Null–Kähler geometry and Twistor Theory

Maciej Dunajski

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

- MD. Null Kähler geometry and isomonodromic deformations. `arXiv: 2010.11216`.
- Tom Bridgeland, MD. Work in progress.

General Relativity and Gravitation, Vol. 7, No. 1 (1976), pp. 31–52

Nonlinear Gravitons and Curved Twistor Theory

ROGER PENROSE

Mathematical Institute, Oxford, England

§(1): *Introduction*

The question of how best to quantize gravity has been the subject of many discussions and arguments over the years. And Peter Bergmann has repeatedly and tirelessly reminded us that gravitational quanta should *not* be described in terms merely of *linearized* gravitation theory. I feel I have been rather slow at coming around to accepting this fully myself. It is, indeed, seductive to attempt to invoke the quantum-mechanical principle of linear superposition as an excuse for putting off, to a second stage of consideration, the complicated nonlinear nature of the gravitational self-interaction—and for putting off, perhaps indefinitely, the daunting encounter between quantum mechanics and the principles of curved-space geometry! If Peter Bergmann has taught us one thing above most others, it is surely that if we remove the life from Einstein's beautiful theory by steam-rollering it first to flatness and linearity, then we shall learn nothing from attempting to wave the magic wand of quantum theory over the resulting corpse.

Let me put things somewhat differently. Consider the common attitude according to which "gravitons" are described by linearized Einstein theory (spin-2 massless Poincaré covariant fields), a perturbative viewpoint being adopted starting from flat Minkowski space. If one such "graviton" is added to the vacuum (Minkowski) state the space remains flat. The null cones do not shift. If a second such "graviton" is added, and a third and a fourth, the space still remains flat, with null cones still locked in their original Minkowskian positions. With such a perturbative viewpoint it is only after an infinite number of "gravitons" have been added that the space can become curved. The situation may be compared with a power-series expansion. For example, with any finite

Nonlinear graviton

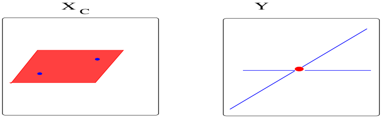
- Holomorphic oriented Riemannian four-manifold $(\mathcal{X}_{\mathbb{C}}, g)$.

Nonlinear graviton

- Holomorphic oriented Riemannian four-manifold $(\mathcal{X}_{\mathbb{C}}, g)$.
- An α–surface is a 2D surface $\xi \subset \mathcal{X}_{\mathbb{C}}$ s. t. $\forall p \in \mathcal{X}_{\mathbb{C}}, T_p\xi$ is a totally null plane with self–dual bi-vector.

- Holomorphic oriented Riemannian four-manifold $(\mathcal{X}_{\mathbb{C}}, g)$.
- An α–surface is a 2D surface $\xi \subset \mathcal{X}_{\mathbb{C}}$ s. t. $\forall p \in \mathcal{X}_{\mathbb{C}}, T_p\xi$ is a totally null plane with self–dual bi-vector.
- **Nonlinear Graviton Theorem** (Penrose 1976). There exists a three parameter family $\mathcal{Y}$ (a twistor space) of α surfaces iff Weyl$_+ = 0$.

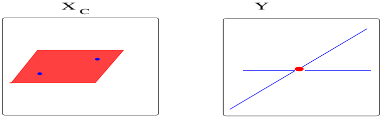
Point $p \in \mathcal{X}_{\mathbb{C}}$	$\longleftrightarrow$	Curve $L_p = \mathbb{CP}^1 \subset \mathcal{Y}$
α-surface	$\longleftrightarrow$	Point.
p_1, p_2 null separated	$\longleftrightarrow$	L_1, L_2 intersect at one point



Nonlinear graviton

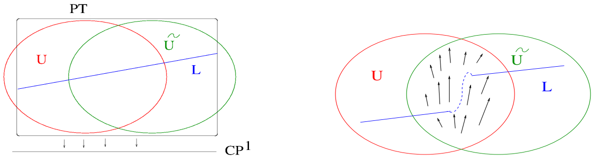
- Holomorphic oriented Riemannian four-manifold $(\mathcal{X}_{\mathbb{C}}, g)$.
- An α–surface is a 2D surface $\xi \subset \mathcal{X}_{\mathbb{C}}$ s. t. $\forall p \in \mathcal{X}_{\mathbb{C}}, T_p\xi$ is a totally null plane with self–dual bi-vector.
- **Nonlinear Graviton Theorem** (Penrose 1976). There exists a three parameter family $\mathcal{Y}$ (a twistor space) of α surfaces iff Weyl$_+ = 0$.

Point $p \in \mathcal{X}_{\mathbb{C}}$	$\longleftrightarrow$	Curve $L_p = \mathbb{CP}^1 \subset \mathcal{Y}$
α-surface	$\longleftrightarrow$	Point.
p_1, p_2 null separated	$\longleftrightarrow$	L_1, L_2 intersect at one point



- More structures on $\mathcal{Y}$ if g Einstein. Reality conditions $(4,0)$ or $(2,2)$.

