MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 2 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 15 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Compute the following limits. If the limit does not exist, state why. Briefly show and explain work.

(a)

$$\lim_{x\to 3} \frac{x^2-6x+7}{x-3}.$$

(Hint: complete the square.)

Solution. Following the hint, the function becomes

$$\lim_{x\to 3} \frac{(x^2-6x+9)-2}{x-3} = \lim_{x\to 3} \left(\frac{(x-3)^2}{x-3} - \frac{2}{x-3} \right).$$

The first term is $\lim_{x\to 3}(x-3) = 0$ and the second term tends to $\pm\infty$ (depending on which side you approach from). So the whole limit $\boxed{\text{does not exist}}$.

Another way to see this is to notice that the numerator as $x \to 3$ is just -2, but the denominator is tending toward 0. So the quotient tends toward $\pm\infty$. In practice, this is how you should think about such limits. Notice that the completing-the-square procedure is a more precise version of this argument.

(b)

$$\lim_{z\to 2} \frac{\sqrt{4z+1}-3}{z-2}$$

Solution. Here you can't directly plug in $z = 2$, because you get the indeterminate form $0/0$. So, rationalize the square root using the trick we discussed in class:

$$\frac{\sqrt{4z+1}-3}{z-2} = \frac{\sqrt{4z+1}-3}{z-2}\frac{\sqrt{4z+1}+3}{\sqrt{4z+1}+3} = \frac{(4z+1)-9}{(z-2)(\sqrt{4z+1}+3)} = \frac{4}{\sqrt{4z+1}+3}.$$

Now to take $\lim_{z\to 2}$ of this we can just plug $z = 2$ in. Doing so, we get the answer $\boxed{2/3}$.

(c)

$$\lim_{t\to -2} \frac{8-|t|^3}{2+t}$$

(Hint: use the identity $x^3+y^3=(x+y)(x^2-xy+y^2)$.)

Solution. Again, you can't directly plug in $t=-2$ because you get the indeterminate form $0/0$. But remember that $|t|$ is a piecewise function. For all t around $t=-2$, we have $|t|=-t$. Making this substitution, the limit becomes

$$\lim_{t\to -2} \frac{8-(-t)^3}{2+t} = \lim_{t\to -2} \frac{8+t^3}{2+t} = \lim_{t\to -2}(4-2t+t^2) = \boxed{12}.$$

The second equality follows from the hint.

(d)

$$\lim_{x\to 0} x\cos\left(\frac{1}{x^2}\right)$$

Solution. Directly plugging in $x=0$ doesn't work because of the $1/x^2$. To deal with the cos term, use the squeeze theorem. Since $-1\le \cos \le 1$,

$$-x \le x\cos\left(\frac{1}{x^2}\right) \le x.$$

But $\lim_{x\to 0} \pm x = 0$. So we have, by the squeeze theorem, that

$$0 \le \lim_{x\to 0} x\cos\left(\frac{1}{x^2}\right) \le 0.$$

Hence the answer to the limit we want is also $\boxed{0}$.

(e)

$$\lim_{z\to 0^+} \frac{1}{1-\ln(z)}$$

Solution. As $z\to 0^+$, the function $\ln(z)$ tends toward $-\infty$. This means that the denominator $1-\ln(z)$ tends toward $+\infty$. Hence the whole fraction tends toward $\boxed{0}$.

(f)

$$\lim_{x\to\infty} \frac{(x+\sin x)(x^2+2)}{(2x+1)^2(x+3)}$$

Solution. This is a limit at infinity. Remember from what we discussed in class that for such limits, only the leading-order term matters. Here you can compute what the leading-order term is without expanding everything: just take the term that grows the fastest in each factor. So the limit becomes

$$\lim_{x\to\infty} \frac{x\cdot x^2}{(2x)^2\cdot x} = \lim_{x\to\infty} \frac{x^3}{4x^3} = \boxed{\frac{1}{4}}.$$

(g)

$$\lim_{t \to -\infty} \frac{e^t - e^{-t}}{e^{2t} + e^{-2t}}.$$

(Hint: as $t \to -\infty$, terms like e^{-t} are the fastest-growing ones. Also, remember that e^{2t} grows much faster than e^t.)

Solution. This is another limit at infinity. However, this time as $t \to -\infty$, the biggest terms are e^{-t} and e^{-2t}. Hence the limit becomes

$$\lim_{t \to -\infty} \frac{-e^{-t}}{e^{-2t}} = \lim_{t \to -\infty} -e^t = \boxed{0}.$$

