Ordinary Differential Equations

Math 3027 (Columbia University Summer 2019)

Henry Liu

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0 Introduction

The concept of an algebraic equation and its solutions is introduced fairly early in a standard mathematics curriculum. We first learn how to solve *linear* equations like $3x + 5 = 2$, and then *quadratic* equations like $x^2 - x - 3 = 0$. Solutions are numbers x which satisfy the equation.

This story can be upgraded to equations whose unknowns are *functions* $y(x)$, instead of numbers x. For example, we could know that a function y satisfies

$$\frac{dy}{dx} - y = 0,$$

and want to solve for the function y. In this case, it is not too hard to guess and check that $y(x) = e^x$ is a solution. The main feature of such equations involving functions is that they usually involve derivatives. If they did not, then we might as well treat f as a number, i.e. a constant, and go through the steps of ordinary algebra.

Definition 0.0.1. A **differential equation (DE)** is an equation relating a function y with its derivatives.

In the same way that ordinary equations becomes harder to solve when there are multiple variables, differential equations become vastly harder when we consider functions of multiple variables. This is mostly because the multivariate setting involves *partial* derivatives.

Definition 0.0.2. An **ordinary differential equation (ODE)** is a DE where the unknown function y is a function of a *single* variable. A **partial differential equation (PDE)** involves functions of multiple variables.

Example 0.0.3. The *heat equation* is a PDE governing the diffusion of some quantity $f(x, y, z, t)$, e.g. heat, over time:

$$\frac{\partial f}{\partial t} = \alpha \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right).$$

Here α is a constant which controls the rate of diffusion.

This course is about ODEs, so from now on the unknown function is always of a single variable. Conventionally, the function is denoted y, and it is usually a function of x.

Definition 0.0.4. The **order** of an ODE is the order of the highest derivative that appears in the equation.

Example 0.0.5. The equation

$$\frac{dy}{dx} - 2y^4 = x^5$$

is a first-order ODE, whereas the equation

$$\frac{d^2y}{dx^2}\frac{dy}{dx}\sin(x) - e^x \left(\frac{dy}{dx}\right)^4 = 0$$

is a second-order ODE.

Usual algebraic equations become harder (if not impossible) to solve as the order grows, so it is not surprising that the theory of ODEs becomes more difficult in higher order. We will start with the theory of first-order ODEs, and then second-order. Unfortunately, even in these two simplest cases, most such ODEs will not be "solvable" in a satisfactory sense.

However, as with algebraic equations, once we obtain a solution f of an ODE, it is very easy to check that f is indeed a solution: just plug it back in. For example, given the first-order ODE

$$\frac{dy}{dx} = 2x,$$

we can easily check that $y(x) = x^2$ is a solution. From calculus we know that $y(x) = x^2 + C$ is a solution for *any* real number c.

Definition 0.0.6. Given an ODE, a **particular solution** is a single function which is a solution of the ODE, like $y = x^2$. A **general solution** is an expression which contains *all* possible solutions of the ODE, like $y = x^2 + C$.

If we specify an additional constraint on the solution $y(x)$, e.g. that $y(0) = 1$, then usually we can solve to get a specific value for c. Such additional constraints are thought of as initial conditions.

Definition 0.0.7. An **initial value problem (IVP)** is a differential equation with an additional *initial value* constraint, e.g. $y(0) = 1$, on solutions.

We will focus mostly on ODEs without initial conditions, but keep in mind that for every ODE it is possible to specify initial conditions to make it into an IVP.

1 First-order equations

Definition 1.0.1. A **first-order** ODE is of the general form

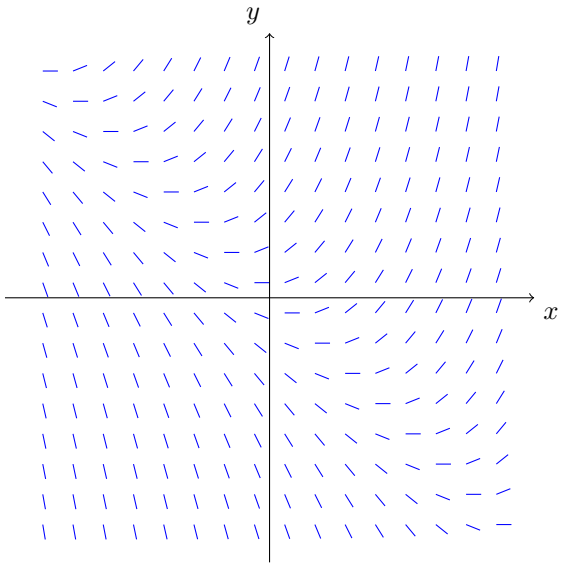
$$\frac{dy}{dx} = f(x, y). \tag{1.1}$$

There are entire books devoted to various fancy techniques and tricks for solving different types of first-order ODEs, in the same way that there are entire books devoted to solving cubic and quartic polynomial equations in one variable. This is *not* the point of the course. We will instead use first-order ODEs to familiarize ourselves with a collection of fundamental ideas and tools in the theory of ODEs in general, and then move on.

1.1 Slope fields

Before delving into the theory of first-order equations, it is best to have some visual representation of them for the sake of intuition. The equation (1.1) says that whatever the function $y(x)$ is, at the point (x, y) it has slope $f(x, y)$. Since tangent lines approximate functions, we may as well get an idea for what $y(x)$ looks like by drawing a line of slope $f(x, y)$ at each point (x, y). Such a diagram is called a **slope field**.

Example 1.1.1. The slope field of the first-order ODE $dy/dx = x + y$ is as follows.



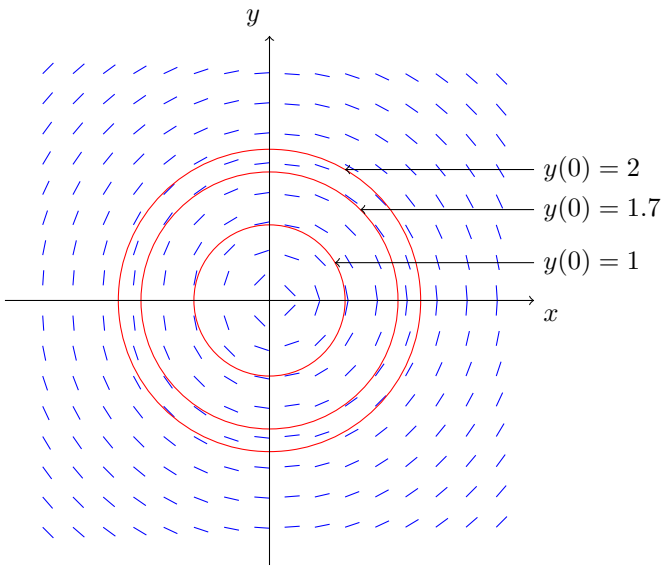
From the slope field, we can "see" different solutions $y(x)$. If we are given an initial condition $y(x_0) = y_0$, we can start at the point (x_0, y_0) and "follow the lines" in either direction. The resulting curve will be a solution $y(x)$ satisfying the initial condition, and is called an **integral curve**. Different initial conditions result in different solutions/integral curves.

Note that a slope field is *not* the same thing as a vector field. A slope field is an assignment of a slope to every point in the plane. A **vector field** is an assignment of a *vector* to every point in the plane. The difference between a slope and a vector is that a vector has a *magnitude* in addition to a direction. Vector fields arise in the study of ODEs when we take an integral curve and find a *parameterization* $(x(t), y(t))$ of it. A single integral curve may be parameterized in many different ways; this is a very important point we will use later. Once we pick a parameterization, we call $(x(t), y(t))$ a **trajectory**.

Example 1.1.2. The slope field (and multiple integral curves) for the first-order ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

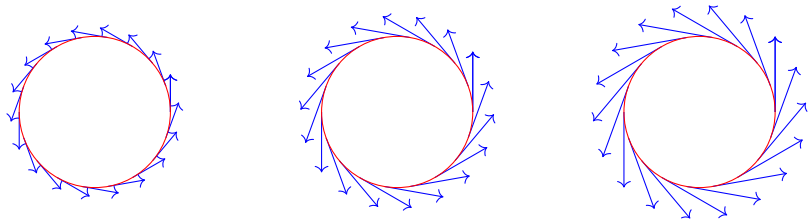
looks like the following.



The integral curves are circles $x^2 + y^2 = R^2$. For a given circle, the standard parameterization is

$$(x(t), y(t)) = (R\cos(t), R\sin(t)),$$

but there are many different parameterizations corresponding to going around the circle at different speeds.



The freedom to parameterize solutions in different ways will be very important later, when we discuss solutions to general first-order linear ODEs.

1.2 Linear equations with constant coefficients

Definition 1.2.1. In (1.1), if $f(x, y)$ depends *linearly* on y, i.e.

$$\frac{dy}{dx} = p(x)y + q(x),$$

then we say the ODE is **linear**.

Definition 1.2.2. In a linear first-order ODE, if the coefficients $p(x)$ and $q(x)$ are actually *independent* of x, then we have a linear first-order ODE with **constant coefficients**. Its general form is

$$\frac{dy}{dx} + ay = b$$

for *constants* a and b.

Example 1.2.3. If we are in the special case that $b = 0$, then the equation is

$$\frac{dy}{dx} + ay = 0.$$

This is often called the **homogeneous form** of the general equation $dy/dx + ay = b$. Assuming $a \neq 0$, it rearranges into

$$\frac{1}{y}\frac{dy}{dx} = -a.$$

To solve this, we can integrate both sides with respect to x. The lhs is the derivative of $\ln|y|$, and the rhs is the derivative of $-ax + C$, so that

$$\ln|y| = -ax + C.$$

Exponentiating both sides yields the general solution

$$y = ce^{-ax}$$

where $c \coloneqq \pm e^C$ is an arbitrary constant. We will see that the exponential function is pervasive in the theory of ODEs.

Many people (including the textbook) like to write

$$\frac{1}{y}\,dy = -a\,dx$$

and then literally integrate both sides. It is fine to treat the symbols dy and dx this way; we will see what it really means later.

Example 1.2.4. In the general (non-homogeneous) case

$$\frac{dy}{dx} + ay = b,$$

one can use a similar procedure to obtain the general solution

$$y = \frac{b}{a} + ce^{-ax}.$$

1.3 Separable equations

The procedure in Example 1.2.3 actually works in more general settings, even for non-linear first-order ODEs. The key point is being able to *separate* x's from y's.

Definition 1.3.1. In (1.1), if $f(x, y)$ factors as $g(x)h(y)$, i.e. the ODE is

$$\frac{dy}{dx} = g(x)h(y)$$

then we say the ODE is **separable**. Here $g(x)$ must be *independent* of y, and $h(y)$ must be *independent* of x. Sometimes people let $M(x) = g(x)$ and $N(y) = 1/h(y)$ and write the general form as

$$N(y)\frac{dy}{dx} = M(x) \quad \text{or} \quad M(x)\,dx + N(y)\,dy = 0.$$

Example 1.3.2. The first-order ODE

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

is separable: $g(x) = x^2$ and $h(y) = 1/(1-y^2)$. By moving all the y's to the lhs, we get

$$(1-y^2)\frac{dy}{dx} = x^2.$$

Now we can integrate both sides to get

$$y - \frac{y^3}{3} = \frac{1}{3}x^3 + C.$$

The general solution is therefore $-x^3 + 3y - y^3 = C$ for an arbitrary constant C.

Example 1.3.3. The first-order ODE

$$\frac{dy}{dx} = \frac{x^2}{x-y^2}$$

is *not* separable, because there is no way to factor the rhs as $g(x)h(y)$. So we cannot solve it by directly integrating.

Since separable first-order ODEs are our first type of ODEs for which we get interesting solutions and are fairly straightforward to analyze, we will deal with them very carefully and rigorously. The first tricky point is the following observation.

1. If for some value y_0 we have $h(y_0) = 0$, then the ODE becomes $dy/dx = 0$ and there is an extra **constant** solution given by $y = y_0$.

Example 1.3.4. The first-order ODE

$$\frac{dy}{dx} = -2xy^2$$

is separable and, by integrating, has general solution

$$y = \frac{1}{x^2 + C} \text{ and } y = 0.$$

Note the *additional* constant solution $y = 0$, since if $y = 0$ then the rhs is zero. This solution does not arise from any finite choice of C, and is, in some sense, the solution obtained in the limit $C \to \infty$.

So we see special behavior when the rhs becomes 0. The other extreme is therefore also possible: the rhs can become ∞ at certain values. Solution functions which "run into" these values break down and become invalid beyond those values.

2. Suppose $y = f(x)$ is a solution. If $y'(x_0) = \infty$, then the **interval of validity** of the solution $y = f(x)$ does not extend beyond x_0.

Example 1.3.5. Recall that in example 1.3.2 we got

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \implies -x^3 + 3y - y^3 = C.$$

Suppose we want a solution function $y = f(x)$ passing through $(1, 0)$, so that $C = -1$. When $y = \pm 1$, the rhs of the ODE becomes ∞. We can compute that

$$y = -1 \implies x = -1, \quad y = 1 \implies x = \sqrt[3]{3} \approx 1.44.$$

Hence the interval of validity for the solution is $(-1, \sqrt[3]{3})$, because $x = 1$ lies in this interval.

One has to be careful when determining intervals of validity: it is possible the given expression for dy/dx is continuous everywhere, but the integral curve has an asymptote and goes off to ∞ anyway (cf. (1.2)). We will not worry too much about these details.

1.4 Existence and uniqueness

We can see already there is a huge difference between linear and non-linear first-order ODEs. At least for constant-coefficient linear ODEs, the solution exists and is valid for all x, since it is (essentially) just some exponential function; we do not have to worry about the interval of validity like we did earlier. This is actually true for *all* linear ODEs.

Theorem 1.4.1 (Existence and uniqueness for linear first-order ODEs)**.** *Consider the IVP*

$$\frac{dy}{dx} + p(x)y = q(x), \quad y(x_0) = y_0.$$

If $p(x)$ and $q(x)$ are continuous on an open interval $x_- < x < x_+$ containing x_0, then on the interval $x_- < x < x_+$ there exists a unique solution to the IVP.

This theorem is conceptually very important. Existence assures us that trying to solve the ODE is not a futile effort, and indicates to us that the only way solutions can break down is when $p(x)$ or $q(x)$ are discontinuous. Uniqueness means if we can guess/find a solution, we have found the one and only possible solution and do not have to keep searching for solutions. This is no longer true for higher-order ODEs. For example, $y'' = -y$ has the two distinct solutions $y = \sin(x)$ and $y = \cos(x)$.

It will become clear later why this theorem is true, when we explicitly solve all linear first-order ODEs via a trick called *integrating factors*.

On the other hand, solutions to non-linear first-order ODEs may not be valid for all values of x, even when everything is nice and continuous. For example, one can easily check that a solution to

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 0 \tag{1.2}$$

is given by the function $y = \tan(x)$ and is only valid on $(-\pi/2, \pi/2)$. It is also not immediately clear whether other solutions exist.

Theorem 1.4.2 (Existence and uniqueness for non-linear first-order ODEs)**.** *Consider the IVP*

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

If f and $\partial f/\partial y$ are continuous on a rectangle $x_- < x < x_+$, $y_- < y < y_+$ containing (x_0, y_0), then on some interval $x_0 - h < x < x_0 + h$ contained in $x_- < x < x_+$, there exists a unique solution to the IVP.

In contrast to the linear case, this theorem only assures us of existence and uniqueness *locally*, i.e. in a small region around the initial value (x_0, y_0).

1.5 Autonomous systems

We will now proceed to solve linear first-order ODEs in general. This requires a small digression.

Definition 1.5.1. In (1.1), if $f(x, y)$ is *independent* of x, i.e. the ODE is

$$\frac{dy}{dx} = f(y),$$

then we say the ODE is **autonomous**.

Autonomous equations are particularly common. If we interpret the variable x as time, then $f(y)$ being time-independent means that the rules governing the system described by the ODE do not change over time. For example, while the trajectory of a cannonball is a time-dependent function, the physical laws which dictate the ODE that its trajectory satisfies are time-independent.

One way that autonomous equations arise for us is as follows. To study general first-order ODEs

$$\frac{dy}{dx} = f(x, y),$$

we can alternatively study the autonomous **system** of ODEs

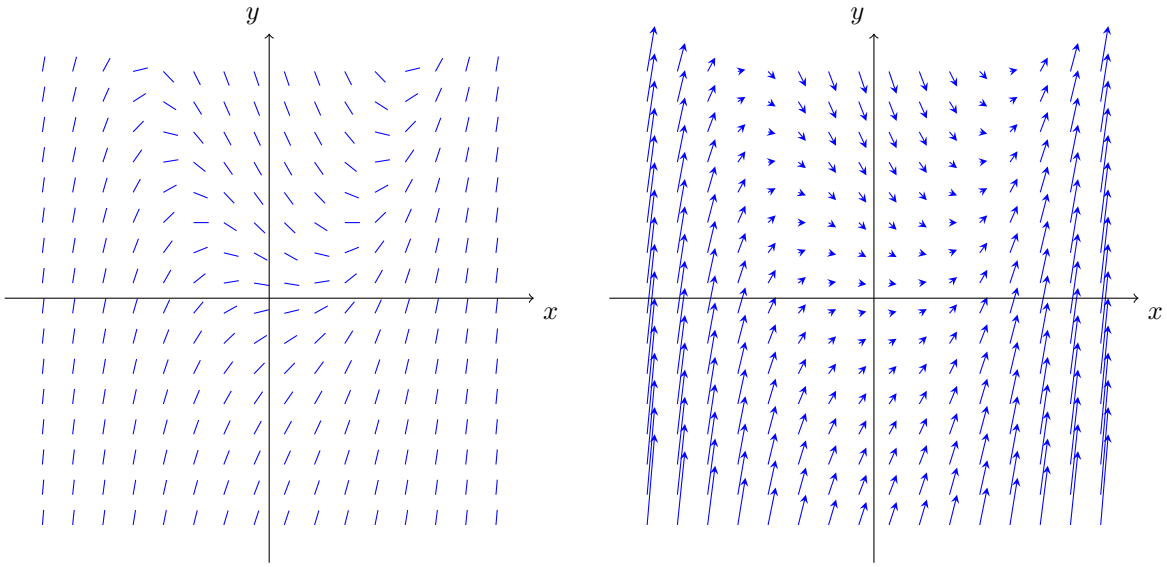
$$\begin{cases} \dfrac{dy}{dt} &= M(x, y) \\ \dfrac{dx}{dt} &= N(x, y) \end{cases} \tag{1.3}$$

where $f(x, y) = M(x, y)/N(x, y)$. Of course any f can be written this way, by taking $M = f$ and $N = 1$. If a solution of the original ODE is written as an integral curve, then a solution $(x(t), y(t))$ to (1.3) is a *parameterization* (in t) of such an integral curve.

Historically, people like to write (1.3) as

$$-M(x, y)\,dx + N(x, y)\,dy = 0.$$

This is an unfortunate choice of notation. We will use this notation to agree with common references (including our textbook), but it is best to keep (1.3) in mind instead. The upside of this notation is that *gradient* vector of the (parameterized) solution passing through (x, y) is given by $(-M(x, y), N(x, y))$. If we plot the vectors (N, M), we get the slope field of the ODE but with actual vectors instead of just slopes.



Alternatively, thinking of the ODE as

$$-M(x,y) + N(x,y)\,\frac{dy}{dx} = 0$$

is also fine, but this forgets the parameterization (and replaces the vector field with the slope field).

1.6 Exact equations and integrating factors

Suppose we have a first-order ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

and we recognize that its vector field (M, N) satisfies the following very special property.

Definition 1.6.1. A vector field is **exact**, or **conservative**, if it is the gradient

$$\nabla\psi = \begin{pmatrix} \dfrac{\partial\psi}{\partial x} \\ \dfrac{\partial\psi}{\partial y} \end{pmatrix}$$

of some (differentiable) function $\psi(x, y)$. If ψ exists, it is called the **scalar potential** of the vector field.

Then we have effectively *solved* the ODE, because

$$M + N\frac{dy}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x,y)$$

and the ODE becomes

$$\frac{d}{dx}\psi(x,y) = 0.$$

This of course has solutions given by the level sets

$$\psi(x,y) = C$$

of the scalar potential $\psi(x, y)$.

Example 1.6.2. The first-order ODE

$$(1 + 2xy^2) + (2x^2y + 2y)\frac{dy}{dx} = 0$$

is neither separable nor linear. But one can check that

$$\frac{d}{dx}(x + x^2y^2 + y^2) = 1 + 2xy^2, \quad \frac{d}{dy}(x + x^2y^2 + y^2) = 2x^2y + 2y$$

so its general solution is $x + x^2y^2 + y^2 = C$.

This whole discussion, however, is not practical unless we have a way to test whether a given vector field (M, N) is exact, and find the scalar potential of an exact vector field. Use the shorthand notation $f_t \coloneqq df/dt$ and multiple subscripts for multiple derivatives. If (M, N) were exact, then

$$M_y = \psi_{xy} = \psi_{yx} = N_x.$$

This is a very special property and is definitely not satisfied by arbitrary pairs of functions (M, N). In fact, it is *equivalent* to exactness.

