Equivariant K-theory: Lecture 4

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3:30 PM

Last time:

$$\begin{tikzcd} K_T(X^T) \arrow[r, "\iota_*"] \arrow[dr, "\chi(X^T,-)"'] & K_T(X) \arrow[d, "\chi(X,-)"] \\ & K_T(pt) \end{tikzcd}$$

$T \simeq (\mathbb{C}^*)^n \curvearrowright X$

can be more general, e.g. (reductive) alg. group. as long as

If $\iota_* : K_T(X^T)_{loc} \to K_T(X)_{loc}$ is an isomorphism,

base change $K_T(pt) \rightsquigarrow K_T(pt)_{loc} := K_T(pt)\left[\frac{1}{1-w} : \begin{matrix} w \neq 1 \text{ is} \\ \text{a } T\text{-weight}\end{matrix}\right]$

$$\Rightarrow \quad \chi(X, \mathcal{F}) = \chi(X^T, (\iota_*)^{-1}\mathcal{F})$$

equivariant localization.

Suppose ι is regular: $\quad \iota^*\iota_*\mathcal{F} = \mathcal{F} \otimes \underbrace{\Lambda^\bullet_{-1}}\, N^\vee_{X/X^T}$

$$= \prod (1 - w^{-1}\mathcal{L}^\vee) \in K_T(pt) \otimes K(X^T)$$

$$\frac{1}{1-w\mathcal{L}} = \frac{\frac{1}{1-w}}{1 + \frac{w(1-\mathcal{L})}{1-w}} = \sum_{k\geq 0} \frac{(-w)^k}{(1-w)^{k+1}} \underline{\underline{(1-\mathcal{L})^{\otimes k}}}$$

Claim: $(1-\mathcal{L})^{\otimes N} = 0 \qquad N \gg 0. \qquad \rightsquigarrow (\iota_*)^{-1} = \dfrac{\iota^*}{\Lambda^\bullet_{-1} N^\vee}$ exists

(in $K_T(-)_{loc}$)

Proof idea: non-equivariant! $\mathcal{L}|_U \simeq \mathcal{O}|_U$ on an open $U \subset X$.

$$\dim \operatorname{supp} (\overset{0}{1 - \mathcal{L}}) \otimes \mathcal{F} < \dim \operatorname{supp} \mathcal{F}$$

($\exists$ a filtration on $K(X)$ by dim supp.)

$\Rightarrow$ so it suffices to invert $\{1-w\}_{w\neq 1 \text{ wt of } T.}$

$\Rightarrow$ if ι is regular,

equivariant localization.

$$\chi(X, \mathcal{F}) = \chi\left(X^T, \frac{\mathcal{F}|_{X^T}}{\Lambda^\bullet_{-1}(N_{X/X^T})^\vee}\right)$$

in localized K-theory $K_T(-)_{loc} = K_T(-)\left[\frac{1}{1-w}\right]$

warning: this removes torsion for $1-w$., e.g.

$$H^0(\mathbb{C}^\times, \mathcal{O}) = \mathbb{C}[x^{\pm}]$$

$\circlearrowright$ $(\mathbb{C}^\times)$ acts with wt 1, i.e. x has wt t

$$= \underbrace{\sum_{n\in\mathbb{Z}} t^n}_{\text{torsion for } (1-t)} \equiv 0 \in K_{\mathbb{C}^\times}(\mathbb{C}^\times)_{loc}$$

Often, we know a priori that the answer we want is torsion-free,

e.g. $\chi(X, \mathcal{F}) \in K_T(pt)$

some polynomial.

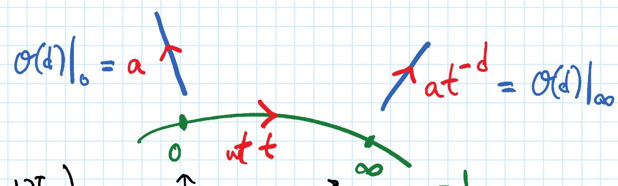
<u>Remark:</u> a general thm of Thomason says

ker & coker $\iota : X^T \hookrightarrow X$ in $K_T(-)$

are <u>torsion</u> $\Rightarrow$ vanish in $K_T(-)_{loc.}$

$\Rightarrow$ ι_* is always an iso in $K_T(-)_{loc}$, but may not have a nice formula.

A <u>guiding example</u>:



$$\chi(\mathbb{P}^1, \mathcal{O}(d)) = \chi\left((\mathbb{P}^1)^T, \frac{\mathcal{O}(d)|_{(\mathbb{P}^1)^T}}{\Lambda^\bullet_{-1} N^\vee}\right)$$

$\mathbb{P}^1$ proper

$H^0 - H^1$

$$= \chi\left(\{0\}, \frac{\mathcal{O}(d)|_0}{\underbrace{\Lambda^\bullet_{-1}(T_0\mathbb{P}^1)^\vee}}\right) + \chi\left(\{\infty\}, \frac{\mathcal{O}(d)|_\infty}{\underbrace{\Lambda^\bullet_{-1}(T_\infty\mathbb{P}^1)^\vee}}\right)$$

$$= \frac{a}{1-t^{-1}} + \frac{at^{-d}}{1-t}$$

overall equivariant wt.

$$= a\,\underbrace{\frac{1-t^{-d-1}}{1-t^{-1}}}$$

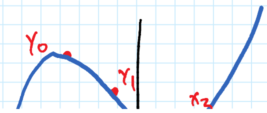
$$\begin{cases} \overbrace{1 + t^{-1} + \cdots + t^{-d}}^{H^0(\mathcal{O}(d))} & d \geq 0 \\ \underbrace{-t - t^2 - \cdots - t^{d-1}}_{-H^1(\mathcal{O}(d))} & d < 0 \end{cases}$$

apparent poles in t <u>cancel</u>.

