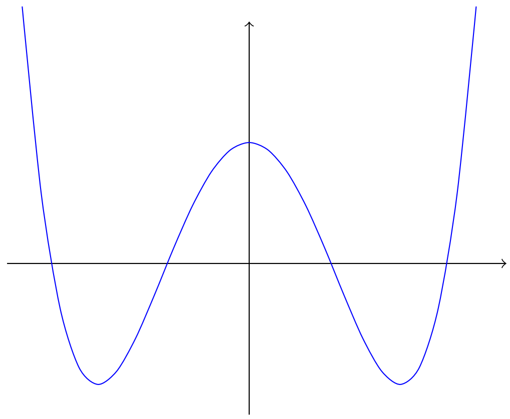
So the critical points are $x = 0$ and $x = \pm 2$, and inflection points are $x = \pm 2/\sqrt{3}$.

(a) In the region $(-\infty, -2/\sqrt{3})$, the second derivative is positive, since $x^2 \to \infty$. So it is concave up here. The critical point $x = -4$ is in this region.

(b) In the region $(-2/\sqrt{3}, 2/\sqrt{3})$, the second derivative is negative, e.g. by plugging in $x = 0$. So it is concave down here.

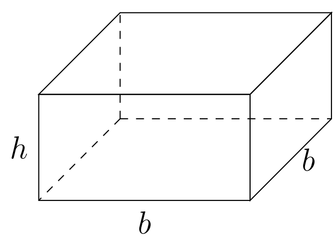
(c) In the region $(2/\sqrt{3}, \infty)$, the second derivative is positive, since $x^2 \to \infty$. So it is concave up here. The critical point $x = 4$ is in this region.

This is enough information to sketch the graph.



(9) (10 points) Now that the course is over, you would like to build a box to hold all your course materials for disposal. You have 10 m^2 of wood to make the box. Because you have accumulated so much material, you would like to maximize the volume of the box. For aesthetic reasons, you want the box to have a square base. What is the maximum possible volume?

Solution. Let b be the width/length of the base, and h be the height of the box.



You want to maximize the volume b^2h. The constraint is that the total surface area $2b^2 + 4hb = 10$. Using the constraint, eliminate the variable h, to get that you want to maximize

$$b^2h = \frac{5}{2}b - \frac{1}{2}b^3.$$

Its derivative is $(5/2) - (3/2)b^2$, so there is a critical point at $b = \pm\sqrt{5/3}$. Since b is a width/length, it must be positive, so

$$b = \sqrt{5/3}.$$

Plug this back into $2b^2 + 4hb = 10$ and solve for h to get

$$h = \sqrt{5/3}.$$

(This makes sense: a cube should maximize volume.) So the maximum possible volume is

$$b^2h = \boxed{(5/3)^{3/2}}\ \mathrm{m}^3.$$