Theorem 1.6.3. *Let (M, N) be a vector field. If $M_y = N_x$, then (M, N) is exact.*

Knowing (M, N) is exact, we can integrate the equation $\psi_x = M$ to get

$$\psi = \int M(x, y)\, dx + C(y)$$

where the "constant" of integration $C(y)$ is independent of x but can be a non-trivial function of y. Then we compute $\psi_y = N$ to solve for $C(y)$.

Example 1.6.4. Return to the earlier example

$$(M, N) = (1 + 2xy^2, 2x^2y + 2y).$$

We can check it is exact by computing $M_y = 4xy = N_x$. Integrating M, we get

$$\psi = x + x^2y^2 + C(y).$$

Then $N = \psi_y = 2x^2y + C'(y)$ so that $C'(y) = 2y$. Hence $C(y) = y^2$. (The constant of integration here is irrelevant.)

Most first-order ODEs $M\, dx + N\, dy = 0$ are not exact. However they can be made exact via the following observation. In the rewriting

$$\frac{dy}{dx} = f(x, y) \rightsquigarrow \begin{cases} \dfrac{dy}{dt} = M(x, y) \\ \dfrac{dx}{dt} = N(x, y) \end{cases}$$

we are always free to multiply $M(x, y)$ and $N(x, y)$ by an arbitrary **integrating factor** $\mu(x, y)$ without changing the original ODE. The new autonomous system

$$\begin{cases} \dfrac{dy}{dt} = \mu(x, y)M(x, y) \\ \dfrac{dx}{dt} = \mu(x, y)N(x, y) \end{cases}$$

will have the same integral curves as the original one, but with different parameterization (i.e. speed of trajectories). Equivalently, one writes

$$\mu M\, dx + \mu N\, dy = 0.$$

The game is to choose the integrating factor q such that the resulting ODE is exact. In general, this is hard, because the exactness condition is $(\mu M)_y = (\mu N)_x$, which is the first-order *partial* differential equation (PDE)

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0.$$

This is in general just as hard to solve as the original equation. Since it is a PDE instead of an ODE, solutions μ may not even be unique. Finding integrating factors in general is hard. However we can find them in special cases.

Example 1.6.5 (Linear first-order ODEs)**.** In the *linear* case $dy/dx + p(x)y = q(x)$, we can take

$$M = p(x)y - q(x), \quad N = 1$$

and therefore we should look for $\mu(x, y)$ such that

$$\mu_x = (\mu M)_y = \mu_y M + \mu p(x).$$

If we assume $\mu(x)$ is actually independent of y, this becomes the equation

$$\mu_x = \mu p(x)$$

which has a solution

$$\mu = \exp \int p(x)\, dx.$$

This is the integrating factor for linear first-order ODEs.

In principle one solves linear first-order ODEs in general using this method. In practice, one rarely needs to do so. However this example does provide essentially the proof of existence of solutions for linear first-order ODEs, as stated in Theorem 1.4.1. Uniqueness follows from a more careful analysis of the integrating factor, which in this case is unique up to a choice of constant of integration. To fully prove uniqueness, one must show this choice does not affect the final solution.

1.7 Changes of variable

Sometimes we can reduce a complicated ODE to the linear or separable case via a change of variables. In practice, it is more common to do this than to use e.g. integrating factors.

Example 1.7.1. Consider the equation

$$\frac{dy}{dx} = 1 + \sqrt{y - x}.$$

The $\sqrt{y-x}$ term is very hard to deal with. If we let $u \coloneqq y - x$, then the equation becomes

$$\frac{du}{dx} = \sqrt{u}.$$

This is now separable. The general solution is $u = (x/2 + c)^2$, i.e. $y = x + (x/2 + c)^2$.

In this example, the equation became much simpler because it was secretly a function of $u = y - x$ and x instead of y and x. In general, to identify useful changes of variables, one must ask if the equation is actually a function of not y but a more complicated combination of y and possibly x. We illustrate this with the following class of functions.

Definition 1.7.2. A function $f(x, y)$ is **homogeneous of degree** d if

$$f(tx, ty) = t^d f(x, y), \quad \text{for all } t.$$

Example 1.7.3. If f only involves monomials of degree d, then f is homogeneous of degree d. For example, there are four monomials in x and y of degree 3, which we can put together into the general form

$$f(x, y) = a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3$$

of a homogeneous degree 3 polynomial.

The interesting class of functions are those which are homogeneous of *degree zero*. This means

$$f(tx, ty) = f(x, y), \quad \text{for all } t. \tag{1.4}$$

Such functions can arise as the quotient of two homogeneous functions of degree d. The simplest (non-constant) such functions are

$$f(x, y) = \frac{x}{y}, \quad f(x, y) = \frac{y}{x}.$$

Theorem 1.7.4. *If $f(x, y)$ is homogeneous of degree 0, then actually it depends only on y/x only.*

Proof. In (1.4), take $t = 1/x$ to get

$$f(1, y/x) = f(x, y).$$

Clearly the rhs is a function of y/x only. □

Example 1.7.5. The function

$$f(x,y) = \frac{3xy^2 + 4y^3}{x^3 - 7x^2y}$$

is a quotient of two homogeneous functions of degree 3, and is therefore homogeneous of degree 0. By the theorem, we know it is secretly a function of just $V := y/x$. So plug in $y = xV$ to get

$$f(x,y) = \frac{3x^3V^2 + 4x^3V^3}{x^3 - 7x^3V} = \frac{3V^2 + 4V^3}{1 - 7V}.$$

This change of variables for degree-0 homogeneous functions helps us solve first-order ODEs of the form

$$\frac{dy}{dx} = f(x,y)$$

where f is homogeneous of degree 0. Since we have identified $y = xV$ as a good substitution, use it to get

$$V + x\frac{dV}{dx} = f(V).$$

This is a separable equation. Rearranging gives

$$\frac{1}{f(V) - V}\frac{dV}{dx} = \frac{1}{x}.$$

Example 1.7.6. Consider the equation

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}.$$

The rhs is homogeneous of degree 0. The substitution $y = xV$ gives

$$V + x\frac{dV}{dx} = \frac{2V}{1 - V^2}.$$

Rearranging into separable form yields

$$\frac{1 - V^2}{V(1 + V^2)}\frac{dV}{dx} = \frac{1}{x}.$$

To integrate the lhs we must do a partial fraction decomposition:

$$\frac{1 - V^2}{V(1 + V^2)} = \frac{1}{V} - \frac{2V}{1 + V^2}.$$

Then the whole equation integrates to

$$\ln|V| - \ln(1 + V^2) = \ln|x| + C.$$

Equivalently, after exponentiating both sides,

$$\frac{V}{1 + V^2} = cx.$$

Substituting back $V = y/x$ and rearranging gives the general solution

$$x^2 + y^2 = cy.$$

2 Second-order linear equations

To go beyond first-order ODEs and still have meaningful theory, we must restrict ourselves to linear equations. In practice, arbitrary ODEs can be well-approximated by linear ones, and it is the linear ODEs which dominate areas like classical physics.

Definition 2.0.1. A **second-order ODE** is of the general form

$$y'' = f(x, y, y'). \tag{2.1}$$

(We switch from writing dy/dx to writing y' for convenience.) It is **linear** if f depends linearly on y and dy/dx, i.e.

$$f\left(x, y, \frac{dy}{dx}\right) = g(x) - p(x)\frac{dy}{dx} - q(x).$$

In this case we usually write

$$y'' + p(x)y' + q(x)y = g(x).$$

As with first-order linear equations, we should begin by considering the associated **homogeneous equation**

$$y'' + p(x)y' + q(x)y = 0.$$

2.1 Structure of solutions

First notice that second-order ODEs in general may have more than one solution. For example, $y'' = -y$ has solutions $y = \sin(x)$ and $y = \cos(x)$. This arises because in some sense we have to "integrate twice", giving *two* constants of integration instead of one. This is very different from the first-order case, where existence and uniqueness guarantees only one solution (up to some constant of integration).

The key property of *linear* ODEs is as follows. Having two different solutions of a linear equation means we can use them to generate other solutions in the following way. (This was irrelevant for first-order ODEs.)

Theorem 2.1.1 (Principle of superposition)**.** *If y_1 and y_2 are solutions to a homogeneous linear ODE*

$$y'' + p(x)y' + q(x)y = 0,$$

then so is $c_1y_1 + c_2y_2$ for any constants c_1, c_2.

Proof. Since we are discussing second-order ODEs, we provide the proof explicitly for second-order linear ODEs. But the same idea works in general, for arbitrary order. We can just plug $c_1y_1 + c_2y_2$ into the equation and use that $(f + g)' = f' + g'$ to get

$$\begin{aligned}&(c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2)\\ &= c_1\left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2\left(y_2'' + p(x)y_2' + q(x)y_2\right) = 0.\end{aligned}$$

□

Linear second-order ODEs also have an existence and uniqueness theorem analogous to that of first-order ODEs (Theorem 1.4.1). The only modification is that we need *two* initial conditions instead of one. The conditions will be of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_0'.$$

In some sense, the second condition fixes the constant of integration from "integrating once", and the first condition fixes the constant of integration from "integrating again".

Theorem 2.1.2 (Existence and uniqueness for linear second-order ODEs)**.** *Consider the IVP*

$$y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = y_0,\ y'(x_0) = y_0'.$$

If $p(x), q(x), g(x)$ are continuous on an open interval I containing x_0, then on the interval I there exists a unique solution to the IVP.

There is one more complication that arises for second-order ODEs. Suppose we have found two solutions y_1 and y_2, and would like to solve an IVP with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_0'.$$

Then superposition tells us we should look for constants c_1, c_2 such that $c_1 y_1 + c_2 y_2$ is the solution we want, i.e.

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0'. \end{cases} \tag{2.2}$$

Here we treat c_1, c_2 as unknowns, and all other quantities as known (i.e. they are just some constants). From linear algebra, there exists a solution for c_1 and c_2 if and only if

$$\det \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \neq 0.$$

Definition 2.1.3. The determinant

$$W(y_1, y_2) \coloneqq \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2$$

is called the **Wronskian**. It is a *function* of x, and one can evaluate it at particular values $x = x_0$:

$$W(y_1, y_2)(x_0) = y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0).$$

Theorem 2.1.4. *Suppose y_1 and y_2 are two solutions of*

$$y'' + p(x)y' + q(x)y = 0.$$

Then it is always possible to choose c_1 and c_2 such that the solution

$$c_1 y_1 + c_2 y_2$$

satisfies prescribed initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_0'$$

if and only if the Wronskian $W(y_1, y_2)$ is not zero at x_0.

It is possible that $W(y_1, y_2)(x_0) = 0$ and no solution exists, despite the existence and uniqueness theorem 2.1.2. This may occur if we chose y_1 and y_2 badly. For example, if $y_1 = y_2$ then we are missing another solution and the general solution will not be just $c_1 y_1 + c_2 y_2 = (c_1 + c_2) y_1$. The Wronskian of y_1 and y_2 measures how "independent" they are.

Example 2.1.5. Take the IVP

$$y'' - 3y' + 2y = 0, \quad y(1) = 5,\ y'(1) = 7.$$

One can verify directly that $y_1 = e^x$ and $y_2 = e^{2x}$ are both solutions. The principle of superposition therefore tells us that

$$c_1 e^x + c_2 e^{2x}$$

is a solution for any constants c_1, c_2. The Wronskian is

$$W(e^x, e^{2x}) = \det \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} = 2e^{3x} - e^{3x} = e^{3x},$$

which is non-zero at $x = 1$. Therefore there are constants c_1, c_2 providing a solution to the IVP. If we had chosen $y_1 = e^x$ and $y_2 = 2e^x$ instead,

$$W(e^x, 2e^x)(1) = e^1 \cdot (2e^1) - e^1 \cdot (2e^1) = 0,$$

telling us that y_1 and y_2 are not independent solutions. This choice of solutions completely misses the solution e^{2x}.

The Wronskian provides an easy test for whether a solution of the form $c_1y_1 + c_2y_2$ exists for the IVP, but does not tell us what exactly c_1 and c_2 are. To actually find their values we must solve the corresponding 2×2 system of equations (2.2).

Example 2.1.6. Continuing with the previous example, we want

$$c_1e^1 + c_2e^2 = 5$$
$$c_1e^1 + 2c_2e^2 = 7.$$

Hence $c_2 = 2e^{-2}$ and $c_1 = 3e^{-1}$. The solution to the IVP is therefore

$$y = 3e^{-1}e^x + 2e^{-2}e^{2x} = 3e^{x-1} + 2e^{2x-2}.$$

Armed with the Wronskian, we can now explain the form of the general solution to a linear second-order ODE. One would like to say all solution are of the form $c_1y_1 + c_2y_2$, as long as the two solutions y_1 and y_2 are "independent".

Theorem 2.1.7. *Suppose y_1 and y_2 are two solutions of*

$$y'' + p(x)y' + q(x)y = 0.$$

Then all solutions are of the form

$$c_1y_1 + c_2y_2$$

for some c_1, c_2 if and only if the Wronskian $W(y_1, y_2)$ has at least one point where it is non-zero.

Proof. Suppose $W(y_1, y_2)(x_0) \neq 0$. Let ϕ be a solution. Look at the IVP with initial conditions

$$y(x_0) = \phi(x_0), \quad y'(x_0) = \phi'(x_0).$$

Clearly ϕ is a solution to the IVP. But by theorem 2.1.4, there exist constants c_1 and c_2 such that $c_1y_1 + c_2y_2$ is also a solution. Uniqueness of solutions implies $\phi = c_1y_1 + c_2y_2$.

Conversely, if $W(y_1, y_2) = 0$ everywhere, then we can pick some initial conditions such that (2.2) has no solutions for c_1 and c_2. But the IVP with these initial conditions must have a solution ϕ, by existence of solutions. Hence ϕ cannot be written as $c_1y_1 + c_2y_2$ for any c_1, c_2. □

In other words, y_1 and y_2 are "independent" as long as $W(y_1, y_2)$ is not the zero function. In this case we call

$$c_1y_1 + c_2y_2$$

the **general solution**, and we say y_1, y_2 form a **fundamental set of solutions**.

2.2 Homogeneous equations with constant coefficients

We apply all this theory to the constant-coefficient case, just like for first-order equations:

$$ay'' + by' + cy = 0 \tag{2.3}$$

for constants a, b, c. A helpful thing to do is to rewrite this ODE in terms of derivative *operators*, as follows.

Definition 2.2.1. Let D denote the operator which differentiates with respect to x, i.e. for a function $f(x)$,

$$Df \coloneqq \frac{df}{dx}.$$

Then (2.3) can be rewritten as

$$(aD^2 + bD + c)y = 0.$$

We can now take the quadratic $aD^2 + bD + c$ and factor it into a product of two linear factors. (This is really a valid thing to do, because the derivative D is a *linear* operator.) So suppose that

$$aD^2 + bD + c = (D - \alpha_1)(D - \alpha_2)$$

for some constants α_1 and α_2. Then we have effectively reduced the second-order ODE into two first-order ones: it will have two solutions y_1 and y_2, given by

$$(D - \alpha_1)y_1 = 0, \quad (D - \alpha_2)y_2 = 0.$$

From section 1.2, we know the (unique) general solutions are:

$$y_1 = c_1 e^{\alpha_1 x}, \quad y_2 = c_2 e^{\alpha_2 x}. \tag{2.4}$$

Hence the general solution to the original second-order ODE is

$$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}.$$

Example 2.2.2. Consider the ODE

$$y'' - 3y' + 2y = 0.$$

In terms of the derivative operator, this is equivalent to

$$(D^2 - 3D + 2)y = 0.$$

Clearly $D^2 - 3D + 2 = (D - 1)(D - 2)$. Hence the general solution is

$$y = c_1 e^x + c_2 e^{2x}.$$

Definition 2.2.3. Sometimes people don't like writing things in terms of the derivative operator D, and instead introduce a new variable r and the **characteristic equation**

$$ar^2 + br + c = 0$$

associated to the ODE $ay'' + by' + cy = 0$. In this language, we have just seen that if $r = \alpha$ is a solution to the characteristic equation, then $y = e^{\alpha x}$ is a solution to the ODE.

However, we must be careful. From the previous section, we know it is important to verify that the two solutions (2.4) we use to form the general solution are actually *independent* solutions. This may not always be the case. For example, if the characteristic equation has a double root $\alpha_1 = \alpha_2$, then we don't actually have two different solutions.

Example 2.2.4. Consider the ODE

$$y'' + 4y' + 4y = 0.$$

Then we factor $D^2 + 4D + 4 = (D + 2)^2$ and get the solutions $y_1 = e^{-2x}$ and $y_2 = e^{-2x}$. Clearly these are not two independent solutions.

From the previous section, we know we can test independence by computing the Wronskian

$$W(e^{\alpha_1 x}, e^{\alpha_2 x}) = \det \begin{pmatrix} e^{\alpha_1 x} & e^{\alpha_2 x} \\ \alpha_1 e^{\alpha_1 x} & \alpha_2 e^{\alpha_2 x} \end{pmatrix} = (\alpha_2 - \alpha_1) e^{(\alpha_2 + \alpha_2)x}.$$

As expected, it is non-zero when $\alpha_1 \neq \alpha_2$, and zero in the case of a double root $\alpha_1 = \alpha_2$. This means we need to manufacture some solution other than $e^{\alpha_1 x}$ somehow.

How do we manufacture this second solution? Let $(D - r)^2 y = 0$ be the ODE. The first solution is $y_1 = e^{rx}$. To get the second solution, one way to proceed is to make an **ansatz**, i.e. an educated guess, that the second solution is of the form

$$y_2 = v(x)e^{rx} \tag{2.5}$$

for some *function* $v(x)$ instead of just a constant. This is just a guess! If it works in the end, we are happy; if it does not, we go back and try something more clever.

To find what $v(x)$ should be, we plug $v(x)e^{rx}$ back into the ODE. This is computationally slightly complicated, so first we compute

$$(D-r)(v(x)e^{rx}) = (v'(x)e^{rx} + rv(x)e^{rx}) - rv(x)e^{rx} = v'(x)e^{rx}.$$

Applying $(D-r)$ again gives

$$(D-r)^2(v(x)e^{rx}) = (v''(x)e^{rx} + rv'(x)e^{rx}) - rv'(x)e^{rx} = v''(x)e^{rx}.$$

For this to be equal to 0, we must have $v''(x) = 0$. Hence $v'(x) = c_1$ and $v(x) = c_1x + c_2$. A solution of the form

$$y = (c_1x + c_2)e^{rx}$$

therefore works! Note that c_2e^{rx} is just some multiple of the first solution y_1, and the truly new solution is

$$y_2 = xe^{rx}.$$

We should check however that $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are truly independent solutions, using the Wronskian. Their Wronskian is

$$\begin{aligned} W(e^{rx}, xe^{rx}) &= \det \begin{pmatrix} e^{rx} & xe^{rx} \\ re^{rx} & (1+rx)e^{rx} \end{pmatrix} \\ &= (1+rx)e^{2rx} - rxe^{2rx} = e^{2rx} \neq 0. \end{aligned}$$

Hence $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are a fundamental set of solutions. This is how we deal with the case of double roots.

To sum up, to solve a constant-coefficient homogeneous linear second-order ODE $ay'' + by' + cy = 0$, we should find the roots α_1, α_2 of its characteristic equation $ar^2 + br + c = 0$.

1. If the roots are distinct, a fundamental system of solutions is

$$y_1 = e^{\alpha_1 x}, \quad y_2 = e^{\alpha_2 x}.$$

2. If the roots are not distinct, a fundamental system of solutions is

$$y_1 = e^{\alpha_1 x}, \quad y_2 = xe^{\alpha_1 x}.$$

The last thing to deal with is that the two solutions α_1 and α_2 may be complex numbers instead of real numbers. In this case it is important to remember Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

Consequently, if we have $\alpha = \lambda + \mu i$, then

$$e^{\alpha x} = e^{\lambda x}e^{i\mu x} = e^{\lambda x}(\cos \mu x + i \sin \mu x).$$

Example 2.2.5. Consider the ODE

$$y'' + y = 0.$$

The solutions to its characteristic equation are $\alpha_1 = i$ and $\alpha_2 = -i$. Hence the two solutions are

$$\begin{aligned} y_1 &= e^{ix} = \cos x + i \sin x \\ y_2 &= e^{-ix} = \cos x - i \sin x. \end{aligned}$$

However this is not exactly desirable, because we started with only real numbers and would like the solutions as real-valued functions. The trick is to "cancel" out the imaginary parts by looking at $y_1 \pm y_2$:

$$y_1 + y_2 = 2\cos x, \quad y_1 - y_2 = 2i \sin x.$$

Since constants are irrelevant, this means we can define a new pair of fundamental solutions

$$\tilde{y}_1 = \cos x, \quad \tilde{y}_2 = \sin x.$$

In general, if a quadratic has complex roots, the two complex roots are conjugate pairs $\lambda \pm \mu i$. Then the two solutions will be

$$y_1 = e^{\lambda x}(\cos \mu x + i \sin \mu x)$$
$$y_2 = e^{\lambda x}(\cos \mu x - i \sin \mu x)$$

and the same argument as in the example shows that we can always define a new pair of fundamental solutions

$$\tilde{y}_1 = e^{\lambda x} \cos \mu x, \quad \tilde{y}_2 = e^{\lambda x} \sin \mu x.$$

2.3 Non-homogeneous equations

Now we return to the general non-homogeneous case

$$y'' + p(x)y' + q(x)y = g(x). \tag{2.6}$$

Suppose that we have already solved the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

and obtained its general solution $c_1 y_1 + c_2 y_2$. The previous section describes how to do this when $p(x)$ and $q(x)$ are *constants*, i.e. actually independent of x. The following observation allows us to use the homogeneous case to get solutions for the original ODE.

