MATH S3027 (SECTION 2)
ORDINARY DIFFERENTIAL EQUATIONS - SUMMER 2019

HOMEWORK 7 (DUE AUG 01)

Each question is worth 10 points. There are 5 questions, for a total of 50 points. You are encouraged to discuss the homework with other students but you must write your solutions individually, in your own words.

(1) The **gamma function** is defined as

$$\Gamma(r) := \int_0^\infty e^{-u} u^{r-1}\, du.$$

Using a change of variables $u = st$, show that

$$\mathcal{L}\{t^{r-1}\} = \frac{\Gamma(r)}{s^r}.$$

Use integration by parts to show that the gamma function satisfies $\Gamma(r+1) = r\Gamma(r)$. Finally, show that $\Gamma(1) = 1$ to conclude that $\Gamma(r) = (r-1)!$ when r is a positive integer.

(2) Using problem 1, solve the IVP

$$y'' - 3y' + 2y = t^2, \quad y(0) = 0, \quad y'(0) = 1.$$

(Please feel free to use a computer to find the partial fraction decomposition, but make sure you know in principle how to do it by hand.)

(3) Find the inverse Laplace transforms of

$$\frac{5e^{-6s} - 11e^{-7s}}{(s-1)(s-2)}, \qquad \frac{2s + e^{-2s}}{s^2 - 1}.$$

(4) Show that if $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{-tf(t)\} = F'(s).$$

Explain why this means $\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s)$. In other words, multiplying by $(-t)^n$ in the time domain is the same thing as taking n derivatives in the frequency domain.

(5) One way to compute the Laplace transform of $f(t)$ is to expand it as a series, and take the Laplace transform term by term. Use problem 1 to do this for

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

to verify that (for $s > 1$)

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}.$$