MATH UN1101

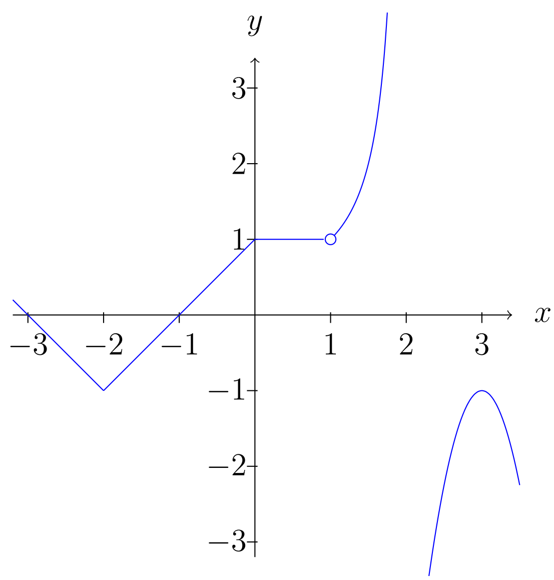
CALCULUS I (SECTION 5) - SPRING 2019

HOMEWORK 8 (DUE MAR 26)

Each part (labeled by letters) of every question is worth 2 points. There are 15 parts, for a total of 30 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) For each of the following functions, find its critical numbers and determine the intervals in the domain on which the function is increasing, and on which it is decreasing. Then, for each critical number, identify whether it is a local maximum or local minimum or neither.

(a) The function given by the graph



Solution. Remember that "critical number" means points $x = c$ where $f'(c) = 0$ *or does not exist*. So there are critical numbers at the following places.

(i) The point $\boxed{x = -1}$, which is a $\boxed{\text{local minimum}}$;

(ii) The whole interval $\boxed{[0, 1)}$. The point 0 is a $\boxed{\text{local maximum}}$, and every other point on this interval is $\boxed{\text{both a local max and a local min}}$, by the definition.

(iii) The point $\boxed{x = 3}$, which is a $\boxed{\text{local maximum}}$.

(Note that $x = 1$ is *not* a critical number, because it is not even in the domain of the function.) Between these critical numbers, the function is:

(i) $\boxed{\text{decreasing}}$ on $(-\infty, -2)$ and $(3, \infty)$;

(ii) $\boxed{\text{increasing}}$ on $(-2, 0)$, $(1, 2)$, and $(2, 3)$.

(b)

$$f(x) = \frac{x^2 - 2}{x^2 + 6}$$

Solution. Solve $f'(x) = 0$ to find critical numbers:

$$f'(x) = \frac{2x \cdot (x^2 + 6) - (x^2 - 2) \cdot 2x}{(x^2 + 6)^2} = \frac{16x}{(x^2 + 6)^2}.$$

The only critical number is therefore $x = 0$.

(i) When $x < 0$, we have $f'(x) < 0$, so f is $\boxed{\text{decreasing on } (-\infty, 0)}$.

(ii) When $x > 0$, we have $f'(x) > 0$, so f is $\boxed{\text{increasing on } (0, \infty)}$.

So $f(0)$ must be a $\boxed{\text{local minimum}}$.

(c)

$$f(x) = \ln(x^3 + 1)$$

Solution. Solve $f'(x) = 0$ to find critical numbers:

$$f'(x) = \frac{1}{x^3 + 1} \cdot 3x^2 = \frac{3x^2}{x^3 + 1}.$$

The only critical number is therefore $x = 0$.

(i) Note that the domain of f is $(-1, \infty)$. So when $x < 0$, we only consider x in $(-1, 0)$. In this case, $f'(x) > 0$ because $x^3 + 1 > 0$. So f is $\boxed{\text{increasing on } (-1, 0)}$.

(ii) When $x > 0$, we have $f'(x) > 0$, so f is $\boxed{\text{increasing on } (0, \infty)}$.

So $f(0)$ is $\boxed{\text{neither a local maximum nor a local minimum}}$.

(2) Your friend leaves your house at 11:00pm, drives to the nearest McDonalds (3 miles away) for a burger, and arrives back at 11:12pm. Prove using the mean value theorem that your friend must have exceeded the 25 miles/hr speed limit at some point during the drive.

Solution. Let $f(t)$ be the distance traveled, where t is the time, in hours, since 11:00pm. So $t = 0$ is 11:00pm, and $t = 0.2$ is 11:12pm. We want to prove that for some c in the interval $(0, 0.2)$, the speed $f'(c)$ exceeded 25. MVT tells us that there exists c in $(0, 0.2)$ such that

$$f'(c) = \frac{f(0.2) - f(0)}{0.2 - 0} = \frac{3 \cdot 2 - 0}{0.2 - 0} = 30.$$

Hence at time c, your friend was driving at 30 miles/hr.

(3) Let $f(x) = x^3 - \cos(x) + r$. Consider the equation $f(x) = 0$ on the interval $[1, \infty)$.

(a) Show that, for *any* real number r, the equation has at most one solution in the given interval.