Theorem 2.3.1. *If Y_1, Y_2 are two solutions of the non-homogeneous equation (2.6), then $Y_1 - Y_2$ is a solution of the associated homogeneous equation.*

Proof. This is a direct check:

$$\begin{aligned}
&(Y_1 - Y_2)'' + p(x)(Y_1 - Y_2)' + q(x)(Y_1 - Y_2) \\
&= (Y_1'' + p(x)Y_1' + q(x)Y_1) - (Y_2'' + p(x)Y_2' + q(x)Y_2) \\
&= g(x) - g(x) = 0.
\end{aligned}$$

□

But we know *all* solutions of the homogeneous equation: they are $c_1 y_1 + c_2 y_2$. So once we can find a single solution Y of the original equation, then all other solutions are of the form

$$Y + c_1 y_1 + c_2 y_2.$$

Definition 2.3.2. Given a linear second-order ODE as in (2.6), the general solution $c_1 y_1 + c_2 y_2$ of its associated homogeneous equation is called the **complementary solution**. A single solution Y of the actual ODE is called a **particular solution**.

It is important to note at this point that a particular solution will *never* be a solution of the homogeneous equation. So particular solutions are distinct from the complementary solution.

Example 2.3.3. Consider the ODE

$$y'' - 5y' + 6y = 2e^x.$$

The complementary solution is $c_1e^{2x} + c_2e^{3x}$. One can check that a particular solution is $y = e^x$. Hence the general solution is

$$y = e^x + c_1e^{2x} + c_2e^{3x}.$$

There are two different ways to find particular solutions that are commonly taught. One way is to simply make an *ansatz*, like we did in (2.5). This is known as the **method of undetermined coefficients**, and works well when the rhs $g(x)$ is of some particular form.

Example 2.3.4. Consider the non-homogeneous linear ODE

$$y'' - 3y' - 4y = 3e^{2x}.$$

The rhs is of a very special form: it is (up to constant factors) an exponential. Since exponentials are indestructible under differentiation, we can make the ansatz that a particular solution is of the form

$$y = Ae^{2x}$$

for some constant A. Plugging this into the ODE gives

$$(4A - 6A - 4A)e^{2x} = 3e^{2x},$$

so that $A = -1/2$. Hence a particular solution is $y = -1/2e^{2x}$. The general solution is therefore

$$y = -\frac{1}{2}e^{2x} + c_1e^{4x} + c_2e^{-x}.$$

Similarly, if the rhs is some multiple of $\sin x$ or $\cos x$, then we should make the ansatz $y = A_1 \sin x + A_2 \cos x$. Here we must involve $\cos x$ because we obtain it upon differentiating $\sin x$. Now we get a 2×2 system of equations for A_1 and A_2 that must be solved. The details in this case reveal no new conceptual insights, so we omit them.

The other commonly-taught method to find particular solutions is called **variation of parameters** and works for arbitrary $g(x)$. However it is made obsolete by the more useful concept of a Green's function, so we will not cover it in this course.

2.4 Higher-order linear equations

All the theory we have developed for second-order linear equations carry over analogously to higher-order. Let's briefly examine the general higher-order case.

Definition 2.4.1. Let $y^{(n)}$ denote the n-th derivative of y. An **n-th order linear ODE** is of the form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x). \tag{2.7}$$

Since now we must "integrate n times" to get a solution, we expect that to get a unique solution one must specify n initial conditions

$$y(x_0) = y_0^{(0)}, \quad y'(x_0) = y_0^{(1)}, \quad \ldots, \quad y^{(n-1)}(x_0) = y_0^{(n-1)} \tag{2.8}$$

where $y_0^{(0)}, y_0^{(1)}, \ldots, y_0^{(n-1)}$ are real constants. Then we have the following.

Theorem 2.4.2 (Existence and uniqueness for n-th order linear ODEs)**.** *If the functions $p_0, \ldots, p_{n-1}$ and g are continuous on an open interval I, then there exists a unique solution to* (2.7) *satisfying the initial conditions* (2.8)*.*

As with the first- and second-order cases, we can consider the associated **homogeneous equation**

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

and its solutions $y_1, \ldots, y_n$. The **general solution** is of the form

$$c_1 y_1 + \cdots + c_n y_n,$$

and there is again a concept of a Wronskian to check whether there exist constants $c_1, \ldots, c_n$ such that the solution satisfies a given set of initial conditions.

Definition 2.4.3. The **Wronskian** $W(y_1, \ldots, y_n)$ is the function given by

$$W(y_1, \ldots, y_n)(x) \coloneqq \det \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}. \tag{2.9}$$

Theorem 2.4.4. *In the conditions of* (2.4.2)*, if* $W(y_1, \ldots, y_n)(x) \neq 0$ *for at least one* x *in the open interval* I*, then all solutions are of the form* $c_1 y_1 + \cdots + c_n y_n$*.*

In general, one can show that the condition on the Wronskian is equivalent to the condition that the functions $y_1, \ldots, y_n$ are *linearly independent*. In the second-order case, this just means y_1 and y_2 are not multiples of each other.

For the non-homogeneous case, Theorem 2.3.1 carries over verbatim, and the general solution is the sum of a particular solution and the complementary solution:

$$y = Y + c_1 y_1 + \cdots + c_n y_n.$$

Again, for some forms of the rhs $g(x)$, one can make an ansatz for the particular solution and proceed. In general, the method of Green's functions is better.

3 Singularities and series solutions

So far we have dealt with ODEs whose solutions are "nice" functions. Most ODEs (even most second-order lienar ones) have solutions which are not expressible in terms of elementary functions. We must resort to finding solutions *locally* around a given point $x = x_0$, expressed as power series. This works well for linear ODEs.

3.1 Review of power series

Definition 3.1.1. A **power series** at $x = x_0$ is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \coloneqq \lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0)^n. \tag{3.1}$$

Sometimes we will only want to write the first few terms. We use the **big-O notation** $O((x - x_0)^N)$ to mean "terms of degree-N or higher".

For most intents and purposes, power series can be thought of as polynomials "with an infinite number of terms". Arithmetic operations and differentiation/integration work as usual. However whenever infinite sums are involved, one must be careful about *convergence* issues. While a sum of finitely many numbers is always well-defined, sums of infinitely many are not always.

Definition 3.1.2. The power series (3.1) **converges at** x if the limit on the rhs exists. It converges **absolutely** if the series

$$\sum_{n=0}^{\infty} |a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

converges.

Every power series has a certain range $x_0 - \rho < x < x_0 + \rho$, which we also write as $|x - x_0| < \rho$, in which it converges and outside which it does not. This range is called the **radius of convergence** of the power series. In the same way that a solution to an ODE may only be valid in a certain region, e.g. Example 1.3.5 and Theorem 1.4.2, a power series is only valid inside its radius of convergence.

Definition 3.1.3. A function f is **analytic at** $x = x_0$ if in an open interval containing x_0, it is equal to a power series with radius of convergence $\rho > 0$.

We think of analytic functions as those which can be expanded as power series. If we already know that a function f is analytic, we can make the ansatz

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

to find its power series. (This ansatz will give a power series expansion around $x = 0$.) For example, from the ansatz, $f(0) = a_0$, so we already know the constant term. Higher-order terms are obtained by the observation that

$$f^{(k)}(x) = \sum_{n=k}^{\infty} a_k k! x^{n-k}$$

and therefore $f^{(k)}(0) = a_k k!$. This results in the **Maclaurin series** expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This is the expansion around $x = 0$. To get an expansion around $x = x_0$, we must replace x^n with $(x - x_0)^n$ in the ansatz, to get the more general **Taylor series** expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n. \tag{3.2}$$

We see that for a series expansion to exist, at least the function f must be infinitely differentiable. Often such functions are called **smooth**.

Example 3.1.4. Let $f(x) = e^x$. Then $f^{(n)}(0) = e^0 = 1$, so we have the series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).$$

Warning: it is possible for a function to be smooth but not analytic. In other words, even if (3.2) exists and has a radius of convergence $\rho > 0$, it may not be equal to the function f. The typical example of such a function is

$$f(x) = 1 - e^{-1/x^2},$$

whose Taylor series has all coefficients equal to zero.

3.2 Ordinary points

We consider only homogeneous equations because the procedure for non-homogeneous equations is very similar. For simplicity, we look at the second-order linear case. The goal is to solve them locally around $x = x_0$.

Definition 3.2.1. Consider the homogeneous second-order linear ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

If around $x = x_0$ the functions $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic, then x_0 is an **ordinary point**. Otherwise it is a **singular point**. At an ordinary point, we can rewrite the ODE as

$$y'' + p(x)y' + q(x)y = 0, \quad p(x) \coloneqq \frac{Q(x)}{P(x)},\ q(x) \coloneqq \frac{R(x)}{P(x)}. \tag{3.3}$$

To find a series solution, it suffices to plug the general form (3.1) into the ODE and solve for the coefficients $\{a_n\}$. These coefficients will satisfy some **recurrence relation**, which relates a_n to a_m for $m < n$.

Example 3.2.2. Consider the first-order ODE

$$y' - y = 0.$$

We already know the solution $y = ce^x$ to this, but instead let's find a series solution around $x = 0$. By uniqueness, the series we get must be equal to $y = ce^x$ around $x = 0$, for some c. We have

$$\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Hence the coefficient of x^n is zero, for every n. Rewrite the first sum so that we can more easily extract this coefficient

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

It follows that

$$a_{n+1}(n+1) - a_n = 0, \quad n = 0, 1, 2, \ldots.$$

This is a recurrence relation for the coefficients $\{a_n\}$. For example, the first few coefficients are

$$a_1 = \frac{a_0}{1} = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \quad \ldots.$$

In general, we can use the recurrence relation to write *all* the coefficients in terms of a_0:

$$a_k = \frac{a_0}{k!}.$$

There are no constraints on what a_0 is; it plays the role of the constant c. The series solution is

$$y = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 e^x,$$

as expected.

There is a deep relationship between ODEs and recurrence relations. A second-order constant-coefficient ODE will produce a recurrence involving a_{n+2} (coming from y'') and a_{n+1} (from y') and a_n (from y). In general, an n-th order constant-coefficient ODE produces an *n-th order* recurrence. Then we are free to choose the first n coefficients $a_0, a_1, \ldots, a_{n-1}$; these act as initial conditions and are analogous to the constants forming the general solution. An easy way to get the i-th solution in a fundamental system of solutions is to set $a_i = 1$ and all other $a_k = 0$.

Example 3.2.3. Consider the second-order linear ODE

$$y'' - y = 0.$$

This produces

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Reindexing, we get

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

For $n \geq 0$, we get the recurrence

$$a_{n+2}(n+2)(n+1) - a_n = 0.$$

It follows that

$$\begin{aligned} a_2 &= \frac{a_0}{2 \cdot 1}, \quad a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \quad \ldots, \\ a_3 &= \frac{a_1}{3 \cdot 2}, \quad a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \quad \ldots. \end{aligned}$$

The general solution is therefore of the form

$$y = a_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)\right) + a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right).$$

These two solutions are known as the *hyperbolic* cosine and sine respectively:

$$\cosh(x) \coloneqq \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad \sinh(x) \coloneqq \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

If we solve the equation using the characteristic equation, we get the general solution $y = c_1 e^x + c_2 e^{-x}$. In terms of this fundamental set,

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

There is a close relation between the *hyperbolic* trig function and usual trig functions. For example, note that

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

Example 3.2.4. Consider the second-order linear ODE

$$y'' - xy = 0$$

called the **Airy equation**. Its two solutions are the Airy functions of the first and second kind, neither of which can be expressed in terms of elementary functions. We get

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

So when $n \geq 1$, we get the recurrence

$$a_{n+2}(n+2)(n+1) - a_{n-1} = 0, \quad n = 1, 2, 3, \ldots,$$

and the special equation $2a_2 = 0$ when $n = 0$. Note that the non-constant coefficient x in the ODE made the recurrence *third-order*. It follows immediately that

$$a_2 = a_5 = a_8 = \cdots = 0.$$

The other coefficients are slightly more complicated:

$$\begin{aligned} a_4 &= \frac{a_1}{4 \cdot 3}, \quad a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}, \quad a_{10} = \cdots, \quad \ldots, \\ a_3 &= \frac{a_0}{3 \cdot 2}, \quad a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, \quad a_9 = \cdots, \quad \ldots. \end{aligned}$$

The general solution is therefore of the form

$$y = a_0 \left(1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + O(x^9)\right) + a_1 \left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + O(x^{10})\right).$$

These series have no closed form.

However it is not always possible to obtain a nice recurrence for all coefficients. This is because coefficients may be complicated functions of x, instead of just constants. Then typically one expands the functions into series, and manually solves for only the first few coefficients in the series solution.

Theorem 3.2.5 (Analyticity of series solutions at ordinary points)**.** *If $x = x_0$ is a ordinary point of*

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

i.e. Q/P and R/P are analytic at x_0, then its general solution is

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2$$

where y_1, y_2 are series solutions analytic at x_0 that form a fundamental set of solutions. Moreover their radius of convergence is at least as large as the minimum of the radii of convergence of Q/P and R/P.

Proof. We omit the proof that y_1, y_2 must be analytic. We will only check that they form a fundamental set of solutions, by computing their Wronskian. In general, regardless of the ODE,

$$y_1 = 1 + O(x^2), \quad y_2 = x + O(x^2).$$

This is because there are never constraints on a_0 and a_1, and all other terms are $O(x^2)$. Then

$$\begin{aligned} W(y_1, y_2) = \det \begin{pmatrix} 1 + O(x^2) & x + O(x^2) \\ O(x) & 1 + O(x) \end{pmatrix} &= (1 + O(x^2))(1 + O(x)) - O(x)(x + O(x^2)) \\ &= (1 + O(x)) - O(x^2) = 1 + O(x) \neq 0. \end{aligned}$$

Hence y_1, y_2 form a fundamental set of solutions. □

3.3 Regular singular points

Series solutions around ordinary points are more well-behaved than those around singular points. ODEs with singular points may have solutions which are not analytic at those points, so series solutions might not exist there. This may occur for fairly innocuous ODEs. However, for a class of *mild* singularities, we can still use the series methods of the previous section, suitably modified.

Definition 3.3.1. Suppose the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ has a singular point $x = x_0$. This means that $Q(x)/P(x)$ and $R(x)/P(x)$ are not analytic at $x = x_0$ (cf. (3.3)). However if both

$$\lim_{x\to x_0} \frac{Q(x)}{P(x)}(x - x_0), \quad \lim_{x\to x_0} \frac{R(x)}{P(x)}(x - x_0)^2 \tag{3.4}$$

exist, then we say $x = x_0$ is a **regular singular point** and we can still find series solutions. Otherwise it is an **irregular singular point**.

In English, (3.4) means that the function $Q(x)/P(x)$ has a pole of order at most one at $x = x_0$, and $R(x)/P(x)$ a pole of order at most two. For example, a *rational* function having a pole of order at most n at $x = x_0$ means it is of the form

$$\frac{g(x)}{(x - x_0)^n}$$

for some function $g(x)$ which is well-defined, i.e. has no pole, at $x = x_0$. (Equivalently, it means that in a series expansion around $x = x_0$, we must include terms of negative order up to $(x - x_0)^{-n}$.)

Example 3.3.2. Consider the **hypergeometric equation**

$$x(1 - x)y'' + (c - (a + b + 1)x)y' - aby = 0.$$

The functions $x(1 - x)$ and $(c - (a + b + 1)x)$ and $-ab$ are all polynomials and therefore analytic. So the only way

$$\frac{Q(x)}{P(x)} = \frac{c - (a + b + 1)x}{x(1 - x)}, \quad \frac{R(x)}{P(x)} = \frac{-ab}{x(1 - x)}$$

can fail to be analytic is if $x = 0$ or $x = 1$. These are two singular points of the equation.

1. At $x = 0$, the two functions

$$\frac{Q(x)}{P(x)}x = \frac{c - (a + b + 1)x}{1 - x}, \quad \frac{R(x)}{P(x)}x^2 = \frac{-abx}{1 - x}$$

   are analytic, by expansion using geometric series. So $x = 0$ is a regular singular point.

2. At $x = 1$, one can check also that the singularity is regular.

So we have found two regular singular points of the equation, at $x = 0, 1$.

Example 3.3.3. Consider the **Cauchy–Euler equation**

$$x^2y'' + \alpha xy' + \beta y = 0. \tag{3.5}$$

One can check that $x = 0$ is the only singular point, and it is regular. In some sense the Cauchy–Euler equation is the simplest ODE with regular singular points. But even in the simplest case

$$x^2y'' + y = 0$$

we cannot find any non-trivial series solutions. The lhs is

$$x^2 \sum_{n=0}^{\infty} a_n n(n - 1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} (n(n - 1) + 1)a_n x^n.$$

Setting this equal to 0, we get

$$a_0 = a_1 = 0, \quad (n(n - 1) + 1)a_n = 0,$$

i.e. all coefficients are zero! Hence the only series solution we can find this way is the trivial solution $y = 0$. The general solution is actually

$$y = c_1 \cos(\ln|x|) + c_2 \sin(\ln|x|).$$

When $x \neq 0$ this is analytic and therefore has a valid power series expansion. At $x = 0$ we run into issues with the $\ln|x|$ terms, and the function is no longer analytic. In general, the ansatz

$$y = x^r, \quad r \in \mathbb{C},$$

solves the Cauchy–Euler equation whenever

$$r(r-1) + \alpha r + \beta = 0.$$

This is called the associated **indicial equation**. In the above case, the two solutions for r are complex, leading to the appearance of sin and cos.

There is a modification of the series method, due to Frobenius, which handles the case of regular singular points. For convenience, suppose the regular singular point is at $x = 0$. Then

$$x\frac{Q(x)}{P(x)} = xp(x) \coloneqq \sum_{n=0}^{\infty} p_n x^n, \quad x^2\frac{R(x)}{P(x)} = x^2 q(x) \coloneqq \sum_{n=0}^{\infty} q_n x^n$$

are both analytic and have valid series expansions. We can rewrite

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \rightsquigarrow \quad x^2y'' + x(xp(x))y' + x^2q(x)y = 0.$$

Writing $p(x)$ and $q(x)$ in their series forms, this is equivalently

$$x^2y'' + x(p_0 + p_1x + O(x^2))y' + (q_0 + q_1x + O(x^2))y = 0. \tag{3.6}$$

If all $p_n = q_n = 0$ except p_0 and q_0, this becomes a Cauchy–Euler equation

$$x^2y'' + p_0xy' + q_0y = 0.$$

So in some sense the general case is "just" a Cauchy–Euler equation but with power series coefficients instead of constant coefficients. Hence we should allow modifications to the solution $y = x^r$ (where r solves the indicial equation) by power series:

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \tag{3.7}$$

where r solves the indicial equation

$$r(r-1) + p_0r + q_0 = 0.$$

We will call this the **Frobenius ansatz**. (There will be one for each solution r.) Using this ansatz to find solutions at a regular singular point is called the Frobenius method.

