MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 5 SOLUTIONS

Each part (labeled by letters) of every question is worth 2 points. There are 5 parts, for a total of 10 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Compute the derivatives of the following functions. You no longer need to state differentiation rules, but do write each step clearly.

(a)
$$h(\theta) = \cot(\theta).$$

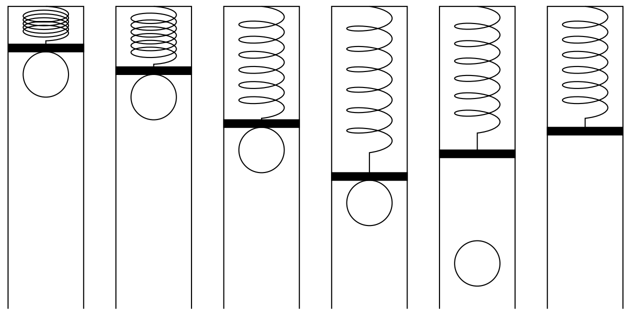
Solution. Remember that
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$
So to differentiate, we need to apply the quotient rule:
$$\begin{aligned} h'(\theta) &= \frac{(\cos\theta)'(\sin\theta) - (\cos\theta)(\sin\theta)'}{\sin^2\theta} \\ &= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} \\ &= \frac{-1}{\sin^2\theta} = \boxed{\csc^2\theta}. \end{aligned}$$

(2) Compute the following limits. Briefly explain your steps.

(a)
$$\lim_{h\to 0} \frac{\sin(\pi/2 + h) - 1}{h}.$$
(Hint: this should involve almost no calculation.)

Solution. The key is to recognize this as the definition of the derivative of $f(x) = \sin(x)$ at $x = \pi/2$, i.e.
$$f'(\pi/2) = \lim_{h\to 0} \frac{f(\pi/2 + h) - f(\pi/2)}{h} = \lim_{h\to 0} \frac{\sin(\pi/2 + h) - 1}{h}.$$
But we know $f'(x) = \cos(x)$. So $f'(\pi/2) = \boxed{0}$.

(3) Annoyed by your calculus homework, you crumple it into a ball and launch it into an infinitely deep hole using your new Spring Launcher Technology™.



At time t (in milliseconds), the end of the spring is at depth $x(t) = 10\sin(t)/t - 15$ (in centimeters).

(a) Compute the function $v(t)$ which describes the instantaneous velocity of the end of the spring at time t.

Solution. Since velocity is the rate of change of position, $v(t) = x'(t)$. Apply the quotient rule to $\sin(t)/t$:

$$\frac{d}{dt}\left(\frac{\sin t}{t}\right) = \frac{(\cos t)\cdot t - (\sin t)\cdot 1}{t^2}.$$

So the entire velocity function is

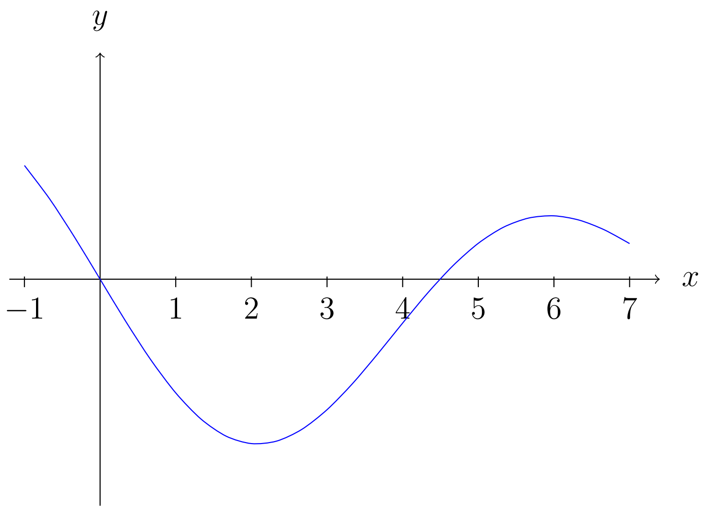
$$v(t) = \boxed{10 \cdot \frac{t\cos t - \sin t}{t^2}}.$$

(b) There are infinitely many times t where the spring will be fully extended (and about to retract back). Briefly explain why at such times, $v(t) = 0$.

Solution. Such times t will be the times where the instantaneous slope of the position function $x(t)$ is zero. If the slope were non-zero, then moving in the direction of increasing slope would increase $x(t)$, and therefore give a time where the spring is extended even farther. But instantaneous slope being zero at time t means $v(t) = x'(t) = 0$, by definition. (Your explanation may differ a little from mine; I'm happy as long as you make the connection between $v(t)$ and slope or rate of change.)

(c) Let $t_0 > 0$ be the smallest positive time such that $v(t_0) = 0$. Find an interval (a, b) that must contain t_0, and prove that it does.

Solution. Since $v(t)$ is continuous for $t > 0$, we can use the intermediate value theorem (IVT). We can look at a plot of the graph of $v(t)$ to see visually that the first t_0 is between π and 2π:



So we take the interval $(\pi, 2\pi)$. Plugging in values,

$$v(\pi) = \frac{\pi\cos(\pi) - \sin(\pi)}{\pi^2} = -\frac{1}{\pi} < 0$$

$$v(2\pi) = \frac{2\pi\cos(2\pi) - \sin(2\pi)}{4\pi^2} = \frac{1}{2\pi} > 0.$$

Hence IVT allows us to conclude that $\boxed{\pi < t_0 < 2\pi}$.