Solution. Compute the derivative:

$$f'(x) = 3x^2 + \sin(x).$$

On the interval $[1, \infty)$, this is always positive because $3x^2 \geq 3$ and $-1 \leq \sin(x) \leq 1$. So $f(x)$ is increasing on the given interval and cannot have more than one zero.

(b) Find a value of r for which there is one solution, and another value of r for which there are no solutions. Give a rigorous justification in both cases (of the existence/non-existence of a solution).

Solution. $\boxed{\text{For } r = -100 \text{ there is a solution}}$. This is because

$$f(1) = 1^3 - \cos(1) - 100 < 0$$
$$f(10) = 10^3 - \cos(10) - 10 > 0,$$

so by IVT a solution exists between 1 and 10. $\boxed{\text{For } r = 100 \text{ there is no solution}}$, because

$$f(1) = 1 - \cos(1) + 100 > 0$$

and f only increases from $x = 1$ onward, by (a).

(c) How many inflection points does f have?

Solution. Take the second derivative, using $f'(x)$ from (a):

$$f''(x) = 6x + \cos(x).$$

Let $g(x) = f''(x)$ for convenience. Inflection points are where $g(x) = 0$. This has at least one solution by IVT, since

$$g(-10) < 0, \quad g(10) > 0.$$

It has at most one solution because $g'(x) = 6 - \sin(x) > 0$, i.e. g is always increasing. So f has exactly $\boxed{\text{one}}$ inflection point.

(4) Find the absolute minimum and absolute maximum values of f on the given interval.

(a)

$$f(x) = \frac{x}{x^2 - x + 1}, \quad [0, 3].$$

Solution. First find critical values. The derivative is

$$f'(x) = \frac{1 \cdot (x^2 - x + 1) - x \cdot (2x - 1)}{(x^2 - x + 1)^2} = \frac{-x^2 + 1}{(x^2 - x + 1)^2}.$$

So $f'(x) = 0$ at $x = \pm 1$. But $x = -1$ is not even in the interval that we are considering, so we forget about it. The remaining critical values and endpoints are have to check are $x = 0, 1, 3$. Their values are:

$$f(0) = 0, \quad f(1) = 1, \quad f(3) = 3/7.$$

So the $\boxed{\text{absolute maximum is } f(1) = 1}$, and the $\boxed{\text{absolute minimum is } f(0) = 0}$.

(b)

$$f(t) = t + \cot(t/2), \quad [\pi/4, 7\pi/4].$$

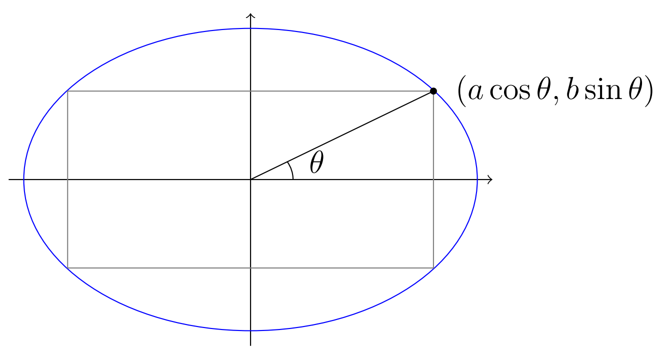
Solution. First find critical values. The derivative is

$$f'(t) = 1 - \csc^2\left(\frac{t}{2}\right) \cdot \frac{1}{2}.$$

If $f'(t) = 0$, then rearranging gives $\sin(t/2) = 1/\sqrt{2}$. The solutions which lie in the given interval are $t = \pi/2$ and $t = 3\pi/2$. So we need to check the values of f at $t = \pi/4, \pi/2, 3\pi/2, 7\pi/4$. This you can do using a calculator if necessary. The $\boxed{\text{absolute maximum is } f(3\pi/2)}$ and the $\boxed{\text{absolute minimum is } f(\pi/2)}$. (There are more clever ways to conclude this, without a calculator, but we won't worry about them.)

(5) Consider the ellipse $x^2/a^2 + y^2/b^2 = 1$. What is the area of the largest rectangle that can be inscribed in it? (Hint: points on this ellipse are of the form $(x, y) = (a\cos\theta, b\sin\theta)$. At some point, to make your life easier, use the double angle formula $2\sin\theta\cos\theta = \sin 2\theta$.)

Solution. This is an optimization problem, so we must write down a function whose maximum value we want to find. This function must produce the area of a rectangle as its output. The area depends on the side lengths of the rectangle.