Example 3.3.4. Consider the hypergeometric equation

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby = 0.$$

We would like to find a series solution around the regular singular point $x = 0$. First rewrite the equation in the form (3.6), by computing

$$x\frac{Q(x)}{P(x)} = \frac{c-(a+b+1)x}{1-x} = \sum_{n=0}^{\infty} cx^n - (a+b+1)\sum_{n=1}^{\infty} x^n = c + O(x)$$
$$x^2\frac{R(x)}{P(x)} = -\frac{abx}{1-x} = 0 + O(x).$$

Hence the associated Cauchy–Euler equation is

$$x^2y'' + cxy' = 0.$$

The indicial equation for r is therefore

$$r(r-1) + cr = 0 \quad \rightsquigarrow \quad r = 0, 1-c.$$

So as long as c is not an integer, the two (distinct) Frobenius ansatzes should be

$$y_1 = \sum_{n=0}^{\infty} a_n x^n, \quad y_2 = x^{1-c} \sum_{n=0}^{\infty} a_n x^n.$$

Note that the first solution y_1 is actually a power series, and therefore analytic by general theory. This solution is generally called the hypergeometric function. Explicitly, the associated recurrence is

$$(n+1)na_{n+1} - n(n-1)a_n + c(n+1)a_{n+1} - (a+b+1)na_n - aba_n = 0, \quad n \geq 0.$$

Rearranging, this becomes

$$a_{n+1} = \frac{(n+a)(n+b)}{(n+1)(n+c)} a_n, \quad n \geq 0.$$

For convenience, we define the **rising factorial**

$$(a)_n \coloneqq a(a+1)(a+2)\cdots(a+n-1).$$

Then from the recurrence, we get

$$a_n = \frac{(a)_n (b)_n}{n!(c)_n} a_0.$$

Definition 3.3.5. The **hypergeometric function** is a three-parameter family of functions defined by

$${}_2F_1(a,b,c;x) \coloneqq \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}.$$

One can check that the other solution of the hypergeometric equation around $x = 0$ is

$$x^{1-c}\, {}_2F_1(1+a-c, 1+b-c, 2-c; x).$$

Hypergeometric functions are generalizations of many special functions. For example,

$$\ln(1+x) = x \cdot {}_2F_1(1,1,2;-x)$$
$$\arcsin(x) = x \cdot {}_2F_1(1/2, 1/2, 3/2; x^2).$$

Many special families of polynomials also arise in this way: Laguerre polynomials, which appear in the study of the hydrogen atom; Hermite polynomials, which appear in the study of the harmonic oscillator; Bessel functions, which appear in the study of heat conduction in rods, etc. We will see in section 3.4 the reason why hypergeometric functions are so ubiquitous.

Now we plug the Frobenius ansatz (3.7) back into the original equation (3.6) in order to analyze properties of the solutions. This yields

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \left(\sum_{n=0}^{\infty} a_n (n+r) x^{n+r}\right)\left(\sum_{n=0}^{\infty} p_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)\left(\sum_{n=0}^{\infty} q_n x^n\right) = 0.$$

We need to extract the coefficient of x^{n+r} from all three terms.

1. The first term is straightforward and gives $(n+r)(n+r-1)a_n$.

2. The second term requires us to multiply power series. Power series multiply just like polynomials. The x^{n+r} term must receive a contribution of x^{k+r} from the first sum and a contribution of x^{n-k} from the second sum, where k is any integer from 0 to n. So the coefficient of x^{n+r} here is

$$\sum_{k=0}^{n} a_k(k+r)p_{n-k} = a_n(n+r)p_0 + \sum_{k=0}^{n-1} a_k(k+r)p_{n-k}.$$

3. Similarly, the third term gives

$$\sum_{k=0}^{n} a_k q_{n-k} = a_n q_0 + \sum_{k=0}^{n-1} a_k q_{n-k}.$$

Putting these together, we get a recurrence. If we put all terms involving a_n on the lhs and all other terms on the rhs, the recurrence is

$$\left[(n+r)(n+r-1) + p_0(n+r) + q_0\right] a_n = -\sum_{k=0}^{n-1} a_k \left[(k+r)p_{n-k} + q_{n-k}\right].$$

To better understand the structure of this recurrence, recall that

$$I(r) \coloneqq r(r-1) + p_0 r + q_0$$

is the indicial equation of the associated Cauchy–Euler problem. The recurrence can be rewritten

$$I(n+r)a_n = -\sum_{k=0}^{n-1} a_k \left[(k+r)p_{n-k} + q_{n-k}\right]. \tag{3.8}$$

In order to solve for a_n for all $n \geq 1$ (in terms of a_0), we therefore require that $I(r+1), I(r+2), \ldots$ are all non-zero.

Suppose the two roots of the indicial equation are r_1 and r_2, with $r_1 \geq r_2$. This means $I(r_1) = I(r_2) = 0$ and $I(x) \neq 0$ for all other x. In particular,

$$I(r_1+1), \quad I(r_1+2), \quad \ldots \quad \neq 0,$$

so we can always find a solution using r_1. However a problem can arise if $r_1 - r_2$ is an *integer* m. Then

$$I(r_2+m) = I(r_1) = 0,$$

and so we won't be able to solve for a_m using the recurrence (3.8). In particular if $r_1 = r_2$ we also have this problem. The following theorem tells us what ansatz to make (for the second solution) in this situation.

Theorem 3.3.6. *When $r_1 - r_2 = m$ is an integer, a fundamental set of solutions is of the form*

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n, \quad y_2 = y_1 \ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n.$$

Proof. Let $y_1 = x^{r_1}\sum a_n x^n$ be the solution coming from r_1. Make the ansatz $y_2 = v(x)y_2$ for some function $v(x)$. If we plug this into the original equation (3.6) and use that y_1 is a solution, we find

$$x^2(v''(x)y_1 + 2v'(x)y_1') + xp(x)v'(x)y_1 = 0.$$

Rearranging, this becomes a *first-order* homogeneous linear equation for $v'(x)$:

$$x^2 y_1 v''(x) + (2x^2 y_1' + xp(x)y_1)v'(x) = 0.$$

This can be solved using the integrating factor

$$\mu(x) = \exp \int \frac{2x^2 y_1' + xp(x)y_1}{x^2 y_1} = \exp \int \left(\frac{2y_1'}{y_1} + \frac{p(x)}{x} \right) = y_1^2 \exp \int \frac{p(x)}{x}.$$

Multiplying by μ changes the equation into $(v'(x)\mu)' = 0$, and therefore the solution (up to constants) is

$$v'(x) = \frac{1}{y_1^2} e^{-\int p(x)/x}.$$

We want to write this as a series. Recall $y_1 = x^{r_1}\tilde{y}_1$ where $\tilde{y}_1$ is analytic. Then

$$v'(x) = \frac{1}{x^{2r_1}\tilde{y}_1} e^{-\int p_0/x + p_1 + O(x)} = \frac{1}{x^{2r_1+p_0}\tilde{y}_1} e^{-p_1 x + O(x^2)} = \frac{1}{x^{2r_1+p_0}} O(1).$$

From the indicial equation we see that $r_1 + r_2 = 1 + p_0$, so

$$2r_1 + p_0 = 1 + r_1 - r_2 = 1 + m.$$

Hence when we integrate $v'(x)$ term by term, we get

$$v(x) = c_0 x^{-m} + \cdots + c_{m-1} x^{-1} + c_m \ln x + c_{m+1} x + O(x^2)$$

for some constants c_i. Importantly, there is a term $c_m \ln x$. It follows that

$$y_2 = v y_1 = c_m y_1 \ln x + x^{r_1} x^{-m} (\text{analytic}).$$

But $r_1 - m = r_2$. Hence indeed y_2 is of the claimed form. □

Theorem 3.3.7 (Analyticity of Frobenius series solutions at regular singular points)**.** *Consider the ODE*

$$x^2 y'' + x(xp(x))y' + x^2 q(x) y = 0$$

where $x = 0$ is a regular singular point. The two solutions arising from the Frobenius method form a fundamental system of solutions. If ρ is the radius of convergence for power series expansions of $xp(x)$ and $x^2 q(x)$, then the general solution has radius of convergence at least ρ.

3.4 Singularities at infinity

So far we have looked at singularities at $x = x_0$ for finite x_0. Whether such points are ordinary or singular points controls the behavior of solutions as $x \to x_0$. Similarly, we can also look at the "point" $x = \infty$. Whether this is an ordinary or singular point controls the behavior of solutions as $x \to \infty$.

In order to investigate behavior at infinity, we need to do a change of variables. If we let

$$\xi \coloneqq \frac{1}{x},$$

then looking at $\xi \to 0$ is equivalent to looking at $x \to \infty$, and we already know what it means to investigate singularities at zero.

Definition 3.4.1. An ODE has a **ordinary/singular point at infinity** if, under the change of variables $\xi = 1/x$, it has an ordinary/singular point at $\xi = 0$.

We must be careful when performing this change of variables. In addition to plugging in $x = 1/\xi$, we need to plug in

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = -\frac{1}{x^2}\frac{d}{d\xi} = -\xi^2\frac{d}{d\xi}. \tag{3.9}$$

Higher-order derivatives are obtained by a composition of first-order derivatives, keeping the product rule in mind, e.g.

$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \left(-\xi^2\frac{d}{d\xi}\right)\left(-\xi^2\frac{d}{d\xi}\right) = 2\xi^3\frac{d}{d\xi} + \xi^4\frac{d^2}{d\xi^2}. \tag{3.10}$$

Example 3.4.2. Consider the hypergeometric equation

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0.$$

For the sake of notation, define

$$Y(\xi) \coloneqq y\left(\frac{1}{\xi}\right) = y(x),$$

so that it is clear Y' means "derivative with respect to ξ". We perform the change of variables, using (3.9) and (3.10):

$$\frac{1}{\xi}\left(1-\frac{1}{\xi}\right)\left(2\xi^3Y' + \xi^4Y''\right) + \left(c - \frac{a+b+1}{\xi}\right)\left(-\xi^2Y'\right) - abY = 0.$$

Simplifying, we get

$$\xi^2(\xi-1)Y'' + \xi\left((2-c)\xi + (a+b-1)\right)Y' - abY = 0.$$

This is the hypergeometric equation "at infinity". In the notation of (3.4), we have

$$\begin{aligned} P(\xi) &= \xi^2(\xi-1) \\ Q(\xi) &= \xi\left((2-c)\xi + (a+b-1)\right) \\ R(\xi) &= -ab, \end{aligned}$$

and we are concerned with the functions

$$\frac{Q(\xi)}{P(\xi)} = \frac{(2-c)\xi + (a+b-1)}{\xi(\xi-1)}, \quad \frac{R(\xi)}{P(\xi)} = \frac{-ab}{\xi^2(\xi-1)}.$$

These are not analytic at $\xi = 0$, so it is a singular point. But

$$\lim_{\xi\to 0}\xi\frac{Q(\xi)}{P(\xi)} = 1-a-b, \quad \lim_{\xi\to 0}\xi^2\frac{R(\xi)}{P(\xi)} = ab$$

both exist, so $\xi = 0$ is a *regular* singular point.

This example, along with Example 3.3.2, shows us that the hypergeometric equation actually has three singular points, at $x = 0, 1, \infty$, all of which are regular singular points. In fact, any second-order ODE with three regular singular points can be converted to the hypergeometric equation via a change of variables. This is the reason that hypergeometric functions are so ubiquitous; many families of polynomials satisfy second-order ODEs, which often have three regular singular points (counting ∞). So they can be rewritten in terms of hypergeometric functions, after some change of variables and possibly some scaling.

Example 3.4.3. Consider the equation

$$(x-1)(x-2)y'' + \alpha xy' + \beta y = 0.$$

It clearly has regular singular points at $x = 1$ and $x = 2$. At $x = \infty$, we get

$$\left(\frac{1}{\xi}-1\right)\left(\frac{1}{\xi}-2\right)\left(2\xi^3Y' + \xi^4Y''\right) + \frac{\alpha}{\xi}\left(-\xi^2Y'\right) + \beta Y = 0.$$

Simplifying, we get

$$(1-\xi)(1-2\xi)\xi^2 Y'' + (1-\xi)(1-2\xi)(2-\alpha)\xi Y' + \beta Y = 0.$$

So $\xi = 0$, i.e. $x = \infty$, is another regular singular point. (Note that in the ξ coordinate, $\xi = 1$ and $\xi = 1/2$ are also regular singular points, but these correspond to $x = 1$ and $x = 2$.) Hence this equation has three regular singular points,at $x = 1, 2, \infty$. If we do a change of variables

$$x \quad \mapsto \quad u \coloneqq x - 1,$$

then in the coordinate u the equation has three regular singular points, at $u = 0, 1, \infty$. Hence it is a hypergeometric equation, with solution

$${}_2F_1(a, b, c; u) = {}_2F_1(a, b, c; x - 1)$$

for some parameters a, b, c. In fact, we can explicitly perform this change of variables, to get

$$u(u-1)y'' + \alpha(u+1)y' + \beta y = 0.$$

The parameters a, b, c therefore need to satisfy

$$-ab = -\beta, \quad c = -\alpha, \quad -(a+b+1) = -\alpha,$$

where the minus signs are because we must flip $u(u-1)$ into $u(1-u)$ by multiplying the whole equation by -1. For example, if $(\alpha, \beta) = (4, 2)$, then $(a, b, c) = (1, 2, 4)$ and one solution to the original equation is

$$y = {}_2F_1(1, 2, 4; x - 1).$$

In general, given any three distinct points x_1, x_2, x_3 on the real line (with ∞ included), there is a change of variables that will take these three points to $0, 1, \infty$ respectively:

$$x \mapsto \frac{(x - x_1)(x_2 - x_3)}{(x - x_3)(x_2 - x_1)}.$$

This is called a **Möbius transformation**. In the example above, we sent $(1, 2, \infty) \mapsto (0, 1, \infty)$ using the map $x \mapsto x - 1$. We could also choose to send $(1, \infty, 2) \mapsto (0, 1, \infty)$ using the map

$$x \mapsto \lim_{x_2 \to \infty} \frac{(x-1)(x_2-2)}{(x-2)(x_2-1)} = \frac{x-1}{x-2}.$$

In this new coordinate $\tilde{u}$, the solution will again be a hypergeometric series, with different parameters $\tilde{a}, \tilde{b}, \tilde{c}$. But in terms of x, it must be equal to the original solution. This is one way to produce non-trivial identities for hypergeometric series.

3.5 Irregular singular points

Series solutions at irregular singular points are hopeless. The following example illustrates why.

Example 3.5.1. Consider the first-order linear ODE

$$x^2 y' + y = 0.$$

It has an *irregular* singular point at $x = 0$. One checks that the general solution is

$$y = ce^{-1/x}.$$

It is impossible to obtain this solution as a series, even using the Frobenius ansatz. This is because

$$e^{-1/x} = 1 - x^{-1} + \frac{1}{2}x^{-2} + \frac{1}{6}x^{-3} + \cdots$$

has terms of arbitrarily negative degree, and therefore cannot be written in the form of the Frobenius ansatz at all. The function $e^{-1/x}$ has an **essential singularity** at $x = 0$. This is a "pole of infinite order".

The general idea for second-order linear equations $y'' + p(x)y' + q(x)y = 0$ is that

1. a regular singularity at $x = 0$ corresponds to $p(x) = O(x^{-1})$ and $q(x) = O(x^{-2})$;
2. an irregular singularity at $x = 0$ corresponds to $p(x) = O(x^{-k-1})$ and $q(x) = O(x^{-2k-2})$ for some integer $k \geq 1$.

Definition 3.5.2. The integer k is called the **rank** of the irregular singularity. Equivalently, it is the smallest integer such that

$$(x - x_0)^{k+1}p(x), \quad (x - x_0)^{2k+2}q(x)$$

are analytic functions at $x = x_0$.

The example suggests that solutions at an irregular singular point of rank k should involve a term $\exp O((x - x_0)^{-k})$. Indeed, solutions are of the form

$$y(x) = \tilde{y}(x) \exp\left(\frac{c_k}{(x - x_0)^k} + \frac{c_{k-1}}{(x - x_0)^{k-1}} + \cdots + \frac{c_1}{x - x_0}\right)$$

where $\tilde{y}(x)$ is some Frobenius series. Usually the constants c_i are determined by analyzing asymptotics near $x = x_0$, and then the Frobenius ansatz is used. We will not go through this procedure.

Example 3.5.3. An irregular singular point can be viewed as the result of two regular singular points "colliding". Consider the **confluent hypergeometric equation**

$$xy'' + (c - x)y' - ay = 0.$$

This has a regular singular point $x = 0$ and an *irregular* singular point $x = \infty$. It is obtained from the usual hypergeometric equation by the change of coordinates $\tilde{x} \coloneqq bx$ and then sending $b \to \infty$. The change of coordinates produces

$$\frac{\tilde{x}}{b}\frac{b - \tilde{x}}{b}(b^2 y'') + \left(c - (a + b + 1)\frac{\tilde{x}}{b}\right)(by') - aby = 0$$

since $d/d\tilde{x} = b(d/dx)$. Divide out a factor of b and simplify a little to get

$$\tilde{x}\frac{b - \tilde{x}}{b}y'' + \left(c - \frac{a + b + 1}{b}\tilde{x}\right)y' - ay = 0.$$

In the limit $b \to \infty$, both fractions become 1 and we get the confluent hypergeometric equation. The singular points $x = 0, \infty$ become singular points $\tilde{x} = 0, \infty$, but the singular point $x = 1$ becomes $\tilde{x} = b$ and tends to ∞ in the limit. It collides with the existing regular singularity to create an irregular singularity at ∞.

Definition 3.5.4. From this analysis, the analytic solution at $x = 0$ of the confluent hypergeometric equation must be the series

$$\lim_{b \to 0} {}_2F_1\left(a, b, c; \frac{x}{b}\right).$$

This is the **confluent hypergeometric series** ${}_1F_1(a, c; x)$.

In the same way that any second-order equation with three regular singular points can be transformed into the hypergeometric equation, any second-order equation with a regular singular point and an irregular singular point can be transformed into the confluent hypergeometric equation. For example, this is true for Laguerre polynomials and Bessel functions.

4 Laplace transform

The Laplace transform is a general tool used to transform *differential* equations into usual algebraic ones. It is usually easier to solve the resulting algebraic equation and do run the Laplace transform backward to get the desired solution to the ODE.

Definition 4.0.1. The **Laplace transform** of a function $f(t)$ is a new function $F(s)$ of a real variable s. The usual notation is

$$\mathcal{L}\{f(t)\} \coloneqq F(s) \coloneqq \int_0^\infty e^{-st} f(t)\,dt. \tag{4.1}$$

We often call e^{-st} the **kernel**. Note that this has nothing to do with the concept of kernel from linear algebra.

Historically, the variable t stood for time, so we often say $f(t)$ is a function in the **time domain**. By analogy with Fourier theory, the variable s represents frequency, and so the corresponding $F(s)$ is in the **frequency domain**.

Example 4.0.2. The simplest possible function we can plug into the Laplace transform is the constant function $f(t) = 1$:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = -\frac{e^{-st}}{s}\bigg|_{t=0}^{t=\infty} = \frac{1}{s}.$$

From this, one can directly see that

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}.$$

One has to be careful when taking Laplace transforms of functions which "grow faster" than e^{at} as $t \to \infty$. This is because in those cases, the integral (4.1) may diverge and the Laplace transform may not exist. The technical condition we need is that

> beyond some T, for all $t > T$ we have $|f(t)| \le Ke^{at}$ for some constants a and K.

This condition is usually abbreviated as $f(t) = O(e^{at})$, similar to the big-O notation we used for series in Definition 3.1.1. In English, it means that $f(t)$ grows at most as fast as some constant multiple of the exponential e^{at}.

Example 4.0.3. All polynomials grow slower than exponentials, so if $f(t)$ is any polynomial of t then $f(t) = O(e^{at})$ for any positive a. On the other hand, for example,

$$e^{t^2} \neq O(e^{at}) \quad \text{for any } a.$$

As long as we avoid such fast-growing functions and functions which are not piecewise continuous, the Laplace transform will always exist. We will not worry about existence much from now on, unless it plays a crucial role in an argument.

Example 4.0.4. Let $f(t) = \sin at$ and $g(t) = \cos at$. Let their Laplace transforms be

$$F(s) = \mathcal{L}\{f(t)\}, \quad G(s) = \mathcal{L}\{g(t)\}.$$

We can compute these two Laplace transforms simultaneously:

$$F(s) = \int_0^\infty e^{-st}\sin at\,dt = -\frac{e^{-st}}{s}\sin at\bigg|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt = \frac{a}{s}G(s)$$
$$G(s) = \int_0^\infty e^{-st}\cos at\,dt = -\frac{e^{-st}}{s}\cos at\bigg|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin at\,dt = \frac{1}{s} - \frac{a}{s}F(s).$$

Solving this system of equations for $F(s)$ and $G(s)$ gives

$$F(s) = \frac{a}{s^2+a^2}, \quad G(s) = \frac{s}{s^2+a^2}.$$

The usual way to compute Laplace transforms of functions is to know (or look up in a table) common pieces, and then to use properties of the Laplace transform. This is analogous to how we usually differentiate functions, by knowing a few common derivatives and applying the sum/product/chain rules. Since we have graduated from basic calculus, we refer to the "sum rule" as the property of *linearity*.

Theorem 4.0.5 (Linearity of the Laplace transform)**.** *For constants c_1, c_2 and functions f_1, f_2,*

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.$$

Example 4.0.6. The Laplace transform of $3e^{-2t} - 7\cos 5t$ is

$$3\mathcal{L}\{e^{-2t}\} + 7\mathcal{L}\{\cos 5t\} = \frac{3}{s+2} - \frac{7s}{s^2+25}.$$

4.1 Solving IVPs

The utility of the Laplace transform for ODEs lies in part (1) of the following theorem, which tells us how to find the Laplace transform of y' once we know the Laplace transform of y. Since the time variable is usually called t, from now on our ODEs have independent variable t instead of x.

