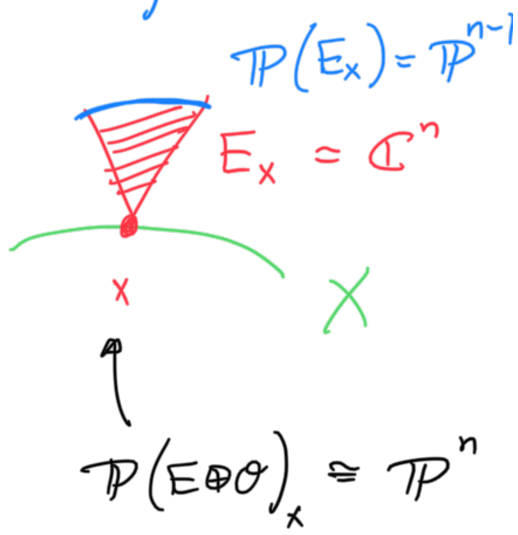
Thom iso thm: $\pi : E \to X$ affine bundle

$$\pi^* : K_G(X) \simeq K_G(E).$$

Pf sketch: Compare with $\mathbb{P} = \mathbb{P}(E \oplus \mathcal{O})$



(π^* surjective). given $\mathcal{F} \in K_G(E)$,

can extend to $\mathcal{F}_{\mathbb{P}} \in K_G(\mathbb{P})$

means $\mathcal{F}_{\mathbb{P}}\big|_E = \mathcal{F}$.

$$\mathcal{F}_{\mathbb{P}} = \sum_{k=0}^{n-1} \mathcal{O}_{\mathbb{P}}(k) \otimes \pi_{\mathbb{P}}^* \underbrace{\mathcal{E}_k}_{\in K_G(X).}$$

restrict back to $E \subset \mathbb{P}$

$$\Rightarrow \quad \mathcal{F} = \sum_{k=0}^{n-1} \mathcal{O}_E \otimes \pi^* \mathcal{E}_k$$

(π^* injective), if E has a section $\iota : X \hookrightarrow E$

$$\iota^* \pi^* \mathcal{F} = \mathcal{F}$$

in particular, π^* is injective.

otherwise, more work (see [CG7]). □.

e.g. $K_G(\mathbb{C}^n) \simeq K_G(pt) = R(G)$.

Excision long exact sequence (Quillen).

actually hard.

$$X \underset{\text{closed}}{\overset{\iota}{\hookrightarrow}} Y \underset{\text{open}}{\overset{j}{\hookleftarrow}} Y/X =: U$$

$\exists$ a long exact seq:

$$\cdots \to K_G^1(U) \to K_G(X) \longrightarrow K_G(Y) \longrightarrow K_G(U) \to 0$$

higher K-theory group. (see e.g. Schlichting "Higher algebraic K-theory").

e.g. view $\mathbb{P}(V) = (V \setminus 0) / \mathbb{C}^\times \underset{\text{open}}{\subset} [V/\mathbb{C}^\times]$ (as a stack).

Fact: $K([X/G]) \simeq K_G(X)$.

Compute: $K_T(\mathbb{P}(V))$ $\quad T = (\mathbb{C}^\times)^{\dim V} \subset GL(V)$ (max. torus).

$$\cdots \to K_{T\times\mathbb{C}^\times}(\{0\}) \xrightarrow{\iota_*} K_{T\times\mathbb{C}^\times}(V) \xrightarrow{j^*} K_T(\mathbb{P}(V)) \to 0$$

$$K_{T\times\mathbb{C}^\times}(\{0\}) \xrightarrow{\iota_*} K_{T\times\mathbb{C}^\times}(\{0\})$$

extra $\mathbb{C}^\times$-equivariance $K_{\mathbb{C}^\times}(pt) = \mathbb{Z}[s^\pm]$

$$\mathcal{O}_0 \longmapsto \iota_*\mathcal{O}_0 = \sum (-1)^i \underset{s^{-i}}{} \Lambda^i V^\vee$$

(Koszul resolution).

$\mathcal{O}_{\mathbb{P}(V)}(1)$ on $\mathbb{P}(V)$.

$$\Rightarrow \quad K_T(\mathbb{P}(V)) = \underbrace{K_T(pt)[s^\pm]}_{K_{T\times\mathbb{C}^\times}(pt)} \Big/ \langle \underbrace{\sum_i (-s)^i \Lambda^i V}_{} = 0 \rangle$$

removed duals for clarity.

$\prod_i (1 - t_i s)$ where $K_T(pt) = \mathbb{Z}[t_1^\pm, \ldots, t_n^\pm]$

$$\Rightarrow \quad K(\mathbb{P}^{n-1}) = \mathbb{Z}[s^\pm] \Big/ \langle (1-s)^n = 0 \rangle$$

e.g. $K(\mathbb{P}^1)$ has a relation $(1-s)^2 = 0$

Euler exact sequence.