From the diagram, the height of the rectangle is $2b\sin\theta$ and the width is $2a\cos\theta$. So the total area is

$$A(\theta) = 4ab\sin\theta\cos\theta = 2ab\sin 2\theta.$$

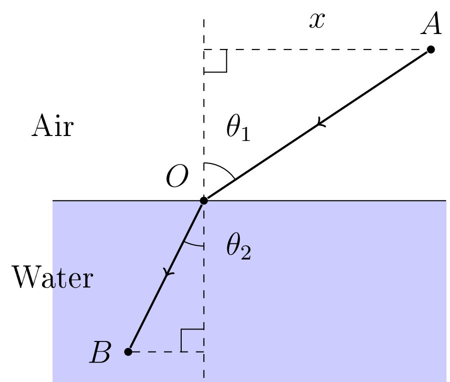
Since we want θ to produce a valid rectangle, θ must be in the interval $(0, \pi/2)$. Now we can compute

$$A'(\theta) = 2ab \cdot \cos(2\theta) \cdot 2 = 4ab\cos(2\theta).$$

So if $A'(\theta) = 0$, then $\theta = \pi/4$ is the only critical point. The maximum area is therefore

$$\boxed{A(\pi/4) = 2ab\sin(\pi/2) = 2ab}.$$

(6) Let v_1 be the speed of light in air, and v_2 be the speed of light in water. Because $v_1 \neq v_2$, light rays *refract* when they enter water from air.



The principle of least action says that the light ray will travel along the path which takes the *least time*.

(a) Define the following constants.
- Let ℓ be the total horizontal distance from A to B.
- Let h_1 be the height of A (above water level).
- Let h_2 be the depth of B (below water level).

Write expressions for the distances AO and OB traveled by the light ray, as functions of x (in the diagram) which involve the constants ℓ, h_1, h_2. What is the total time needed for the light ray to get from A to B?

Solution. Using the Pythagorean theorem, we have

$$AO = \sqrt{(h_1)^2 + x^2}, \quad OB = \sqrt{(h_2)^2 + (\ell - x)^2}.$$

On AO, the velocity is v_1, so the time it takes for the light to travel from A to O is AO/v_1. Similarly, it takes a time of OB/v_2 for the light to travel from O to B. The total time is

$$\boxed{\frac{1}{v_1}\sqrt{(h_1)^2 + x^2} + \frac{1}{v_2}\sqrt{(h_2)^2 + (\ell - x)^2}}.$$

(b) The total time in (a) is a function of x. Call it $T(x)$. If $x = x_0$ is the value of x for the actual path that the light ray takes, explain why the principle of least action implies $T'(x_0) = 0$.

Solution. Since the actual path of the light *minimizes* the total time taken, $x = x_0$ is a minimum value for the function $T(x)$. So x_0 is a critical value for T, i.e. $T'(x_0) = 0$.

(c) Using (b), show that the actual path taken by the light ray must satisfy

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

This is known as Snell's law. The ratio v_1/v_2 is called the *index of refraction*; for water and air, it is approximately $4/3$.

Solution. If we compute $T'(x)$, we get

$$T'(x) = \frac{1}{v_1}\frac{2x}{2\sqrt{(h_1)^2 + x^2}} + \frac{1}{v_2}\frac{-2(\ell - x)}{2\sqrt{(h_2)^2 + (\ell - x)^2}}.$$

Note that

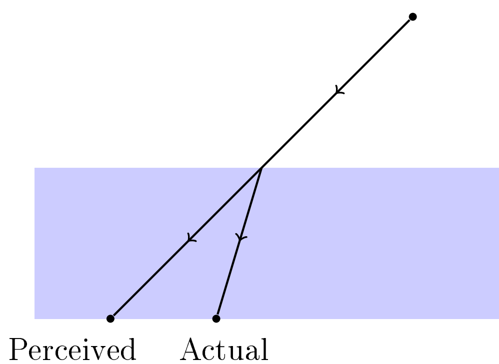
$$\sin\theta_1 = \frac{x}{\sqrt{(h_1)^2 + x^2}}, \quad \sin\theta_2 = \frac{\ell - x}{\sqrt{(h_2)^2 + (\ell - x)^2}}.$$

Substituting these in, we find

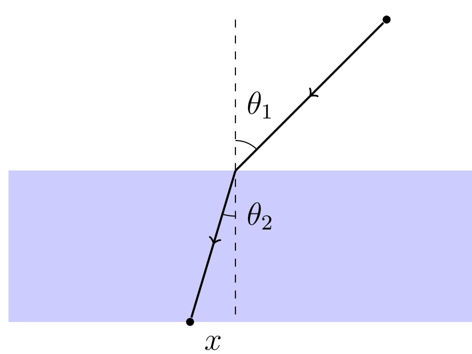
$$T'(x) = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}.$$

So if $T'(x_0) = 0$, then $\sin(\theta_1)/v_1 = \sin(\theta_2)/v_2$.

(d) You are standing in a pond which is one meter deep. For simplicity, you are two meters tall. You look downward at a 45° angle and see a little crab crawling at the bottom of the pond. Taking refraction into account, how far is the crab from your feet? (Non-exact answers are fine.)



Solution. The relevant part of the diagram is:



We know $\theta_1 = \pi/4$. So using (c),

$$\frac{\sin\pi/4}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{4}{3}.$$

Solving, $\theta_2 = \arcsin(3/(4\sqrt{2})) \approx 0.559$. The length x is therefore

$$\frac{x}{1} = \tan(\theta_2) \approx 0.626.$$

The crab is approximately $\boxed{1.626 \text{ meters}}$ from your feet.

(7) Annoyed by your calculus homework, you crumple it into a ball and throw it into an infinitely deep hole. Because it is spring break, this action gets you a free 2 points. Go enjoy your break!