Theorem 4.1.1 (Properties of the Laplace transform (part 1))**.**

1. *(Time derivative) Suppose $f(t) = O(e^{at})$ for some constant a. Then*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{4.2}$$

2. *(Frequency derivative) Let $\mathcal{L}\{f(t)\} = F(s)$. Then*

$$\mathcal{L}\{-tf(t)\} = F'(s). \tag{4.3}$$

Proof. We only prove (1). Integrating by parts gives

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt.$$

Since $f(t) = O(e^{at})$,

$$\lim_{t\to\infty} |e^{-st}f(t)| \le K \lim_{t\to\infty} e^{-(s-a)t} = 0.$$

Hence the first term becomes $-f(0)$, and the second term is $s\mathcal{L}\{f(t)\}$, and we are done. □

Assuming $f'(t)$ satisfies the same condition as $f(t)$, we can repeatedly apply (4.2). For example,

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

In other words, the Laplace transform of *any* derivative of $f(t)$ can be related to the Laplace transform of $f(t)$ itself.

Example 4.1.2. Consider the IVP

$$y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 0.$$

Let $Y(s) \coloneqq \mathcal{L}\{y\}$. Using the given initial conditions, the Laplace transform of the lhs is

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \big(s^2Y(s) - s - 0\big) - 3\,(sY(s) - 1) + 2Y(s) = (s^2 - 3s + 2)Y(s) - (s-3).$$

Setting this equal to the rhs $\mathcal{L}\{0\} = 0$, it follows that

$$Y(s) = \frac{s-3}{(s-1)(s-2)}.$$

If we can find a function $y(t)$ whose Laplace transform is exactly $Y(s)$, then this function $y(t)$ is the (unique) solution to the IVP!

The operation of obtaining the original function y from its Laplace transform $Y(s) = \mathcal{L}\{y\}$ is called the **inverse Laplace transform**. We often write

$$y = \mathcal{L}^{-1}\{Y(s)\}.$$

Computing inverse Laplace transforms in general is hard, just like integrating arbitrary functions is hard. But the functions for which we will want the inverse Laplace transform are usually recognizable combinations of basic pieces, especially if we keep partial fraction decomposition in mind.

Example 4.1.3. Continuing with the previous example, we would like the inverse Laplace transform of

$$Y(s) = \frac{s-3}{(s-1)(s-2)}.$$

Using partial fractions, we write

$$Y(s) = \frac{a}{s-1} + \frac{b}{s-2}$$

for some constants a and b which satisfy

$$a(s-2) + b(s-1) = s+1 \quad \rightsquigarrow \quad \begin{cases} a+b=1 \\ -2a-b=-3 \end{cases}.$$

Solving, $a = 2$ and $b = -1$. Hence

$$Y(s) = \frac{2}{s-1} - \frac{1}{s-2} = 2\mathcal{L}\{e^t\} - \mathcal{L}\{e^{2t}\}.$$

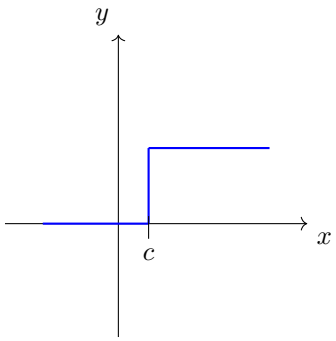
Hence $\mathcal{L}^{-1}\{Y(s)\} = 2e^t - e^{2t}$. One can check this is the solution to the IVP.

We can summarize the logic behind using the Laplace transform to solve IVPs using the following diagram:

$$\begin{array}{ccc} \text{Linear ODE for } y(t) & \xrightarrow{\text{hard}} & \text{Solution } y(t) \\ \Big\downarrow{\scriptstyle \text{Laplace transform } \mathcal{L}\{-\}} & & \Big\uparrow{\scriptstyle \text{inverse Laplace transform } \mathcal{L}^{-1}\{-\}} \\ \text{Algebraic equation for } Y(s) & \xrightarrow{\text{algebra}} & \text{Solution } Y(s) = \mathcal{L}\{y(t)\}. \end{array}$$

4.2 Piecewise continuous forcing functions

The true power of the Laplace transform arises when the rhs of a (non-homogeneous) linear ODE has *discontinuities*. This rhs function is often called the **forcing function**. Perhaps the ODE models a physical phenomenon which is *abrupt*, e.g. flicking on a switch to complete an electrical circuit. The building block for such discontinuous behavior is the **Heaviside step function**



$$u_c(t) \coloneqq \begin{cases} 0 & t < c \\ 1 & t \geq c. \end{cases}$$

Any piecewise continuous function can be written in terms of step functions and continuous functions, as follows.

1. If we want $f(t)$ on (c, ∞), use $f(t)u_c(t)$.
2. If we want $f(t)$ on $(-\infty, c)$, use $f(t)(1 - u_c(t))$.
3. If we want $f(t)$ on (c, d), use $f(t)(u_c(t) - u_d(t))$.

Whether or not endpoints are included makes no difference when integrating, so we neglect the difference between e.g. (c, ∞) and $[c, \infty)$.

Theorem 4.2.1 (Properties of the Laplace transform (part 2))**.** *Let $F(s) = \mathcal{L}\{f(t)\}$.*

1. *(Frequency shift)*
$$\mathcal{L}\{e^{at}f(t)\} = F(s - a). \tag{4.4}$$
2. *(Time shift)*
$$\mathcal{L}\{u_c(t)f(t - c)\} = e^{-cs}F(s). \tag{4.5}$$

In particular, if we apply the time shift property (4.5) to the constant function $f(t) = 1$, we get the Laplace transform of a single step function:

$$\mathcal{L}\{u_c(t)\} = e^{-cs}\mathcal{L}\{1\} = \frac{e^{-cs}}{s}.$$

Example 4.2.2. Consider the IVP

$$y' + y = g(t), \quad y(0) = 0$$

where the forcing function is

$$g(t) = \begin{cases} 0 & t < 2 \\ 1 & 2 \le t < 5 \\ 0 & t \ge 5. \end{cases}$$

This represents e.g. the charge stored in a capacitor where a constant 1 volt is applied between times $t = 2$ and $t = 5$. First note we can rewrite

$$g(t) = u_2(t) - u_5(t).$$

Take the Laplace transform to get

$$sY(s) + Y(s) = \frac{e^{-2s} - e^{-5s}}{s}.$$

Hence we need to compute the inverse Laplace transform of

$$Y(s) = \frac{e^{-2s} - e^{-5s}}{s(s + 1)}.$$

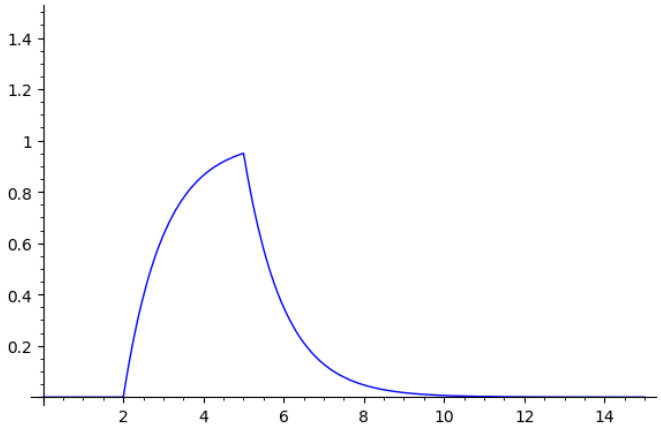
Note that we can use the time shift property (4.5) backward for each of the two terms in the rhs here. So it suffices to compute the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t},$$

and then use the time shift property (4.5) to get the solution

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_2(t)(1 - e^{-(t-2)}) - u_5(t)(1 - e^{-(t-5)}).$$

This solution, when graphed, looks like



In general, piecewise continuous forcing functions often represent a system being "switched on". A common situation is a piecewise *constant* forcing function, like a sum of step functions. The solution will then have two components:

1. a **transient** response, which is present for a short time immediately after the system is switched on;
2. a **steady-state** response, which is the long-term (asymptotic) behavior of the solution.

In the example above, the forcing function caused the capacitor to charge almost up to 1 volt, and then discharge. This is the transient behavior. After a long time, the capacitor will have total charge approaching zero, which is the steady-state behavior.

Example 4.2.3. Steady-state behavior need not be constant. The IVP

$$y'' + y = u_2(t), \quad y(0) = y'(0) = 0$$

has Laplace transform

$$s^2Y(s) + Y(s) = \frac{e^{-2s}}{s},$$

and hence the solution is

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+1)}\right\} = u_2(t)(1 - \cos(t-2)).$$

This has oscillatory steady-state behavior.

In the case when the steady-state response is constant, i.e. $\lim_{t\to\infty} y(t)$ exists, we can compute its value without necessarily finding the solution y. This is possible because the Laplace transform $\mathcal{L}\{y\}$ already contains this information.

Theorem 4.2.4. *Let $\mathcal{L}\{f(t)\} = F(s)$. Assume f is bounded on $(0,\infty)$.*

1. *(Initial value)* $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$ *whenever both limits exist.*
2. *(Final value)* $\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s)$ *whenever both limits exist.*

Proof. Start with the equation (4.2)

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Take the limit $s \to \infty$ of both sides. On the lhs, this limit can be exchanged with the integral. (We ignore all convergence issues here and assume this is always a valid operation.) So

$$\lim_{s\to\infty} \mathcal{L}\{f'(t)\} = \int_0^\infty \lim_{s\to\infty} e^{-st} f'(t)\, dt = \int_0^\infty 0 \cdot f'(t)\, dt = 0.$$

Hence we get $0 = \lim_{s\to\infty} sF(s) - f(0)$, which is the statement of the initial value theorem.

For the final value theorem, we start with the same equation and take the limit $s \to 0^+$ of both sides. On the lhs,

$$\lim_{s\to 0^+} \mathcal{L}\{f'(t)\} = \int_0^\infty \lim_{s\to 0^+} e^{-st} f'(t)\,dt = \int_0^\infty f'(t)\,dt = \lim_{t\to\infty} f(t) - f(0).$$

Comparing this to the rhs, we get the statement of the final value theorem. □

It is possible that $\lim_{s\to 0^+} sF(s)$ exists but $\lim_{t\to\infty} f(t)$ does not. For example, f could have oscillatory steady-state behavior. In practice, there is a useful condition for the Laplace transform $Y(s)$ to ensure that the steady-state limit $\lim_{t\to\infty} y(t)$ exists. We state but do not prove it.

Theorem 4.2.5. *If all the poles of $sF(s)$ have negative real parts, then $\lim_{t\to\infty} f(t)$ exists.*

Example 4.2.6. In Example 4.2.2, we computed that

$$Y(s) = \frac{e^{-2s} - e^{-5s}}{s(s+1)}.$$

This function satisfies the conditions of the theorem, because $sY(s)$ has a single pole at $s = -1$. So, without computing the inverse Laplace transform, we know that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0^+} sY(s) = \lim_{s\to 0^+} \frac{e^{-2s} - e^{-5s}}{s+1} = 0.$$

This agrees with what we get by actually finding $y(t)$ and taking the limit $t \to \infty$.

Example 4.2.7. In Example 4.2.3, we computed that

$$Y(s) = \frac{e^{-2s}}{s(s^2+1)}.$$

Then $sY(s)$ has two poles, at $s = 0 \pm i$, which do not have *negative* real parts. So we are not guaranteed that $\lim_{t\to\infty} f(t)$ exists, and in fact we know from the earlier example that it does not; the steady-state behavior is oscillatory. On the other hand,

$$\lim_{s\to 0^+} sF(s) = \lim_{s\to 0^+} \frac{e^{-2s}}{s^2+1} = 1.$$

4.3 Impulsive forcing functions

Another common type of forcing function is **impulses**: short-time high-magnitude spikes. These are modeled by functions $g(t)$ which are non-zero only in a small interval (t_0, t_1), and the *total* impulse imparted is the quantity

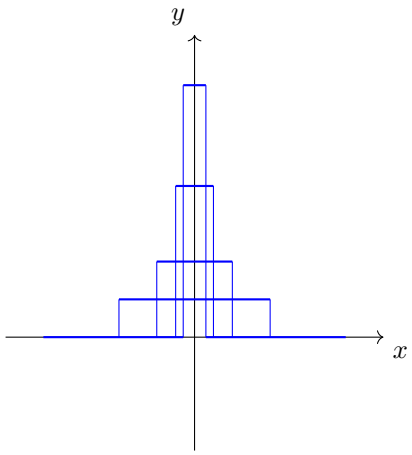
$$\int_{t_0}^{t_1} g(t)\,dt.$$

In the same way we build discontinuous functions out of a canonical step function $u_c(t)$ which steps up exactly by 1 unit, we will only consider functions $g(t)$ whose total impulse is 1.

Example 4.3.1. Consider the family of functions

$$D_\tau(t) = \begin{cases} 1/(2\tau) & -\tau < t < \tau \\ 0 & \text{otherwise.} \end{cases}$$

These all have total impulse 1. For small τ, these are good functions for modeling impulses.



In the limit $\tau \to 0$, we get an *idealized* impulse $\delta(t)$. The total impulse is still 1, but we should think that $\delta(t) = 0$ for every $t \neq 0$. There is no actual function δ that satisfies these requirements. To be precise, such an object δ exists only as a **distribution**, or **density function**, but not as an actual function. It does not make sense to ask for the "value" of a density function at a point t; it only makes sense to ask for its value when integrated over an interval $[a, b]$.

Definition 4.3.2. The **Dirac delta function**, also known as the **unit impulse**, is the distribution satisfying

$$\int_a^b \delta(t)\,dt = \begin{cases} 1 & [a,b] \text{ contains } 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for any $\epsilon > 0$ we have $\int_\epsilon^\infty \delta(t)\,dt = 0$, and similarly for $(-\infty, -\epsilon)$. So we lose nothing by thinking of δ as a "function" such that

$$\delta(t) = 0 \quad \text{for all } t \neq 0.$$

One can derive many properties of the delta function with arguments like this. However these arguments and calculations will not be completely mathematically rigorous. In the same way that for improper integrals $\int_a^\infty$ we think of "plugging in ∞", when we work with delta functions we tend to think of them as actual functions. In reality, for improper integrals there is a limit $\lim_{A\to\infty} \int_a^A$ hiding in the background. For delta functions, the analogous thing to keep in mind is

$$\delta(t) = \lim_{\tau \to 0} D_\tau(t).$$

Example 4.3.3. Let f be any continuous function. Take an interval $[a, b]$ containing 0. Then since $\delta(t) = 0$ except at $t = 0$,

$$\int_a^b f(t)\delta(t)\,dt = \int_a^b f(0)\delta(t)\,dt = f(0)\int_a^b \delta(t)\,dt = f(0). \tag{4.6}$$

To really make this calculation precise, we should think about

$$\lim_{\tau\to 0} \int_a^b f(t) D_\tau(t)\,dt,$$

which we won't do in detail.

Note that we can shift the impulse from $t = 0$ to an arbitrary $t = t_0$ by using $\delta(t - t_0)$. Usually we will take $t_0 > 0$. The formula (4.6) yields its Laplace transform:

$$\mathcal{L}\{\delta(t - t_0)\} = \int_0^\infty e^{-st}\delta(t - t_0)\, dt = e^{-st_0}. \tag{4.7}$$

Example 4.3.4. Consider the IVP from Example 4.2.2, but now with an impulse at $t = 2$ as forcing function:

$$y' + y = \delta(t - 2), \quad y(0) = 0.$$

Then the Laplace transform is

$$sY(s) + Y(s) = e^{-2s},$$

so that the solution to the IVP is

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+1}\right\} = u_2(t)e^{-(t-2)}.$$

4.4 Convolution and impulse response

Now that we are used to using the Laplace transform to solve IVPs involving complicated forcing functions, we can ask a very general question: to what extent does the solution actually depend on the forcing function? We can try leaving the forcing function $g(t)$ unspecified, to see if we can still get some sort of solution which depends on $g(t)$ in some way.

Example 4.4.1. Consider the IVP

$$ay' + by = g(t), \quad y(0) = y_0. \tag{4.8}$$

Let $G(s) \coloneqq \mathcal{L}\{g(t)\}$ be the Laplace transform of the rhs. Then

$$a(sY(s) - y_0) + bY(s) = G(s).$$

Rearranging, we can write

$$Y(s) = \underbrace{\frac{y_0}{as + b}}_{\Phi(s)} + \underbrace{\frac{G(s)}{as + b}}_{\Psi(s)}.$$

By linearity, the solution to the original IVP is

$$y(t) = \underbrace{\mathcal{L}^{-1}\{\Phi(s)\}}_{\phi(s)} + \underbrace{\mathcal{L}^{-1}\{\Psi(s)\}}_{\psi(s)}.$$

1. $\Phi(s)$ is what we would get if we set the forcing function to zero, i.e. solved the associated homogeneous equation. In other words, $\phi = \mathcal{L}^{-1}\{\Phi(s)\}$ solves

$$ay' + by = 0, \quad y(0) = y_0. \tag{4.9}$$

2. $\Psi(s)$ is what we would get if we set the initial condition to zero. In other words, $\psi = \mathcal{L}^{-1}\{\Psi(s)\}$ solves

$$ay' + by = g(t), \quad y(0) = 0.$$

So $\Phi(s)$ is independent of the forcing function $g(t)$, and we should focus only on $\Psi(s)$. In general,

$$\Psi(s) = H(s)G(s)$$

where $H(s) = (as + b)^{-1}$ is a function which is also independent of $g(t)$. Then we would like to be able to write

$$\psi = \mathcal{L}^{-1}\{\Psi(s)\} = \text{some function of } \mathcal{L}^{-1}\{H(s)\} \text{ and } g = \mathcal{L}^{-1}\{G(s)\}.$$

The function $H(s)$ is called the **transfer function**.

In other words, we want to know what multiplication of functions in the frequency domain corresponds to in the time domain. For example, if $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, it is certainly not true that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t)g(t),$$

since this is already false for $f(t) = g(t) = 1$, where $F(s) = G(s) = 1/s$. The actual formula is more subtle.

Theorem 4.4.2. *If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then*

$$F(s)G(s) = \mathcal{L}\{h(t)\}$$

where the function h is the **convolution**

$$h(t) \coloneqq \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

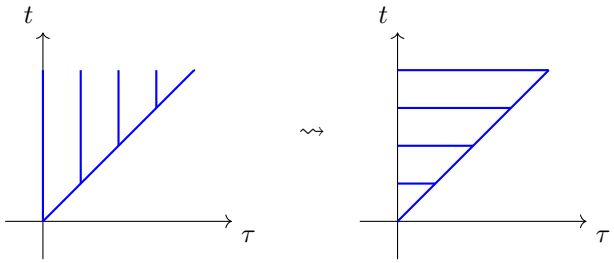
Proof. This is a slightly tricky direct calculation. Note that we can rewrite

$$F(s)G(s) = \int_0^\infty \int_0^\infty e^{-s(\zeta+\tau)} f(\zeta)g(\tau)\,d\zeta\,d\tau,$$

which suggests a change of variables $\zeta \rightsquigarrow t \coloneqq \zeta + \tau$ for the inner integral. Then

$$F(s)G(s) = \int_0^\infty \int_\tau^\infty e^{-st} f(t-\tau)g(\tau)\,dt\,d\tau.$$

Now we change the order of integration from $dt\,d\tau$ to $d\tau\,dt$.



This only changes the bounds of integration and not the integrand:

$$F(s)G(s) = \int_0^\infty \int_0^t e^{-st} f(t-\tau)g(\tau)\,d\tau\,dt.$$

Pull the e^{-st} out to get

$$F(s)G(s) = \int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)g(\tau)\,d\tau\right)dt = \mathcal{L}\{h(t)\}. \qquad \square$$

Definition 4.4.3. We view the **convolution** of f and g as a new type of product of functions. The resulting function is usually denoted

$$(f * g)(t) \coloneqq \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Example 4.4.4. Now we can continue with Example 4.4.1. Using the convolution product,

$$\mathcal{L}^{-1}\{\Psi(s)\} = \mathcal{L}^{-1}\{H(s)G(s)\} = h(t) * g(t)$$

where $h(t) \coloneqq \mathcal{L}^{-1}\{H(s)\}$ is the inverse Laplace transform of the transfer function $H(s)$. So once we know what $h(t)$ is, we are done. In our case,

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{as+b}\right\} = \frac{1}{a}e^{-\frac{b}{a}t}.$$

Hence the solution to (4.8) is given by

$$y(t) = \phi(t) + h(t) * g(t)$$

where $\phi(t)$ is the solution of the associated homogeneous equation (4.9) and $h(t)$ is given as above.

In general, we have the following interpretation of $h(t)$. In frequency domain, recall that $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\} = \mathcal{L}^{-1}\{H(s)G(s)\}$ is the solution of

$$ay' + by = g(t), \quad y(0) = y_0.$$

If we set $G(s) = 1$, then the solution will be $\mathcal{L}^{-1}\{H(s)\} = h(t)$. In the time domain, (4.7) tells us that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{1\} = \delta(t).$$

It follows that the solution to

$$ay' + by = \delta(t), \quad y(0) = y_0$$

is precisely the function $h(t)$ we wanted.

