MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

HOMEWORK 10 (DUE AUG 12)

Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) Find a fundamental set of solutions for

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \mathbf{x}.$$

Check using the Wronskian that it is indeed a fundamental set.

(2) Find the general solution to

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}.$$

(3) Let $\mathbf{A}$ be an $n \times n$ matrix. Show, using the series definition of the matrix exponential, that

$$e^{\mathbf{A}t} e^{\mathbf{A}s} = e^{\mathbf{A}(t+s)} \qquad (\star)$$

for all real numbers s, t.

(4) Now we use a trick to prove equation ($\star$) more easily. Show that both

$$e^{\mathbf{A}t} e^{\mathbf{A}s} \quad \text{and} \quad e^{\mathbf{A}(t+s)}$$

are (matrix) solutions to the first-order IVP

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = e^{\mathbf{A}s},$$

where we fix s and treat t as a variable. Explain why this immediately implies ($\star$) holds for all s, t. (Hint: matrix solutions to IVPs satisfy the same theorems as usual solutions.)

(5) Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ has two eigenvalues: $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Describe all critical points of the system and roughly sketch the phase diagram for the system.