- Penrose/Sparling: $\mathcal{Y}$ as a deformation of $\mathbb{CP}^3 - \mathbb{CP}^1$.

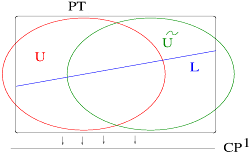


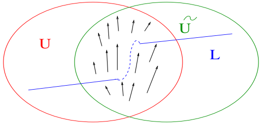
Kodaira theorems: Normal bundle $N(L_p) \equiv T(\mathcal{Y}_c)|_{L_p}/TL_p$

$$H^1(L_p, N(L_p)) = 0, \quad H^0(L_p, N(L_p)) \cong T_p\mathcal{X}_{\mathbb{C}}.$$

Twistor spaces

- Penrose/Sparling: $\mathcal{Y}$ as a deformation of $\mathbb{CP}^3 - \mathbb{CP}^1$.



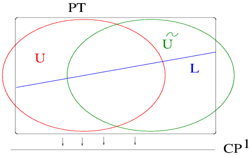


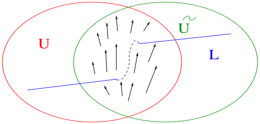
Kodaira theorems: Normal bundle $N(L_p) \equiv T(\mathcal{Y}_c)|_{L_p}/TL_p$

$$H^1(L_p, N(L_p)) = 0, \quad H^0(L_p, N(L_p)) \cong T_p\mathcal{X}_{\mathbb{C}}.$$

- Hitchin/Kronheimer: $\mathcal{Y}$ as a hypersurface in the total space of $\mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus \mathcal{O}(m_3) \to \mathbb{CP}^1$.

- Penrose/Sparling: $\mathcal{Y}$ as a deformation of $\mathbb{CP}^3 - \mathbb{CP}^1$.





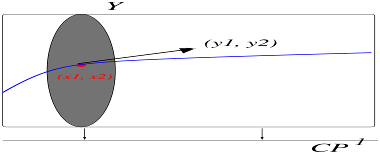
Kodaira theorems: Normal bundle $N(L_p) \equiv T(\mathcal{Y}_c)|_{L_p}/TL_p$

$$H^1(L_p, N(L_p)) = 0, \quad H^0(L_p, N(L_p)) \cong T_p\mathcal{X}_{\mathbb{C}}.$$

- Hitchin/Kronheimer: $\mathcal{Y}$ as a hypersurface in the total space of $\mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus \mathcal{O}(m_3) \to \mathbb{CP}^1$.
- Hard part: find the twistor lines.

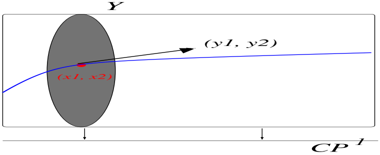
Heavenly equations

- $\mu : \mathcal{Y} \to \mathbb{CP}^1$. Parametrise L_p by its intersection with $\mathbb{C}^2 = \mu^{-1}(0)$ (coordinates (x^1, x^2)), and a direction (coordinates (y^1, y^2)).



Heavenly equations

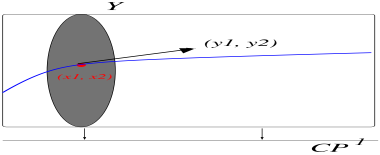
- $\mu : \mathcal{Y} \to \mathbb{CP}^1$. Parametrise L_p by its intersection with $\mathbb{C}^2 = \mu^{-1}(0)$ (coordinates (x^1, x^2)), and a direction (coordinates (y^1, y^2)).



- $\mathcal{O}(2)$–valued symplectic form on fibres of μ: $\exists\Theta = \Theta(x^1, x^2, y^1, y^2)$

$$
\begin{aligned}
\omega^1 &= x^1 + \lambda y^1 - \lambda^2 \Theta_{y^2} - \lambda^3 \Theta_{x^2} + \dots, \\
\omega^2 &= x^2 + \lambda y^2 + \lambda^2 \Theta_{y^1} + \lambda^3 \Theta_{x^1} + \dots, \quad \text{where} \quad \Theta_{x^1} = \partial_{x^1}\Theta.
\end{aligned}
$$

- $\mu : \mathcal{Y} \to \mathbb{CP}^1$. Parametrise L_p by its intersection with $\mathbb{C}^2 = \mu^{-1}(0)$ (coordinates (x^1, x^2)), and a direction (coordinates (y^1, y^2)).