Definition 4.4.5. The function $h(t)$ is called the **impulse response**.

All this work yields a very general conceptual framework for handling forcing functions. Given a linear IVP with forcing function $g(t)$, the solution is given by the following procedure.

1. Find the solution ϕ to the associated homogeneous equation.
2. Find the impulse response h of the equation, i.e. the solution when $g(t) = \delta(t)$.
3. The actual solution is given by
$$y(t) = \phi(t) + h(t) * g(t).$$

We say that such ODEs are *characterized by their impulse response.*

5 First-order systems

Now we return to the study of first-order equations, but in *systems*. In the same way that one can have a system of algebraic equations, one can have a system of ODEs. We start with first-order systems, which are the most important case.

Definition 5.0.1. A **first-order system** of ODEs is a system of equations of the form

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots, x_n). \end{aligned} \tag{5.1}$$

The associated IVP has initial conditions $x(t_0) = x_0$ for each x_i.

First-order systems arise very naturally in the study of ODEs. Given an arbitrary n-th order equation $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$, define

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \cdots, \quad x_n = y^{(n-1)}.$$

Then the n-th order equation becomes equivalent to the first-order system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= F(t, x_1, x_2, \dots, x_n). \end{aligned}$$

Initial conditions for the n-th order equation, e.g. $y'(0) = y_0'$, become conditions for the first-order equations, e.g. $x_2(0) = y_0'$. Hence there is a close relation between a *first-order system* of n ODEs and a *single n-th order* ODE. For linear equations, all of the general theory from section 2.4 carries over, as we will see.

Definition 5.0.2. The first-order system (5.1) is a **linear** system if each $F_i(t, x_1, x_2, \dots, x_n)$ is a linear function of $x_1, x_2, \dots, x_n$. In other words, (5.1) takes the form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t). \end{aligned} \tag{5.2}$$

When all the g_i are zero, the system is **homogeneous**.

Theorem 5.0.3 (Existence and uniqueness for linear first-order systems)**.** *If the functions p_{ij} and g_i are continuous on an open interval I containing t_0, then there exists a unique solution to* (5.2) *satisfying the initial conditions*

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \cdots, x_n(t_0) = x_n^0. \tag{5.3}$$

It is cumbersome to work with systems of ODEs in this form. A better way is to use the language of *linear algebra*, and write everything in terms of matrices and vectors. This is not just for the sake of notation; tools from linear algebra will prove very helpful in analyzing systems of ODEs.

5.1 Review of linear algebra

We briefly review the most important tools we will need from linear algebra, and introduce some new notation coming from ODEs.

Definition 5.1.1. Our **vectors** and **matrices** consist of *functions*, and are written

$$\mathbf{x}(t) \coloneqq \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A}(t) \coloneqq \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

All relevant concepts are defined on *component-wise* unless stated otherwise. For example, $\mathbf{x}$ is **continuous** if each component x_i is continuous, and **derivatives** and **integrals** are defined by differentiating and integrating each component individually.

As usual, we use the notation $\mathbf{x}'$ instead of $d\mathbf{x}/dt$ whenever there is no ambiguity. In matrix notation, the general form (5.2) of a linear first-order system of ODEs, with initial conditions (5.3), can be written as

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

The usual rules of calculus work with matrices as well, and can be checked by writing everything out in components. However since matrices do not *commute* with each other, i.e. $\mathbf{AB} \neq \mathbf{BA}$, one must be careful about the ordering of terms in usual calculus rules.

Example 5.1.2. We verify the matrix version of the product rule:

$$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\mathbf{B}. \tag{5.4}$$

The ij-th component of the lhs is

$$\left(\frac{d}{dt}(\mathbf{AB})\right)_{ij} = \frac{d}{dt}\sum_{k=1}^{n} A_{ik}B_{kj}$$

by the definition of matrix multiplication. Since the components A_{ik} and B_{kj} are ordinary functions, we can proceed with the product rule from ordinary calculus, to get

$$\left(\frac{d}{dt}(\mathbf{AB})\right)_{ij} = \sum_{k=1}^{n} \left(A_{ik}B'_{kj} + A'_{ik}B_{kj}\right).$$

Working backward now, the pieces on the rhs are exactly the ij-th components of the rhs in (5.4).

Definition 5.1.3. Let $\mathbf{P}$ be a matrix. If a vector $\mathbf{v}$ satisfies

$$\mathbf{Pv} = \lambda\mathbf{v}$$

for some constant λ, then we say it is an **eigenvector** of $\mathbf{P}$ with **eigenvalue** λ. Eigenvalues are solutions of the **characteristic equation** (or characteristic **polynomial**)

$$\det(\mathbf{P} - \lambda\mathbf{I}) = 0.$$

If we can form a basis of eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, usually called an **eigenbasis**, then in that basis the matrix $\mathbf{P}$ is a *diagonal* matrix. In equations,

$$\mathbf{P} = \mathbf{S}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{S} \coloneqq \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix}.$$

In such situations we call $\mathbf{P}$ **diagonalizable**. We like working with diagonalizable matrices because in the eigenbasis they are diagonal, and diagonal matrices and scalars behave very similarly. Unfortunately, while most matrices are diagonalizable once we work over the *complex* numbers, not all of them are.

Example 5.1.4. Consider the two matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The first matrix $\mathbf{A}$ is already diagonalized, so its eigenbasis is the standard basis

$$\mathbf{v}_1 = \mathbf{e}_1 \coloneqq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \mathbf{e}_2 \coloneqq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

consisting of two eigenvectors with eigenvalue 2. The second matrix $\mathbf{B}$ is not diagonalized. Its eigenvalues are the zeros of the characteristic polynomial

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2.$$

So just as with $\mathbf{A}$ we expect two eigenvectors of eigenvalue 2. If $\mathbf{v} = (v_1, v_2)$ is such an eigenvector then

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix}.$$

The general solution to this system is spanned by the *single* vector $\mathbf{e}_1$. So $\mathbf{B}$ only has one actual eigenvector of eigenvalue 2, even though $\lambda = 2$ is a zero of multiplicity two.

So in the case of a shortage of eigenvectors one cannot fully diagonalize a matrix $\mathbf{A}$. However it can be transformed into a *nearly* diagonal form called the **Jordan form**. The Jordan form is *block-diagonal*, i.e. of the form

$$\mathbf{A} = \begin{pmatrix} \boxed{\mathbf{B}_1} & & & \\ & \boxed{\mathbf{B}_2} & & \\ & & \ddots & \\ & & & \boxed{\mathbf{B}_m} \end{pmatrix}.$$

The $\mathbf{B}_i$ are called **Jordan blocks** and are of the form

$$\mathbf{B}_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix},$$

where λ_i is the eigenvalue associated to the Jordan block. The corresponding basis for a Jordan block no longer consists of eigenvectors: associated to each Jordan block is only one actual eigenvector.

Example 5.1.5. Consider the three matrices

$$\begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix}.$$

All three have characteristic polynomial $(2-\lambda)^3 = 0$, i.e. a single eigenvalue 2 of multiplicity three.

1. The first matrix has three 1×1 Jordan blocks. It therefore has three linearly independent eigenvectors (and is evidently diagonalizable).
2. The second matrix has a 2×2 Jordan block and a 1×1 Jordan block. It therefore has *two* linearly independent eigenvectors.
3. The third matrix has a single 3×3 Jordan block. It therefore has only a single linearly independent eigenvector.

The question now is what is being used as a basis within each Jordan block. The eigenvalue associated to a Jordan block has a deficiency of eigenvectors, which is why the block cannot be fully diagonalized.

Definition 5.1.6. Let λ be an eigenvalue of $\mathbf{A}$. A vector $\mathbf{v}$ is a **generalized eigenvector** associated to λ if

$$(\mathbf{A} - \lambda \mathbf{I})^k \mathbf{v} = 0$$

for some positive integer k. The smallest k such that this equation holds for $\mathbf{v}$ is called the **rank** of $\mathbf{v}$.

Theorem 5.1.7 (Jordan normal form)**.** *Every (complex) matrix* $\mathbf{A}$ *can be put into Jordan normal form. An* $r \times r$ *Jordan block with eigenvalue* λ *corresponds to a rank-r generalized eigenvector* $\mathbf{v}$ *such that a basis for that Jordan block is given by*

$$\mathbf{v}, \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}, \quad \cdots, \quad (\mathbf{A} - \lambda\mathbf{I})^{r-1}\mathbf{v}.$$

Actual eigenvectors are generalized eigenvectors of rank 1. Note that the last vector in this basis is an actual eigenvector, because if we denote it by $\mathbf{w} \coloneqq (\mathbf{A} - \lambda\mathbf{I})^{r-1}\mathbf{v}$, then

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = (\mathbf{A} - \lambda\mathbf{I})^{r}\mathbf{v} = 0$$

by the definition of $\mathbf{v}$. This yields a procedure for computing generalized eigenvectors for eigenvalue λ.

1. Find all ordinary eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ for a given eigenvalue λ.
2. For each ordinary eigenvector $\mathbf{v}_i =: \mathbf{v}_i^{(1)}$, solve the equations

   $$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_i^{(2)} = \mathbf{v}_i^{(1)}, \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_i^{(3)} = \mathbf{v}_i^{(2)}, \quad \cdots$$

   as far as possible, to get a sequence of generalized eigenvectors

   $$\mathbf{v}_i^{(1)}, \mathbf{v}_i^{(2)}, \ldots, \mathbf{v}_i^{(r)}. \tag{5.5}$$

   Then $\mathbf{v}_i^{(j)}$ is a rank-j generalized eigenvector.

For an $r \times r$ Jordan block, its generalized eigenvectors (5.5) form a basis in which the matrix $\mathbf{A}$ is in Jordan form. This means

$$\begin{aligned} \mathbf{A}\mathbf{v}_i^{(1)} &= \lambda\mathbf{v}_i^{(1)} \\ \mathbf{A}\mathbf{v}_i^{(2)} &= \mathbf{v}_i^{(1)} + \lambda\mathbf{v}_i^{(2)} \\ \mathbf{A}\mathbf{v}_i^{(3)} &= \mathbf{v}_i^{(2)} + \lambda\mathbf{v}_i^{(3)} \\ &\vdots \\ \mathbf{A}\mathbf{v}_i^{(r)} &= \mathbf{v}_i^{(r-1)} + \lambda\mathbf{v}_i^{(r)}. \end{aligned} \tag{5.6}$$

5.2 Homogeneous systems with constant coefficients

In the language of linear algebra, we usually write a system of n first-order equations in the form

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t),$$

for an $n \times n$ matrix $\mathbf{P}(t)$. An initial condition would be of the form $\mathbf{x}(t_0) = \mathbf{x}_0$. The associated **homogeneous** equation is

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}. \tag{5.7}$$

Theorem 5.2.1 (Structure of solutions to homogeneous systems)**.** *The general solution of* (5.7) *is of the form*

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

for a fundamental set of solutions $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$.

As we did for first- and second-order ODEs, the simplest case to begin with is when all coefficients are constants. In the context of systems, this means that in

$$\mathbf{x}' = \mathbf{P}\mathbf{x}, \tag{5.8}$$

the components of the matrix $\mathbf{P}$ are all constants, i.e. independent of t. The theory of such constant-coefficient systems of n ODEs parallels the general theory of single linear n-th order ODEs from section 2.4.

Example 5.2.2. Suppose $\mathbf{P}$ is a diagonal matrix, i.e. that

$$\mathbf{P} = \begin{pmatrix} p_{11} & & & \\ & p_{22} & & \\ & & \ddots & \\ & & & p_{nn} \end{pmatrix}$$

where all off-diagonal entries are zero. Then the system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ is **decoupled**, in the sense that it is now a collection of first-order equations that have *nothing to do with each other*:

$$\begin{aligned} x_1' &= p_{11}x_1 \\ &\vdots \\ x_n' &= p_{nn}x_n. \end{aligned}$$

Each equation is the simplest kind of first-order ODE. The general solution is $x_i(t) = c_i e^{p_{ii}t}$.

The idea for solving the general case is to make the ansatz $\mathbf{x}(t) = \mathbf{v}e^{at}$ for some constant a, and some constant vector $\mathbf{v}$. Plugging this into (5.8) yields

$$a\mathbf{v}e^{at} = \mathbf{P}\mathbf{v}e^{at}.$$

Since $e^{at} \neq 0$, we can cancel it from both sides. Then

$$a\mathbf{v} = \mathbf{P}\mathbf{v},$$

i.e. $\mathbf{v}$ is an eigenvector for $\mathbf{P}$ with eigenvalue a. This should not be surprising: a general $\mathbf{P}$ becomes diagonal in the eigenbasis, and therefore decoupled.

Example 5.2.3. The second-order homogeneous equation $y'' - 3y' + 2y = 0$ is equivalent to a system of 2 first-order equations:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors of this matrix are

$$\lambda_1 = 1,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda_2 = 2,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Hence the general solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{5.9}$$

We can compare this solution to our earlier one in Example 2.2.2. In the context of the second-order equation, $x_1 = y$ and $x_2 = y'$. So (5.9) says

$$\begin{aligned} x_1 = y &= c_1 e^t + c_2 e^{2t} \\ x_2 = y' &= c_1 e^t + 2c_2 e^{2t}, \end{aligned}$$

which makes sense and agrees with our earlier solution to the second-order equation.

Definition 5.2.4. To test independence of solutions, the **Wronskian** of n solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is

$$W(\mathbf{x}_1, \ldots, \mathbf{x}_n)(t) \coloneqq \det \begin{pmatrix} | & & | \\ \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \\ | & & | \end{pmatrix}.$$

This is compatible with the previous definition (2.9) for an n-th order ODE, because the i-th derivative of a solution there becomes the i-th entry of a solution vector here. Note that, in this language, it is clear that the Wronskian tests linear independence of the solutions, as vectors. However, we notice a problem: not every matrix $\mathbf{P}$ is diagonalizable, so there will not always be n linearly independent eigenvectors to provide n independent solutions. We saw this phenomenon already in the second-order case, in section 2.2, where the characteristic equation could have a repeated root. There the two independent solutions were

$$y_1 = e^{\alpha t}, \quad y_2 = te^{\alpha t}.$$

In this case, the resulting matrix $\mathbf{P}$ is *not diagonalizable*. However it can still be put into Jordan form, by Theorem 5.1.7. Then the following formula yields a fundamental set of solutions; we will see in the next section where it comes from.

Theorem 5.2.5. *Suppose $\mathbf{P}$ is a single $r \times r$ Jordan block of eigenvalue λ. Let $\mathbf{v}_i$ be a generalized eigenvector of rank i, for $i = 1, \dots, r$. Then the r independent solutions of*

$$\mathbf{x}' = \mathbf{P}\mathbf{x}$$

are given by

$$\mathbf{x}^{(k)} = e^{\lambda t}\left(\mathbf{v}_k + t\mathbf{v}_{k-1} + \frac{t^2}{2!}\mathbf{v}_{k-2} + \cdots + \frac{t^{k-1}}{(k-1)!}\mathbf{v}_1\right) \tag{5.10}$$

for $k = 1, \dots, r$.

Proof. Using the product rule, compute that

$$\begin{aligned}
(\mathbf{x}^{(k)})' &= \lambda e^{\lambda t}\left(\mathbf{v}_k + t\mathbf{v}_{k-1} + \frac{t^2}{2!}\mathbf{v}_{k-2} + \cdots + \frac{t^{k-1}}{(k-1)!}\mathbf{v}_1\right) + e^{\lambda t}\left(\mathbf{v}_{k-1} + t\mathbf{v}_{k-2} + \cdots + \frac{t^{k-2}}{(k-2)!}\mathbf{v}_1\right) \\
&= e^{\lambda t}\left((\mathbf{v}_{k-1} + \lambda\mathbf{v}_k) + t(\mathbf{v}_{k-2} + \lambda\mathbf{v}_{k-1}) + \cdots + \frac{t^{k-2}}{(k-2)!}(\mathbf{v}_1 + \lambda v_2) + \frac{t^{k-1}}{(k-1)!}\lambda\mathbf{v}_1\right) \\
&= e^{\lambda t}\left(\mathbf{P}\mathbf{v}_k + t\mathbf{P}\mathbf{v}_{k-1} + \cdots + \frac{t^{k-2}}{(k-2)!}\mathbf{P}v_2 + \frac{t^{k-1}}{(k-1)!}\mathbf{P}\mathbf{v}_1\right) \\
&= \mathbf{P}e^{\lambda t}\left(\mathbf{v}_k + t\mathbf{v}_{k-1} + \cdots + \frac{t^{k-2}}{(k-2)!}v_2 + \frac{t^{k-1}}{(k-1)!}\mathbf{v}_1\right) = \mathbf{P}\mathbf{x}^{(k)}.
\end{aligned}$$

Here we used that $\mathbf{v}_1, \dots, \mathbf{v}_r$ form the basis in which $\mathbf{P}$ is in Jordan normal form, so that (5.6) holds. □

Example 5.2.6. Consider the system

$$\mathbf{x}' = \mathbf{P}\mathbf{x}, \quad \mathbf{P} \coloneqq \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

Then $\mathbf{P}$ has eigenvalue $\lambda = 1$ with multiplicity 2, but the only eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So we look for a generalized eigenvector $\mathbf{v}_2$ arising from this eigenvector, by solving

$$(\mathbf{P} - I)\mathbf{v}_2 = \mathbf{v}_1.$$

One possible solution is

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then Theorem 5.2.5 tells us that a fundamental set of solutions is

$$\mathbf{x}^{(1)} = e^t \mathbf{v}_1 = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$
$$\mathbf{x}^{(2)} = e^t(\mathbf{v}_2 + t\mathbf{v}_1) = \begin{pmatrix} te^t \\ e^t + te^t \end{pmatrix}.$$

This makes sense, because the system actually corresponds to the second-order equation $y'' - 2y' + y = 0$, for which a fundamental set of solutions is $y_1 = e^t$ and $y_2 = te^t$.

Note that generalized eigenvectors are not unique, even up to scaling. For example, in the preceding example, we could have taken

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The resulting fundamental set of solutions is

$$\tilde{\mathbf{x}}^{(1)} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad \tilde{\mathbf{x}}^{(2)} = \begin{pmatrix} -e^t + te^t \\ te^t \end{pmatrix},$$

which is also perfectly valid. It is related to the other fundamental set of solutions by

$$\mathbf{x}^{(1)} = \tilde{\mathbf{x}}^{(1)}, \quad \mathbf{x}^{(2)} = \tilde{\mathbf{x}}^{(2)} + \tilde{\mathbf{x}}^{(1)}.$$

Example 5.2.7. In the case of complex eigenvalues, we extract *real*-valued solutions in the usual way. Consider

$$\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors for this system are

$$\lambda_1 = -\frac{1}{2} + i, \ \mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \lambda_2 = -\frac{1}{2} - i, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Note that eigenvalues being complex conjugates means eigenvectors are also complex conjugates. A fundamental set of solutions is therefore

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t} = \begin{pmatrix} e^{-1/2}(\cos t + i \sin t) \\ e^{-1/2}(-\sin t + i \cos t) \end{pmatrix}$$
$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t} = \begin{pmatrix} e^{-1/2}(\cos t - i \sin t) \\ e^{-1/2}(-\sin t - i \cos t) \end{pmatrix}.$$

These are *complex*-valued solutions. One can use the usual trick of looking at

$$\frac{1}{2}(\mathbf{x}^{(1)}(t) + \mathbf{x}^{(2)}(t)), \quad \frac{1}{2i}(\mathbf{x}^{(1)}(t) - \mathbf{x}^{(2)}(t)),$$

but instead, we should use the following more general observation to get real-valued solutions.

Theorem 5.2.8. *If $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$ is a complex-valued solution to the homogeneous system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, then its real and imaginary parts $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are real-valued solutions.*

Proof. Substituting $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ into the equation,

$$(\mathbf{u}' - \mathbf{P}\mathbf{u}) + i(\mathbf{v}' - \mathbf{P}\mathbf{v}) = 0.$$

A complex number is zero only when both its real and imaginary parts are zero, so

$$\mathbf{u}' = \mathbf{P}\mathbf{u}, \quad \mathbf{v}' = \mathbf{P}\mathbf{v}$$

and both are solutions. But both are also real. □

Example 5.2.9. Continuing with the example, we can separate the real and imaginary parts of $\mathbf{x}^{(1)}(t)$ to get

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{-1/2}\cos t \\ -e^{-1/2}\sin t \end{pmatrix} + i \begin{pmatrix} e^{-1/2}\sin t \\ e^{-1/2}\cos t \end{pmatrix}.$$

Similarly,

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{-1/2}\cos t \\ -e^{-1/2}\sin t \end{pmatrix} - i \begin{pmatrix} e^{-1/2}\sin t \\ e^{-1/2}\cos t \end{pmatrix}.$$

The real and imaginary parts of either solution are the same, and form a fundamental set of *real*-valued solutions.