One major use case of equiv. K-theory is invariants.

i.e. proper.

$T \curvearrowright X$

a torus $\cong (\mathbb{C}^\times)^n$

$$\int_X \alpha \quad \rightsquigarrow \quad \chi(X, \mathcal{F})$$

$\int_X \alpha \in H^*_T(pt)$, $\quad \chi(X,\mathcal{F}) \in K_T(pt)$.

a deformation-invariant of X !

(vs. individual $\underline{H^i(X, \mathcal{F})}$. in $D^b \mathrm{Coh.}$)

not deformation invariant.

For many applications, need to compute χ.

Equivariant localization: having $T \curvearrowright X$ simplifies the computation of χ:

T-fixed locus.

$$\begin{tikzcd} K_T(X^T) \arrow[r, hook, "\iota_*"] \arrow[dr, "\chi(X^T, -)"'] & K_T(X) \arrow[d, "\chi(X,-)"] \\ & K_T(pt) \end{tikzcd}$$

usually much simpler

e.g. $X^T = \bigsqcup$ pts.

If ι_* were an isomorphism,

$$\Rightarrow \quad \chi(X, \mathcal{F}) = \chi(X^T, \underline{(\iota_*)^{-1}} \mathcal{F})$$

want this to exist.

Let's study ι_*.

If ι is a regular embedding,

normal bundle to $\iota: X^T \hookrightarrow X$

$$\iota^* \iota_* \mathcal{E} = \mathcal{E} \otimes \Lambda^\bullet_{-1} N^\vee_{X/X^T}$$

$\mathcal{E} \in K_T(X^T)$

Koszul resolution. $(\iota_* \mathcal{O}_0 = \pi^* \Lambda^\bullet_{-1} \underline{V}^\vee)$

$= T_0 V$

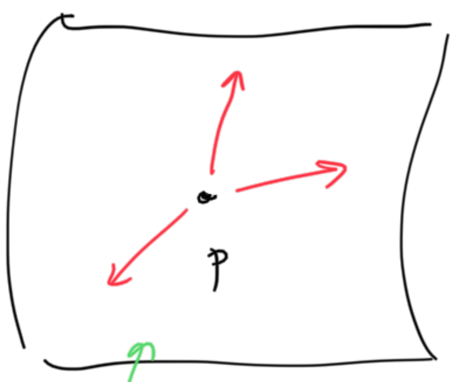
$$\Rightarrow \text{formally,} \quad (\iota_*)^{-1} = \frac{\iota^*}{\quad}$$

$\underbrace{\Lambda_{-1}^{\bullet} N_{X/X^T}^{\vee}}$

what does it look like?

V 

Suppose for simplicity that $X^T = \{p\}$. $p \in X$.



X

no tangent weight can be trivial.

$$\Rightarrow \Lambda_{-1}^{\bullet} N_{X/X^T}^{\vee} = \Lambda_{-1}^{\bullet} (\underbrace{T_p X})^{\vee}$$

as a T-module $= w_1 + \cdots + w_n$

$$= \prod_{w \in T_p X} (1 - w^{-1})$$

notation for

$\Rightarrow$ need inverses for $\{1-w\}_{\substack{w \text{ is a wt} \\ \text{of } T}}$ $w \neq 1$

More generally,

using that $T \curvearrowright X^T$ is trivial.

$$N_{X/X^T} \in K_T(X^T) = K(X^T) \otimes K_T(pt).$$

K-theoretic "Chern roots"

$$\Rightarrow \Lambda_{-1}^{\bullet} N_{X/X^T}^{\vee} = \prod_{w\mathcal{L} \in N_{X/X^T}} (1 - \underset{\substack{\cap \\ K_T(pt)}}{w^{-1}} \underset{\substack{\cap \\ \mathrm{Pic}(X^T)}}{\mathcal{L}^{\vee}})$$

K-theoretic splitting principle. (exercise.)

$$\left(\text{Cf. in } H_T^* \text{ where this is } e(N_{X/X^T}) = \prod_{e^{s_i}\mathcal{L}_i \in N_{X/X^T}} (s_i + c_1(\mathcal{L}_i)) \right)$$

Can expand:

$$\frac{1}{1 - \underset{\text{wt}}{w}\underset{\in \mathrm{Pic}}{\mathcal{L}}} = \frac{1/1-w}{1 + \frac{(1-\mathcal{L})w}{1-w}} = \sum_{k \geq 0} \frac{(-w)^k}{(1-w)^{k+1}} \underline{(1-\mathcal{L})^{\otimes k}}$$

Claim: on schemes X, operators $\mathcal{L} \otimes \curvearrowright K(X)$ are unipotent

$$\Rightarrow (1-\mathcal{L})^{\otimes N} = 0, \quad N \gg 0.$$

$$\Rightarrow \quad \frac{1}{1 - w\mathcal{L}} \quad \text{exists as long as we invert } 1 - w.$$