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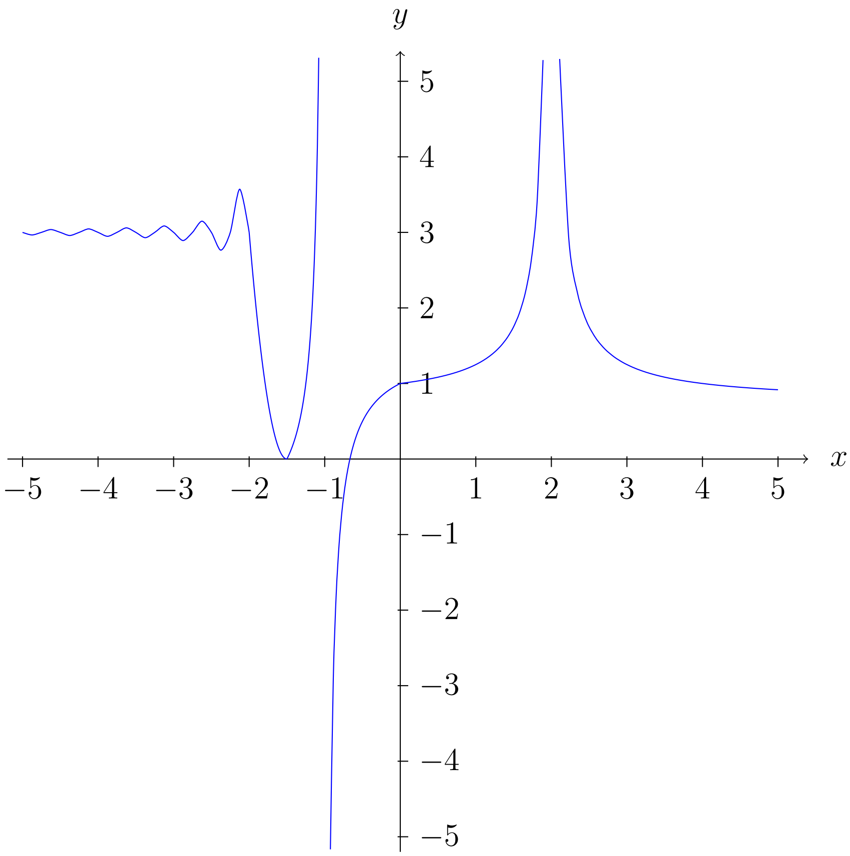
(2) For the following function f, whose graph is given, state the following.



(a) The equations of all horizontal asymptotes.

Solution. As $x \to +\infty$, the graph approaches $y = 1$. As $x \to -\infty$, the graph approaches $y = 3$. So the two horizontal asymptotes are $\boxed{y = 1 \text{ and } y = 3}$.

(b) The equations of all vertical asymptotes, and their left and right limits.

Solution. There are clearly two vertical asymptotes, at $\boxed{x = -1 \text{ and } x = 2}$. The limits are

$$\boxed{\lim_{x \to -1^+} f(x) = -\infty, \quad \lim_{x \to -1^-} f(x) = +\infty, \quad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = +\infty.}$$

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(3) Annoyed by your calculus homework, you crumple it into a ball and throw it into an infinitely deep hole. To develop a mathematical model for its speed $v(t)$ as a function of the time t since you threw it, you have placed highly accurate sensors at every meter of the hole, which tell you the exact time your homework passes them.

Depth (m)	Time (s)
0.000	0.0000
1.000	0.4508
2.000	0.8408
3.000	1.2022
4.000	1.5497
5.000	1.8902
6.000	2.2272
7.000	2.5624
8.000	2.8967
9.000	3.2305

(Note: unlike in the last homework, this data comes from a physically realistic model.)

(a) Using the data above and each of the following secant lines, estimate the speed of your homework the instant it reaches depth 1.000m.

(i) The secant line between depths 1.000m and 9.000m.

Solution. Speed is change in distance over change in time. So the speed is the slope of the given secant line:

$$\frac{9.000 - 1.000}{3.2305 - 0.4508} \approx \boxed{2.878}.$$

(ii) The secant line between depths 1.000m and 5.000m.

Solution. Repeating the same procedure, we get

$$\frac{5.000 - 1.000}{1.8902 - 0.4508} \approx \boxed{2.779}.$$

(iii) The secant line between depths 1.000m and 2.000m.

Solution. Repeating the same procedure, we get

$$\frac{2.000 - 1.000}{0.8408 - 0.4508} \approx \boxed{2.564}.$$

(b) Which of the above do you expect to be the most accurate estimation of the speed of your homework the instant it reaches depth 1.000m? Briefly explain why.

Solution. The instantaneous speed of your homework is the slope of the tangent line. Secant lines approximate tangent lines more and more accurately as the two points move closer and closer. The two points for the secant line $\boxed{\text{are closest in (c)}}$, so we expect it to be the most accurate.

(c) Using the data above and a secant line, give the most accurate estimate possible for how deep your homework is at time 1.0000s. Briefly explain how you got the answer.

Solution. We would like to write depth x as a function of the time t using the secant line between 2.000m and 3.000m, because the homework is somewhere inbetween those two depths at time $t = 1.000$. In other words, we want the equation of the line passing through $(0.8408, 2.000)$ and $(1.2022, 3.000)$, so that we can plug in $t = 1$.

To find the equation $x = mt + b$ of the line, we find its slope m and then b. The slope is

$$m \approx \frac{3.000 - 2.000}{1.2022 - 0.8408} \approx 2.767.$$

So $x = 2.767t + b$. Plugging in $(t, x) = (0.8408, 2.000)$ gives $b \approx -0.326$. Finally, plug in $t = 1.000$ to get the estimate

$$x \approx 2.767 \cdot 1 - 0.326 = \boxed{2.441}.$$

(d) Describe the speed of your homework as a function of time. What is the initial speed? How does the speed increase/decrease as time goes on? (Don't worry too much about the exactness/precision of your answer; a qualitative description is fine.)

Solution. The initial speed is close to

$$\frac{1.000 - 0.000}{0.4508 - 0.000} \approx \boxed{2.218}.$$

To see roughly what happens to the speed as time goes on, you can try computing the slopes of a few more secant lines (of points next to each other). Here I will list all of them, though you are certainly *not* expected to have computed them all!

Secant line	Speed (m/s)
0.000 to 1.000	2.218
1.000 to 2.000	2.564
2.000 to 3.000	2.767
3.000 to 4.000	2.878
4.000 to 5.000	2.937
5.000 to 6.000	2.967
6.000 to 7.000	2.983
7.000 to 8.000	2.991
8.000 to 9.000	2.996

$\boxed{\text{Speed increases quickly at first, but increases less and less quickly as time goes on}}$. As time goes on, the speed seems to tend toward 3m/s. (Any answer that computes a few of the above secant line slopes and contains a roughly correct description of what the results of the computations mean will receive full marks.)