5.3 Matrix exponentials

There is a more systematic approach to solutions of homogeneous constant-coefficient linear systems, which in particular will explain how to obtain the formula (5.10) for general solutions associated to a Jordan block.

Example 5.3.1. Suppose we tried to obtain a series solution for the IVP

$$\mathbf{x}' = \mathbf{P}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{5.11}$$

Then the ansatz should be

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{x}_1 + t^2\mathbf{x}_2 + \cdots = \sum_{k=0}^{\infty} \mathbf{x}_k t^k$$

for unknown *vector* coefficients $\mathbf{x}_k$. Plugging this into the equation gives

$$\sum_{k=0}^{\infty} k\mathbf{x}_k t^{k-1} - \mathbf{P}\sum_{k=0}^{\infty} \mathbf{x}_k t^k = 0.$$

From this we get the recursion

$$\mathbf{x}_k = \frac{1}{k}\mathbf{P}\mathbf{x}_{k-1}, \quad i = 1, 2, \ldots.$$

The solution to this recursion is

$$\mathbf{x}_k = \frac{1}{k!}\mathbf{P}^k\mathbf{x}_0.$$

It follows that we can write the solution as

$$\mathbf{x}(t) = \left(\sum_{k=0}^{\infty} \frac{\mathbf{P}^k}{k!} t^k\right)\mathbf{x}_0$$

by factoring out $\mathbf{x}_0$ on the right.

Definition 5.3.2. Let $\mathbf{A}$ be a matrix. The **matrix exponential** $e^{\mathbf{A}}$ is the matrix defined by the series

$$e^{\mathbf{A}} := \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}.$$

Using this notation, the solution to the IVP (5.11) is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0.$$

This should be compared to the series solution in Example 3.2.2, and also to the non-matrix version

$$y' = ay, \quad y(0) = y_0,$$

which has solution $y(t) = y_0 e^{at}$. The only problem now is how to compute $e^{\mathbf{A}t}$ for the matrix case. One has to be careful because while scalars are commutative, matrices are not.

Theorem 5.3.3 (Properties of the matrix exponential)**.** *Let* $\mathbf{A}, \mathbf{B}$ *be* $n \times n$ *matrices.*

1. *If* $\mathbf{A}$ *is a diagonal matrix, then*
$$e^{\mathbf{A}} = \begin{pmatrix} e^{a_{11}} & & \\ & \ddots & \\ & & e^{a_{nn}} \end{pmatrix}.$$
2. *If* $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$ *(i.e. a change of basis), then*
$$e^{\mathbf{A}} = \mathbf{S}e^{\mathbf{B}}\mathbf{S}^{-1}.$$
3. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *commute, then*
$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}.$$

Proof. We show (2). Since $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$,
$$\mathbf{A}^2 = (\mathbf{S}\mathbf{B}\mathbf{S}^{-1})(\mathbf{S}\mathbf{B}\mathbf{S}^{-1}) = \mathbf{S}\mathbf{B}^2\mathbf{S}^{-1},$$
and in general we have $\mathbf{A}^k = \mathbf{S}\mathbf{B}^k\mathbf{S}^{-1}$, so that
$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{S}\mathbf{B}^k\mathbf{S}^{-1}}{k!} = \mathbf{S}\left(\sum_{k=0}^{\infty} \frac{\mathbf{B}^k}{k!}\right)\mathbf{S}^{-1} = \mathbf{S}e^{\mathbf{B}}\mathbf{S}^{-1}. \qquad \square$$

If $\mathbf{P}$ is diagonalizable, then we can write $\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ for a diagonal matrix $\mathbf{D}$. Then we can use (1) and (2) to compute $e^{\mathbf{P}t}$. If it is not diagonalizable, we use generalized eigenvectors to form $\mathbf{S}$, to put $\mathbf{P}$ into Jordan normal form. When $\mathbf{P}$ is a single Jordan block of eigenvalue λ, this means
$$\mathbf{S}^{-1}\mathbf{P}\mathbf{S} = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix} = \lambda\mathbf{I} + \mathbf{N}$$
where $\mathbf{N}$ is the matrix with 1's in the first diagonal above the main diagonal. Since $\lambda\mathbf{I}$ commutes with any matrix,
$$e^{\mathbf{P}t} = \mathbf{S}e^{\lambda\mathbf{I}t}e^{\mathbf{N}t}\mathbf{S}^{-1}. \tag{5.12}$$
Clearly $e^{\lambda\mathbf{I}t} = e^{\lambda t}\mathbf{I}$. It remains to compute $e^{\mathbf{N}t}$.

Theorem 5.3.4. *If* $\mathbf{N}$ *is* $n \times n$*, then* $\mathbf{N}^n = 0$*. So*
$$e^{\mathbf{N}t} = \mathbf{I} + \mathbf{N}t + \mathbf{N}^2\frac{t^2}{2!} + \cdots + \mathbf{N}^{n-1}\frac{t^{n-1}}{(n-1)!} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & t \\ & & & & 1 \end{pmatrix}. \tag{5.13}$$

Example 5.3.5. Consider the linear system given by
$$\mathbf{x}' = \mathbf{P}\mathbf{x}, \quad \mathbf{P} = \begin{pmatrix} 6 & -2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}.$$

One can check that its Jordan normal form is given by

$$\mathbf{S}^{-1}\mathbf{P}\mathbf{S} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Working in this basis, we have

$$\mathbf{S}^{-1}e^{\mathbf{P}t}\mathbf{S} = e^{3t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows that

$$e^{\mathbf{P}t} = e^{3t}\mathbf{S}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\mathbf{S}^{-1} = e^{3t}\begin{pmatrix} \frac{t^2}{2}+3t+1 & -\frac{t^2}{2}-2t & -t \\ \frac{t^2}{2}+3t & -\frac{t^2}{2}-2t+1 & -t \\ \frac{t^2}{2}+2t & -\frac{t^2}{2}-t & -t+1 \end{pmatrix}.$$

General solutions are obtained by letting the initial conditions be arbitrary constants. Hence the general solution is

$$\mathbf{x}(t) = e^{\mathbf{P}t}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

This means $\mathbf{x}(t)$ is c_1 times the first column of $e^{\mathbf{P}t}$, plus c_2 times the second column of $e^{\mathbf{P}t}$, etc. A fundamental set of solutions is therefore given by the columns of $e^{\mathbf{P}t}$.

Definition 5.3.6. The matrix $e^{\mathbf{P}t}$ is called the **fundamental matrix**. Its columns form a fundamental set of solutions.

Using this, we can understand where the formula (5.12) comes from: the r solutions given by that formula are exactly the r columns in the matrix in (5.13). For example, the formula (5.10) for a fundamental system of solutions corresponding to a single Jordan block comes from reading off the columns of the matrix exponential (5.13), keeping in mind that that calculation was done in the Jordan basis (of generalized eigenvectors).

5.4 Non-homogeneous systems

Now we tackle the general non-homogeneous case

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t). \tag{5.14}$$

Theorem 5.4.1. *The general solution to* (5.14) *is*

$$\mathbf{x} = \mathbf{v}(t) + c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

where $\mathbf{v}(t)$ *is a specific solution, and* $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ *is the solution of the associated homogeneous system.*

To find the specific solution $\mathbf{v}(t)$, possible strategies parallel the strategies we used in the non-system case in section 2.3; we can use the method of undetermined coefficients *for systems*, and also variation of parameters, and even the Laplace transform if we are given an IVP. However, in the case that $\mathbf{P}$ is constant, i.e. independent of t, there is a better approach.

Suppose for simplicity that $\mathbf{P}$ can be diagonalized as $\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$. Then we can work in the eigenbasis to decouple the equations. Concretely this means we define

$$\mathbf{y} \coloneqq \mathbf{S}^{-1}\mathbf{x}$$

and rewrite the equation (5.14) in terms of $\mathbf{y}$:

$$\mathbf{S}\mathbf{y}' = \mathbf{P}\mathbf{S}\mathbf{y} + \mathbf{g}(t).$$

Multiplying by $\mathbf{S}^{-1}$ on both sides gives

$$\mathbf{y}' = \mathbf{S}^{-1}\mathbf{P}\mathbf{S}\mathbf{y} + \mathbf{S}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{S}^{-1}\mathbf{g}(t). \tag{5.15}$$

Since $\mathbf{D}$ is diagonal, this system is decoupled and we can solve each individual non-homogeneous first-order equation using any method we wish. Then the original solution is recovered by

$$\mathbf{x} = \mathbf{S}\mathbf{y}.$$

If $\mathbf{P}$ is constant but not diagonalizable, we can put it into Jordan form. Then in (5.15), the resulting matrix $\mathbf{D}$ is in Jordan form instead of being diagonal. The resulting system is not completely decoupled; each Jordan block looks like

$$\begin{pmatrix} y_1' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} h_1(t) \\ \vdots \\ h_{n-1}(t) \\ h_n(t) \end{pmatrix}$$

where $\mathbf{h}(t) \coloneqq \mathbf{S}^{-1}\mathbf{g}(t)$. Note however that the last equation in this system

$$y_n' = \lambda y_n + h_n(t)$$

is independent of all other y_i, and can be solved on its own. Then once we know y_n, we can plug it into the second last equation

$$y_{n-1}' = \lambda y_{n-1} + y_n + h_n(t),$$

and solve this on its own. Repeating this procedure until we get to the first equation yields a solution $\mathbf{y}$.

Example 5.4.2. Consider the system

$$\begin{aligned} x_1' &= 3x_1 + 2x_2 + 2e^t \\ x_2' &= -\frac{1}{2}x_1 + x_2 + e^{4t}, \end{aligned}$$

so that in matrix form

$$\mathbf{P} = \begin{pmatrix} 3 & 2 \\ -1/2 & 1 \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}.$$

One can check that the Jordan normal form of $\mathbf{P}$ is

$$\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}, \quad \mathbf{D} \coloneqq \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{S} \coloneqq \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix},$$

so that the associated system (5.15) is

$$\mathbf{y}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} e^t \\ e^t + e^{4t} \end{pmatrix}.$$

Hence we should first solve

$$y_2' - 2y_2 = e^t + e^{4t}.$$

This can be done using the method of undetermined coefficients, with the ansatz $y_2 = Ae^t + Be^{4t}$. The solution is $y_2 = -e^t + (1/2)e^{4t}$. Now we plug this in to the first equation, which becomes

$$y_1' = 2y_1 + (-e^t + \frac{1}{2}e^{4t}) + e^t = 2y_1 + \frac{1}{2}e^{4t}.$$

The ansatz $y_1 = Ae^{4t}$ yields the solution $y_1 = (1/4)e^{4t}$. Hence a particular solution to the original system is

$$\mathbf{v} = \mathbf{S}\begin{pmatrix} \frac{1}{4}e^{4t} \\ -e^t + \frac{1}{2}e^{4t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{4t} \\ -e^t + \frac{1}{4}e^{4t} \end{pmatrix}.$$

The general solution is therefore

$$\mathbf{x} = \mathbf{v} + e^{\mathbf{P}t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{4t} + c_1e^{2t} + c_2te^{2t} \\ -e^t + \frac{1}{4}e^{4t} + c_2e^{2t}. \end{pmatrix}$$

5.5 Phase plane and stability

For a single ODE, we can plot the solution $x(t)$ as a graph to get a visual idea of its behavior. For a *system* of ODEs, we should plot the entire vector $\mathbf{x}(t)$. This is difficult to understand visually except in low dimensions, so now we will narrow our focus to systems of *two* linear first-order equations, i.e. when $\mathbf{P}(t)$ is a 2×2 matrix.

Definition 5.5.1. The x_1x_2-plane is the **phase plane**, and the plot of solutions on it is the **phase diagram** of the system of ODEs.

It is enlightening to see, visually, what different kinds of solutions look like for the case of a 2×2 constant-coefficient homogeneous system

$$\mathbf{x}' = \mathbf{P}\mathbf{x}.$$

We can classify different types of such systems, by diagonalizing (or putting into Jordan form) the system and examining the system in that basis. Assume for simplicity that $\mathbf{P}$ is full rank, so no eigenvalues are 0.

Example 5.5.2 (Diagonalizable $\mathbf{P}$)**.** Suppose $\mathbf{P}$ is diagonalizable, with eigenvalues λ_1, λ_2. In the eigenbasis, we may as well think about the phase diagram of the system

$$\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{x}.$$

Letting $(x, y) = (x_1, x_2)$, this autonomous system is related to the first-order *equation*

$$\frac{dy}{dx} = \frac{\lambda_2}{\lambda_1}\frac{y}{x}$$

as described in section 1.5. The general solution to this equation is

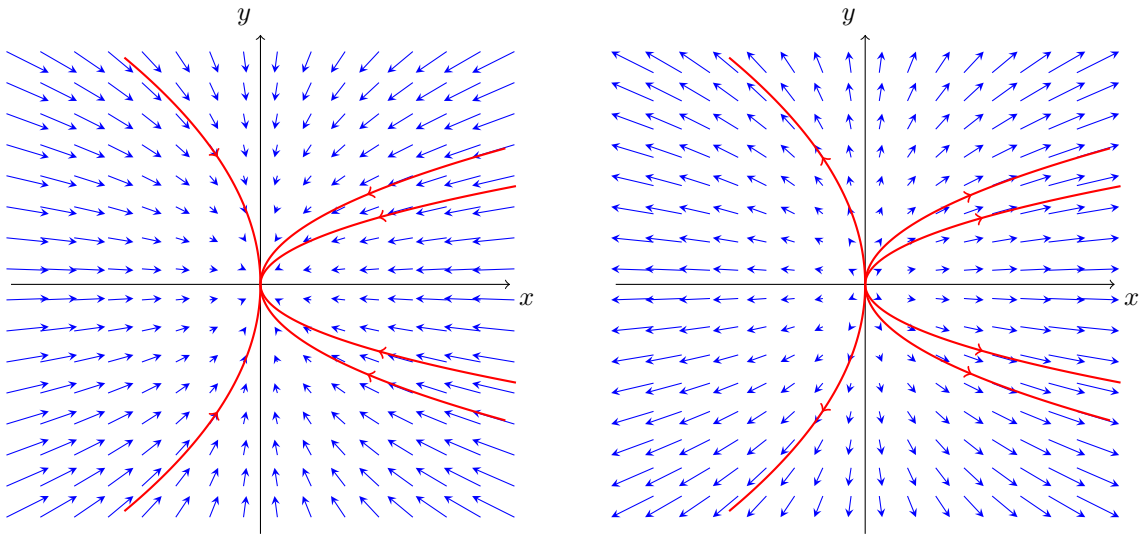
$$y = cx^{\lambda_2/\lambda_1}.$$

This is the *unparameterized* versions of solutions to the original system, and therefore will at least tell us what the integral curves look like. For example, consider $\lambda_2/\lambda_1 = 1/2$. Then integral curves are $y = c\sqrt{x}$.

1. If $(\lambda_1, \lambda_2) = (-2, -1)$, then the *parameterized* solution is $\mathbf{x} = (e^{-2t}, e^{-t})$. In this case, trajectories flow *toward* the origin as time increases.

2. If $(\lambda_2, \lambda_2) = (2, 1)$, then the *parameterized* solution is $\mathbf{x} = (e^{2t}, e^t)$. In this case, trajectories flow *away* from the origin as time increases.

The vector field (and some trajectories) of the original system in these two cases is as follows. Note that everything stays the same except the direction of flow.



The origin $\mathbf{x} = (0, 0)$ is special in this case. It is an *equilibrium*, in the sense that a trajectory $\mathbf{x}(t)$ which starts at $(0, 0)$ remains stationary.

Definition 5.5.3. An **equilibrium** or **critical point** of a first-order system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is a point $\mathbf{x}$ where

$$\mathbf{F}(\mathbf{x}) = 0.$$

Such a point is:

1. **asymptotically stable** if all sufficiently nearby points flow *toward* the critical point;
2. **stable** if all sufficiently nearby points stay within a bounded region around the critical point;
3. **unstable** if it is not stable, i.e. if trajectories flow far away from the critical point.

Example 5.5.4. So far we only looked at *linear* homogeneous systems $\mathbf{x}' = \mathbf{P}\mathbf{x}$ with $\mathbf{P}$ invertible. Then

$$\mathbf{P}\mathbf{x} = 0 \iff \mathbf{x} = 0.$$

So the origin is the only critical point. If $\mathbf{P}$ is not invertible, everything in the kernel of $\mathbf{P}$ is a critical point.

Example 5.5.5 (Nodal sources/sinks)**.** Let $\mathbf{P}$ be diagonalizable with eigenvalues λ_1, λ_2 of the *same sign*.

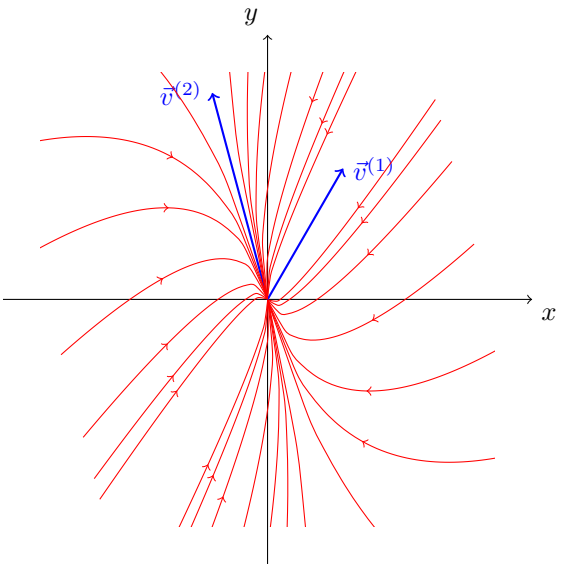
1. If $\lambda_1, \lambda_2 < 0$, then all trajectories flow toward the origin, and so the origin is an asymptotically stable critical point. Such a point is usually called a **sink**.
2. If $\lambda_1, \lambda_2 > 0$, then all trajectories flow away from the origin, and so it is an unstable point. Such a point is usually called a **source**.

In either case, all trajectories around the critical point behave the same way; such a critical point is a **node**.

Let $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ be the eigenvectors corresponding to the eigenvalues λ_1, λ_2. Then the phase diagram for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ is related to the phase diagrams in the previous Example 5.5.2 by a change of coordinates back to the original basis from the eigenbasis. For example, the phase diagram for the system where

$$\mathbf{P} = \mathbf{S}\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{S} = \begin{pmatrix} | & | \\ \mathbf{v}^{(1)} & \mathbf{v}^{(2)} \\ | & | \end{pmatrix}$$

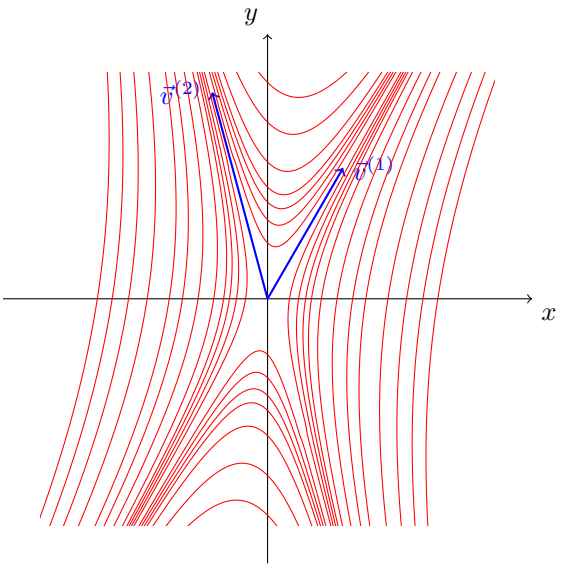
looks like the following.



Example 5.5.6 (Saddle points)**.** If in Example 5.5.2 the diagonalizable matrix $\mathbf{P}$ has two eigenvalues λ_1, λ_2 of *different* sign, then the critical point at $(0,0)$ is not a node anymore. This is because the solution

$$\mathbf{x} = (e^{\lambda_1 t}, e^{\lambda_2 t})$$

has one component flowing away from the origin and another component flowing toward the origin. The critical point in this case is a **saddle point**. Note that it is *unstable*.