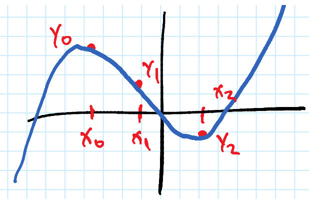
a geometric principle for certain types of pole cancellation!

e.g. the Lagrange interpolation

depends on x



e.g. the Lagrange interpolation



depends on x

$$1 = \sum_{i=0}^{n} 1 \cdot \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$\in \mathbb{Q}$ a priori

$\|$ is exactly $\int_{\mathbb{P}^n} h^n \in H_T^*(pt)$ via localization (exercise)

Lie grp (reductive), e.g. $GL(V)$

Another example : $X = G/B$ ← Borel   e.g. $G \supset B \supset T$ max. torus.

flag variety. $T \circlearrowright X$

1. $t \cdot (gB) = gB$ means

$$g^{-1} t g \in B$$

semisimple elements

$$\Rightarrow g^{-1} t g \in T \subset B$$

$$\Rightarrow g \in N(T)/T = \text{Weyl group } W.$$

normalizer

$\Rightarrow$ fixed points $X^T \simeq W$

Lie alg. of G

2. $T_g X = g^* T_e X = g^* \left(\mathfrak{g}/\mathfrak{b} \right)$ Lie alg. of B

identity in G

Recall, as T-module,

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha \qquad \mathfrak{b} \overset{\simeq \text{ by def.}}{=} \bigoplus_{\alpha \geq 0} \mathfrak{g}_\alpha$$

wts for T, write its K-theoretic wt as e^α

formal symbol, reminds us of multiplicativity.

Let $\mathcal{L}_\lambda := G \times_B \mathbb{C}_\lambda$ ← trivial T-irrep of wt $e^{-\lambda}$

(line bundle)

$$\downarrow$$

$$G \times_B pt = G/B$$

$= w \cdot \mathcal{L}_\lambda|_{eB} = w \cdot e^{-\lambda}$

$$\Rightarrow \chi(G/B, \mathcal{L}_\lambda) = \sum_{w \in W} \frac{\mathcal{L}_\lambda|_{wB}}{\Lambda_{-1}^{\bullet}\left(T_{wB}\, G/B\right)^\vee} \leftarrow w \cdot \prod_{\alpha > 0} (1 - e^\alpha)$$

$$\overset{\star}{=} \sum_{w \in W} \frac{e^{-w\lambda}}{\prod_{\alpha > 0} (1 - e^{w\alpha})}$$

rhs of Weyl character formula

formula $\alpha > 0$
for G-irrep of highest wt λ.

Fact (Borel-Weil-Bott): LHS is concentrated in a single cohom. degree & is exactly the G-irrep of highest wt λ.

$\Rightarrow$ we proved Weyl character formula modulo BWB.

A general example: Let X be smooth & proper, $T \curvearrowright X$

$$\chi\left(X, \Lambda^{\bullet}_{-y}(T_X^{\vee})\right) \underset{\substack{\text{assume} \\ X^T = \sqcup\,\text{pts}}}{=} \sum_{p \in X^T} \frac{\prod_{w \in T_p X}(1 - y w^{-1})}{\prod_{w \in T_p X}(1 - w^{-1})} \leftarrow \Lambda^{\bullet}_{-1}(T_X^{\vee})$$

$\Lambda^{\bullet}_{-y}$ means $\sum_k (-y)^k \Lambda^k$

$$\| \quad \sum_{p,q} (-1)^p (-y)^q \underbrace{H^p(X, \Lambda^q \Omega_X)}_{\text{are } H^{p,q}(X)}$$

Known as the "χ_y-genus" of X

by Hodge theory, $H^{p,q}(X) \subset H^{p+q}(X, \mathbb{C})$ is a trivial! T-module.

$\Rightarrow$ rhs must be constant in $K_T(pt)$.

To see this, use a:

Rigidity argument: given a rational function $f(\vec{x})$ ← from localization

if: 1. know $f(\vec{x})$ is a Laurent poly ← from coherence.

2. all limits $\vec{x} \to 0, \infty$ exist

$\Rightarrow$ f is constant in $\vec{x}$.

$$\lim_{w \to 0, \infty} \frac{1 - y/w}{1 - 1/w} = \begin{cases} y & w \to 0 \\ 1 & w \to \infty \end{cases} \quad \text{exists!}$$

basically our only tool for controlling equivariant dependence from localization, but require very special integrands!