MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

# (BONUS) HOMEWORK 11 (DUE AUG 16, 2PM)

Each question is worth 3 points. There are 5 questions, for a total of 15 points. The grade you obtain on this homework will be **added directly** to your total assignment grade, with **no penalties**. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words. Note that questions are harder than usual and may involve material from slightly beyond the scope of this course.

(1) Use the Laplace transform to solve the IVP

$$\begin{aligned} x_1' &= x_1 + x_2 + 2 \qquad & x_1(0) &= 1 \\ x_2' &= -x_1 + t & x_2(0) &= -1. \end{aligned}$$

(2) Find the Green's function corresponding to the operator

$$D = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2.$$

(This is a **damped harmonic oscillator**, e.g. a mass on a spring taking friction into account, with undamped frequency $\omega_0$ and damping factor $\gamma$.)

(3) Boundary value problems of the form

$$y'' + \alpha y = g(t), \quad y(0) = y(L) = 0$$

can be solved using the **Fourier sine series** ansatz

$$y(t) = \sum_{n=0}^{\infty} a_n \sin\frac{\pi n t}{L}.$$

Explain why this is a reasonable ansatz for the given boundary conditions. Then use it to solve the boundary value problem when

$$g(t) = \sin\frac{\pi t}{L} + \sin\frac{2\pi t}{L}.$$

(4) Consider the non-linear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ (of possibly more than two variables). Suppose it has a critical point at $\mathbf{x}^{(0)}$.
   (a) Explain how to shift the critical point to the origin $\mathbf{0}$.
   (b) Explain how to use **linear approximation** for $\mathbf{f}$ in order to obtain a new *linear* system

$$\mathbf{u}' = \mathbf{P}\mathbf{u} + \mathbf{g}$$

   which best approximates the shifted system around $\mathbf{0}$.

What are the conditions on the original system so that the resulting linear system has an *isolated* critical point at $\mathbf{0}$?

(5) Find the general solution to $2yy'' + 2(y' + 2y)y' + y^2 = t$.