In general, for the diagonalizable case, note that as $t \to \pm\infty$ one of the solutions $\mathbf{v}e^{\lambda t}$ will grow faster than the other. We say the faster-growing solution *dominates* the slower-growing one. This is reflected in the phase diagram: as we follow the trajectories forward/backwards, they eventually become parallel to the relevant eigenvector. For example, for a nodal source, if $\lambda_1 > \lambda_2 > 0$ then:

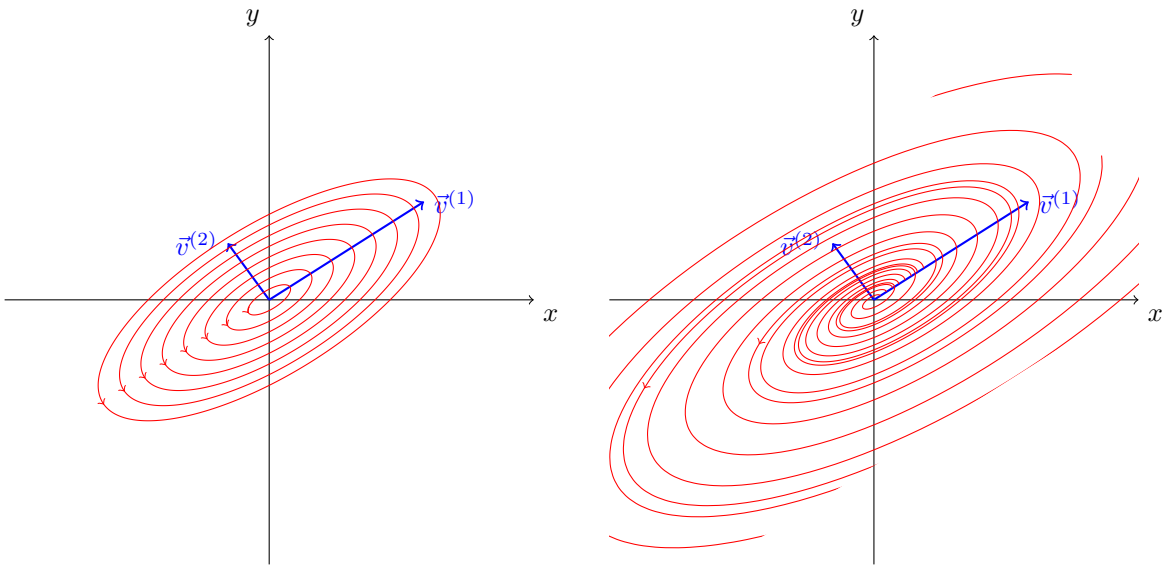
1. $\mathbf{v}^{(1)}e^{\lambda_1 t}$ dominates as $t \to \infty$, so eventually trajectories all become parallel to $\mathbf{v}^{(1)}$;
2. $\mathbf{v}^{(2)}e^{\lambda_2 t}$ dominates as $t \to -\infty$, so near the critical point $(0,0)$ all trajectories become parallel to $\mathbf{v}^{(2)}$.

The same observations hold in general for nodal sinks and saddle points too. However it is not true in all cases that the steady-state solution tends toward eigenvectors.

Example 5.5.7 (Complex eigenvalues)**.** If $\mathbf{P}$ is diagonalizable but has complex eigenvalues $\lambda \pm i\mu$, then neither eigenvector dominates the other and there is oscillatory behavior in the solutions. In the eigenbasis, a fundamental set of solutions is

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}\begin{pmatrix}\cos \mu t \\ -\sin \mu t\end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = e^{\lambda t}\begin{pmatrix}\sin \mu t \\ \cos \mu t\end{pmatrix}.$$

When $\lambda = 0$, these are parameterized circles; when $\lambda \neq 0$, these circles will expand/shrink as t changes, and form spirals instead.



Note that here the critical point is stable but *not* asymptotically stable.

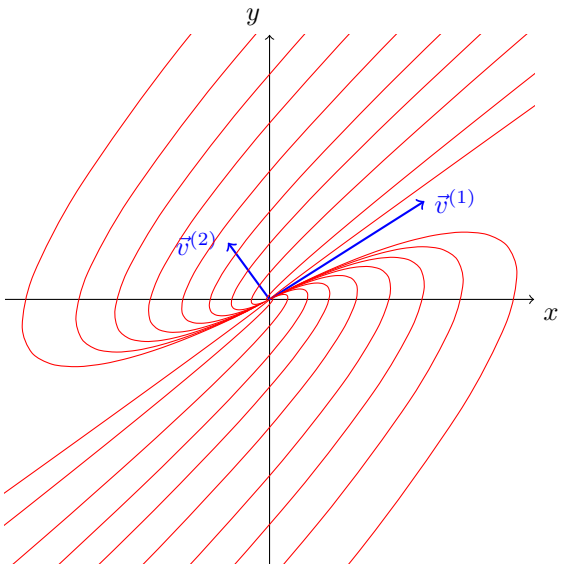
Example 5.5.8 (Non-diagonalizable $\mathbf{P}$)**.** If $\mathbf{P}$ is not diagonalizable, then its Jordan normal form is a single 2×2 Jordan block. In other words, there is a basis of generalized eigenvectors where the system becomes

$$\mathbf{x}' = \begin{pmatrix}\lambda & 1 \\ 0 & \lambda\end{pmatrix}\mathbf{x}$$

with solutions $\mathbf{v}^{(1)}e^{\lambda t}$ and $e^{\lambda t}(\mathbf{v}^{(1)}t + \mathbf{v}^{(2)})$. Unfortunately here the associated first-order equation

$$-\lambda y + (\lambda x + y)\frac{dy}{dx} = 0$$

has no nice solution, so the integral curves cannot be described using elementary functions. However note that as $t \to \pm\infty$ the term $\mathbf{v}^{(1)}te^{\lambda t}$ dominates, so trajectories are parallel to the (only) eigenvector $\mathbf{v}^{(1)}$ both near the origin and also near infinity.



The origin here is also a node, and is sometimes called a **degenerate node**.

A similar kind of analysis can be performed for higher-dimensional systems, especially for diagonalizable systems. For example, for 3×3 matrices, having all three positive eigenvalues gives a nodal source where eventually trajectories become parallel to the eigenvector with largest eigenvalue. However, higher-dimensional Jordan blocks become more complicated. Nodes can be degenerate in multiple ways, depending on the number and size of Jordan blocks.

6 Boundary value problems

In the final part of this course, we study a class of differential equations which is more general than IVPs. Recall that an n-th order IVP (whether with scalar or vector-valued solutions) has initial conditions of the form

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \cdots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

Importantly, these initial conditions have all been at the same point, namely the initial point t_0. For such IVPs, we have a very general existence and uniqueness theorem 2.4.2 on an open interval I which contains the initial point. Often the initial point is the left endpoint of I, especially when the independent variable t represents time. However, instead of constraining the solution at a single point, we can imagine putting **boundary conditions** on solutions in I by constraining how they behave on the *boundary* of I, i.e. *both* endpoints of I.

Definition 6.0.1. An n-th order **boundary value problem (BVP)** on an interval $I = (\alpha, \beta)$ is an n-th order equation with boundary conditions

$$\alpha_0 y(\alpha) + \alpha_1 y'(\alpha) + \cdots + \alpha_{n-1} y^{(n-1)}(\alpha) = 0$$
$$\beta_0 y(\beta) + \beta_1 y'(\beta) + \cdots + \beta_{n-1} y^{(n-1)}(\beta) = 0.$$

In principle the rhs could be arbitrary constants y_α and y_β, but they can always be made zero after a change of variables.

The above is the most general boundary condition. In practice, there are only two important kinds of boundary conditions; most other boundary conditions are combinations of these two.

1. (**Dirichlet boundary condition**) Fix the value of the solution at the boundary, i.e. impose

$$y(\alpha) = 0, \quad y(\beta) = 0.$$

This can represent, for example, the behavior of an elastic which is pinned down at two points.

2. (**Neumann boundary condition**) Fix the *rate of change* of the solution at the boundary, i.e. impose

$$y'(\alpha) = 0, \quad y'(\beta) = 0.$$

This can represent, for example, the heat distribution in a 1-dimensional rod whose endpoints are being constantly heated/cooled.

The type of boundary condition on each endpoint may be different, in which case we say the boundary condition is **mixed**.

Example 6.0.2. Consider the BVP

$$y'' + y = 0, \quad y(0) = y(\pi) = 0.$$

This is a Dirichlet boundary condition. Without the boundary condition, the general solution is

$$\tilde{y}(t) = c_1 \cos t + c_2 \sin t. \tag{6.1}$$

But then $y(0) = c_1$ and $y(\pi) = c_1$, so the boundary condition forces $c_1 = 0$ with no constraint on c_2. The solution is therefore $y = c \sin t$ for an arbitrary constant c. In particular there are infinitely many solutions.

Example 6.0.3. Consider the BVP

$$y'' + y = 0, \quad y'(0) = y'(\pi) = 0.$$

This is a Neumann boundary condition. It forces $c_2 = 0$ in the general solution (6.1), and so the solution is $y = c \cos t$ for an arbitrary constant c.

Knowing the general solution to the equation $y'' + y = 0$, it is not hard to see that we can produce boundary conditions such that there are no solutions, one solution, or infinitely many solutions. In general, establishing existence and/or uniqueness theorems for various classes of BVPs is very difficult.

6.1 Green's functions

A Green's function is a more general version of the impulse response, and is used to find solutions to BVPs whenever they exist. They are constructed out of solutions to the associated homogeneous equation. For simplicity, in this section we focus on second-order equations.

Definition 6.1.1. On an interval $[\alpha, \beta]$, consider the second-order equation $Dy = g(t)$ where

$$D \coloneqq a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t)$$

is an arbitrary linear second-order differential operator with *continuous* coefficients a, b, c. (Later we'll need to assume $a(t)$ has at most finitely many zeros.) Impose general boundary conditions at α and β:

$$\begin{aligned} \alpha_0 y(\alpha) + \alpha_1 y'(\alpha) &= 0 \\ \beta_0 y(\beta) + \beta_1 y'(\beta) &= 0. \end{aligned} \tag{6.2}$$

A **Green's function** for the operator D is a solution $G(t, s)$ to the equation

$$DG(t, s) = \delta(t - s)$$

satisfying the above boundary conditions in the variable t.

Just like impulse responses, Green's functions only depend on the homogeneous part of the equation. If we know $G(t,s)$, then the solution to the original, non-homogeneous equation $Dy = g(t)$ is

$$y(t) = \int_\alpha^\beta G(t,s)g(s)\,ds. \tag{6.3}$$

If G doesn't actually depend on t and s individually, but rather only depends on their difference $t-s$, then $\int_\alpha^\beta G(t-s)g(s)\,ds$ is just a convolution, and in this case G is exactly the impulse response. This is true for IVPs. But for BVPs, G will actually depend on both t and s, instead of just $t-s$.

Theorem 6.1.2. *The function* (6.3) *solves* $Dy = g(t)$ *with the prescribed boundary conditions* (6.2)

Proof. First we check that it solves the equation $Dy = g(t)$. This is a computation using the delta function:

$$Dy = D\int_\alpha^\beta G(t,s)g(s)\,ds = \int_\alpha^\beta DG(t,s)g(s)\,ds = \int_\alpha^\beta \delta(t-s)g(s)\,ds = g(t).$$

Next we check that the solution satisfies the boundary conditions (6.2). At $t=\alpha$,

$$\alpha_0 y(\alpha) + \alpha_1 y'(\alpha) = \int_a^b \left(\alpha_0 G(\alpha,s) + \alpha_1 \frac{\partial G}{\partial t}(\alpha,s)\right) g(s)\,ds = \int_a^b 0\cdot g(s)\,ds = 0$$

because by definition G satisfies the same boundary condition in the t variable. The same thing is true at $t=\beta$. □

So we should figure out how to construct Green's functions. The idea is to impose the boundary conditions one at a time for the *homogeneous* equation. This makes sense because for $t<s$ and also $t>s$ we have $\delta(t,s)=0$, and therefore we should consider the equation $DG(t,s)=0$ on both intervals separately. For example, on the interval $[\alpha,s)$ we should only impose the boundary condition

$$\alpha_0 y(\alpha) + \alpha_1 y'(\alpha) = 0. \tag{6.4}$$

Note that we can write an associated homogeneous IVP

$$Dy = 0, \quad y(\alpha) = -\alpha_1,\ y'(\alpha) = \alpha_0.$$

Solutions y will of course also satisfy the boundary condition (6.4). So we have written down an IVP, whose solution y_1 exists and is unique and solves "half" the BVP. Similarly, we can get a solution y_2 for the IVP associated to the boundary condition at $t=\beta$. These solutions y_1 and y_2 are usually independent, i.e. are not scalar multiples of each other. To understand why this is the case, and also how y_1 and y_2 combine to give the Green's function, we need the following observation.

Theorem 6.1.3. *A function* $u(t)$ *is a solution to*

$$Dy = 0, \quad \alpha_0 y(\alpha) + \alpha_1 y'(\alpha) = 0 \tag{6.5}$$

if and only if $u = cy_1$ *for some constant* c.

Proof. If $u = cy_1$ clearly it satisfies the equation. Conversely, if u satisfies the equation, then

$$0 = \alpha_0 u(\alpha) + \alpha_1 u'(\alpha) = -y_1'(\alpha)u(\alpha) + y_1(\alpha)u'(\alpha) = W(y_1,u)(\alpha).$$

So u and y_1 are not independent solutions, i.e. $u = cy_1$. □

From this theorem we can immediately make two observations. First, as long as y_2 doesn't satisfy the boundary condition (6.4) at $t = \alpha$ corresponding to y_1, they are independent solutions. Second, the Green's function $G(t, s)$ itself satisfies the IVP (6.5) associated to y_1, and also the IVP associated to y_2, and therefore

$$G(t, s) = A(s)y_1(t) \quad \text{for } t \in [\alpha, s) \tag{6.6}$$
$$G(t, s) = B(s)y_2(t) \quad \text{for } t \in (s, \beta] \tag{6.7}$$

for some unknown coefficients $A(s)$ and $B(s)$ which are *independent* of t. To figure out what they are, we need to consider how these two solutions in different intervals are "glued together" at $t = s$. In other words, we need to find how

$$G(t, s)|_{t=s^\pm} \coloneqq \lim_{t \to s^\pm} G(t, s)$$

(and their derivatives) are related to each other.

Theorem 6.1.4. *The functions* (6.6) *and* (6.7) *should be glued according to the conditions:*

1. *(continuity at $t = s$)*
$$G(t, s)|_{t=s^-} = G(t, s)|_{t=s^+};$$

2. *(jump in derivative at $t = s$)*
$$\left.\frac{\partial G(t, s)}{\partial t}\right|_{t=s^+} - \left.\frac{\partial G(t, s)}{\partial t}\right|_{t=s^-} = \frac{1}{a(s)}. \tag{6.8}$$

Proof. The Green's function satisfies the second-order equation

$$\frac{\partial^2 G}{\partial t^2} + \frac{b(t)}{a(t)}\frac{\partial G}{\partial t} + \frac{c(t)}{a(t)}G = \frac{1}{a(t)}\delta(t - s).$$

We can integrate both sides with respect to t, i.e. apply $\int_\alpha^t$ to both sides. Observe the following.

1. Assume $a(t)$ has finitely many zeros, so that for a given s there is some interval I containing $t = s$ on which $a(t)$ is never zero. Then all coefficients are continuous and bounded on I. We work on I from now on.

2. On the rhs, we may replace $a(t)$ with $a(s)$ since the delta function is zero everywhere else.

Then the integral gives

$$\frac{\partial G}{\partial t} + (\text{some continuous functions}) = \frac{1}{a(s)}u_s(t).$$

Continuous functions have the same left-handed and right-handed limits, but the step function $u_s(t)$ does not. Taking left/right handed limits of this equation yields (6.8). Integrating this equation again with respect to t yields

$$G + (\text{some continuous function}) = \frac{1}{a(s)}(t - s)u_s(t),$$

and now the rhs is continuous too. Hence G itself is a continuous function. □

If we plug (6.6) and (6.7) into these conditions, we get that

$$A(s)y_1(s) = B(s)y_2(s)$$
$$B(s)y_2'(s) - A(s)y_1'(s) = \frac{1}{a(s)}.$$

The solution to this system of equations for A, B is

$$A(s) = \frac{y_2(s)}{a(s)W(s)}, \quad B(s) = \frac{y_1(s)}{a(s)W(s)}$$

where $W = W(y_1, y_2)$ is the Wronskian. Hence the Green's function is

$$G(t,s) = \begin{cases} \dfrac{y_1(t)y_2(s)}{a(s)W(s)} & t \le s \\ \dfrac{y_2(t)y_1(s)}{a(s)W(s)} & s \le t. \end{cases} \tag{6.9}$$

The solution (6.3) to the original BVP is

$$\begin{aligned} y(t) &= \int_\alpha^t \frac{y_2(t)y_1(s)}{a(s)W(s)}\,ds + \int_t^\beta \frac{y_1(t)y_2(s)}{a(s)W(s)}\,ds \\ &= y_2(t)\int_\alpha^t \frac{y_1(s)}{a(s)W(s)}\,ds + y_1(t)\int_t^\beta \frac{y_2(s)}{a(s)W(s)}\,ds. \end{aligned}$$

Example 6.1.5. Consider the BVP

$$y'' + y = g(x), \quad y(0) = y(\pi/2) = 0.$$

To find its Green's function, we write the associated IVPs for y_1, y_2:

$$\begin{aligned} y'' + y = 0, \quad y(0) = 0,\ y'(0) = 1 \\ y'' + y = 0, \quad y(\pi/2) = 0,\ y'(\pi/2) = 1. \end{aligned}$$

Hence $y_1 = \sin t$ and $y_2 = -\cos t$. Their Wronskian is the constant function $W(y_1, y_2) = 1$. It follows that the Green's function is

$$G(t,s) = \begin{cases} -\sin t \cos s & t \le s \\ -\cos t \sin s & s \le t. \end{cases}$$

The solution to the original BVP is

$$y(t) = -\cos(t)\int_0^t g(s)\sin(s)\,ds - \sin(t)\int_t^{\pi/2} g(s)\cos(s)\,ds.$$

Green's functions may not always exist. This is because if the Green's function exists for a BVP then (6.3) immediately gives a solution to the BVP, but we know there are BVPs with no solutions. Actually it is possible that the BVP has a solution yet a Green's function still does not exist! One way this can happen is if the two solutions y_1 and y_2 forming the Green's function are not actually independent.

Example 6.1.6 (1d Laplacian with Neumann boundary conditions)**.** Consider the BVP

$$y'' = 0, \quad y'(-1) = y'(1) = 0.$$

It clearly has solutions $y = c$ for any constant c. We can try to compute its Green's function. The associated IVPs for y_1 and y_2 are

$$\begin{aligned} y'' = 0, \quad y(-1) = -1,\ y'(-1) = 0 \\ y'' = 0, \quad y(1) = -1,\ y'(1) = 0. \end{aligned}$$

The solutions are $y_1 = y_2 = -1$, which are not independent. Then $W(y_1, y_2) = 0$ and the Green's function (6.9) is undefined. It is not our construction that is flawed; indeed, we can show that that there cannot be a Green's function for this BVP as follows. If G did exist, then

$$\int_{-1}^{1} \frac{\partial^2}{\partial t^2} G(t,s)\, dt = \int_{-1}^{1} \delta(t-s)\, dt = 1.$$

On the other hand, by the fundamental theorem of calculus

$$\int_{-1}^{1} \frac{\partial^2 G}{\partial t^2}(t,s)\, dt = \frac{\partial G}{\partial t}(1,s) - \frac{\partial G}{\partial t}(-1,s).$$

The boundary conditions for G say that each term on the rhs is 0, a contradiction. If we want to solve this BVP by Green's function methods, it turns out the appropriate modification is to look for G satisfying

$$DG(t,s) = \delta(t-s) - \frac{1}{2}.$$

Such a solution exists and is given by

$$G(t,s) = \frac{|t-s|}{2} - \frac{t^2+s^2}{4}.$$

Green's functions can even be used when the boundary conditions are at $\pm\infty$. This does not give rise to proper IVPs for y_1 and y_2, but even so we can find Green's functions and therefore a solution. Such BVPs are common in physics, where we often assume that some physical phenomenon (e.g. electic/magnetic fields) die out and vanish as we approach $\pm\infty$.

Example 6.1.7 (1d Helmholtz equation)**.** Consider the BVP

$$y'' - k^2 y = g(t), \quad y(\pm\infty) = 0.$$

The associated IVPs for y_1 and y_2, from our construction of G, are

$$y'' = 0, \quad y(-\infty) = 0,\ y'(-\infty) = 1$$
$$y'' = 0, \quad y(\infty) = 0,\ y'(\infty) = 1.$$

One can check that no solution of the form $c_1 e^{kt} + c_2 e^{-kt}$ satisfies these IVPs. However we can drop the conditions on y'; they are superfluous if we just want to satisfy the boundary conditions $y(\pm\infty) = 0$. Then

$$y_1 = c_1 e^{kt}, \quad y_2 = c_2 e^{-kt},$$

and the Green's function is

$$G(t,s) = \begin{cases} A(s)e^{kt} & t \le s \\ B(s)e^{-kt} & s \le t. \end{cases}$$

Theorem 6.1.4 gives conditions that A and B must satisfy. Solving,

$$A(s) = -\frac{1}{2k} e^{-ks}, \quad B(s) = -\frac{1}{2k} e^{ks}.$$

The entire Green's function can be written

$$G(t,s) = -\frac{1}{2k} e^{-k|t-s|},$$

and so the general solution is

$$y(t) = -\frac{1}{2k} \int_{-\infty}^{\infty} g(s) e^{-k|t-s|}\, ds.$$

Often, especially in physics, the boundary conditions are implicitly taken to be $y(\pm\infty) = 0$. Then we speak about the Green's function corresponding to the differential operator itself. For example, this example worked out the Green's function for the 1d Helmholtz operator.