MATH UN1101

CALCULUS I (SECTION 5) - SPRING 2019

# HOMEWORK 9 (DUE APR 04)

Each part (labeled by letters) of every question is worth 2 points. There are 5 parts, for a total of 10 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Evaluate the following limits.

(a)

$$\lim_{x\to 0} \frac{x - \sin x}{x - \tan x}$$

**Solution.** Plugging in $x = 0$ gives the indeterminate form $0/0$. We can directly apply l'Hôpital's to this to get

$$\lim_{x\to 0} \frac{x - \sin x}{x - \tan x} = \lim_{x\to 0} \frac{1 - \cos x}{1 - \sec^2 x}.$$

This is *still* an indeterminate form $0/0$, so we have to apply l'Hôpital's again to get

$$\lim_{x\to 0} \frac{1 - \cos x}{1 - \sec^2 x} = \lim_{x\to 0} \frac{\sin x}{-2\sec x(\sec x)'}.$$

Compute that the derivative of $\sec x$ is $\sec x \tan x$. Then

$$\frac{\sin x}{-2\sec x(\sec x)'} = \frac{\sin x}{-2\sec^2 x \tan x} = -\frac{1}{2}\cos^3 x.$$

It follows that the limit is $\lim_{x\to 0}(1/2)\cos^3 x = \boxed{1/2}$.

(b)

$$\lim_{t\to 1} \frac{\ln t}{t - e^t}$$

**Solution.** Plugging in $t = 1$ gives $0/(1-e)$, which is not an indeterminate form. So the limit is just $\boxed{0}$.

(c)

$$\lim_{x\to\infty} x^{1/4}\ln x$$

**Solution.** This is not an indeterminate form: both pieces tend to $\infty$ as $x \to \infty$. So the final answer is also $\boxed{\infty}$.

(d)

$$\lim_{x\to\infty} x^{1/x}$$

**Solution.** This is the indeterminate form $\infty^0$. Rewrite the function as

$$x^{1/x} = \exp\left(\ln x^{1/x}\right) = \exp\left(\frac{1}{x}\ln x\right).$$

Then

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} \exp\left(\frac{1}{x}\ln x\right) = \exp\left(\lim_{x\to\infty} \frac{1}{x}\ln x\right).$$

This new limit is of the indeterminate form $\infty/\infty$, so we apply l'Hôpital's to it to get

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Hence the final answer is $e^0 = \boxed{1}$.

(2) The sum of two positive numbers is 12. What is the smallest possible value of the sum of their squares?

**Solution.** Let the two positive numbers be $x$ and $y$. Then $x + y = 12$, i.e. $y = 12 - x$, and we want to minimize $x^2 + y^2$. Substituting, we want to minimize

$$f(x) = x^2 + (12 - x)^2.$$

Solve for its critical point(s) by computing its first derivative

$$f'(x) = 2x + 2(12 - x)\cdot(-1) = 4x - 24$$

and solving $f'(x) = 0$. This gives $x = 6$, and therefore $y = 6$. Check that this is a minimum:

$$f''(x) = 4 \quad \rightsquigarrow \quad f''(6) = 4 > 0.$$

Hence the smallest possible value for $f(x)$ is $6^2 + 6^2 = \boxed{72}$.