- $\mathcal{O}(2)$–valued symplectic form on fibres of μ: $\exists \Theta = \Theta(x^1, x^2, y^1, y^2)$

$$\begin{aligned} \omega^1 &= x^1 + \lambda y^1 - \lambda^2 \Theta_{y^2} - \lambda^3 \Theta_{x^2} + \dots, \\ \omega^2 &= x^2 + \lambda y^2 + \lambda^2 \Theta_{y^1} + \lambda^3 \Theta_{x^1} + \dots, \quad \text{where} \quad \Theta_{x^1} = \partial_{x^1}\Theta. \end{aligned}$$

- ASD Ricci–flat (complex hyper–Kähler) metric

$$g = dy^1 dx^2 - dy^2 dx^1 + \Theta_{y^1 y^1}(dx^1)^2 + 2\Theta_{y^1 y^2} dx^1 dx^2 + \Theta_{y^2 y^2}(dx^2)^2,$$

where $\Theta_{x^1 y^2} - \Theta_{x^2 y^1} + \Theta_{y^1 y^1}\Theta_{y^2 y^2} - (\Theta_{y^1 y^2})^2 = 0$.
Heavenly equation (Plebański 1975, MD+Lionel Mason 2001).

Parallel real spinor

- Forget ASD, and Ricci flat. What is special about $(2, 2)$ metrics of the form

$$g = dy^1dx^2 - dy^2dx^1 + \Theta_{y^1y^1}(dx^1)^2 + 2\Theta_{y^1y^2}dx^1dx^2 + \Theta_{y^2y^2}(dx^2)^2 \quad (\star).$$

Parallel real spinor

- Forget ASD, and Ricci flat. What is special about $(2,2)$ metrics of the form

$$g = dy^1dx^2 - dy^2dx^1 + \Theta_{y^1y^1}(dx^1)^2 + 2\Theta_{y^1y^2}dx^1dx^2 + \Theta_{y^2y^2}(dx^2)^2 \quad (\star).$$

- **Theorem A** (MD 2002) Let $(\mathcal{X}, g)$ be a $(2,2)$ signature Riemannian metric which admits a parallel section of $\mathbb{S}_+$, where $T\mathcal{X} \cong \mathbb{S}_+ \otimes \mathbb{S}_-$. Then
  1. Locally there exist coordinates (x^i, y^i), and a function Θ such that g is of the form $(\star)$, and conversely $(\star)$ admits a parallel spinor for any Θ.

Parallel real spinor

- Forget ASD, and Ricci flat. What is special about $(2,2)$ metrics of the form

$$g = dy^1 dx^2 - dy^2 dx^1 + \Theta_{y^1y^1}(dx^1)^2 + 2\Theta_{y^1y^2} dx^1 dx^2 + \Theta_{y^2y^2}(dx^2)^2 \quad (\star).$$

- **Theorem A** (MD 2002) Let $(\mathcal{X}, g)$ be a $(2,2)$ signature Riemannian metric which admits a parallel section of $\mathbb{S}_+$, where $T\mathcal{X} \cong \mathbb{S}_+ \otimes \mathbb{S}_-$. Then
  1. Locally there exist coordinates (x^i, y^i), and a function Θ such that g is of the form $(\star)$, and conversely $(\star)$ admits a parallel spinor for any Θ.
  2. If g is additionally ASD, then Θ satisfies a 4th order integrable PDE

$$\begin{aligned} f &= \Theta_{x^1y^2} - \Theta_{x^2y^1} + \Theta_{y^1y^1}\Theta_{y^2y^2} - (\Theta_{y^1y^2})^2 \\ \Delta_g f &:= f_{x^1y^2} - f_{x^2y^1} + \Theta_{y^2y^2} f_{y^1y^1} + \Theta_{y^1y^1} f_{y^2y^2} - 2\Theta_{y^1y^2} f_{y^1y^2} = 0 \end{aligned}$$

Parallel real spinor

- Forget ASD, and Ricci flat. What is special about $(2,2)$ metrics of the form

$$g = dy^1 dx^2 - dy^2 dx^1 + \Theta_{y^1y^1}(dx^1)^2 + 2\Theta_{y^1y^2} dx^1 dx^2 + \Theta_{y^2y^2}(dx^2)^2 \quad (\star).$$

- **Theorem A** (MD 2002) Let $(\mathcal{X}, g)$ be a $(2,2)$ signature Riemannian metric which admits a parallel section of $\mathbb{S}_+$, where $T\mathcal{X} \cong \mathbb{S}_+ \otimes \mathbb{S}_-$. Then
  1. Locally there exist coordinates (x^i, y^i), and a function Θ such that g is of the form $(\star)$, and conversely $(\star)$ admits a parallel spinor for any Θ.
  2. If g is additionally ASD, then Θ satisfies a 4th order integrable PDE

$$\begin{aligned} f &= \Theta_{x^1y^2} - \Theta_{x^2y^1} + \Theta_{y^1y^1}\Theta_{y^2y^2} - (\Theta_{y^1y^2})^2 \\ \Delta_g f &:= f_{x^1y^2} - f_{x^2y^1} + \Theta_{y^2y^2} f_{y^1y^1} + \Theta_{y^1y^1} f_{y^2y^2} - 2\Theta_{y^1y^2} f_{y^1y^2} = 0 \end{aligned}$$

  3. In this case $\mathcal{Y}$ admits a preferred section of $\kappa^{-1/4}$ (where κ is a holomorphic canonical bundle of $\mathcal{Y}$), preserved by an anti-holomorphic involution fixing a real equator of each rational curve.

Null–Kähler structures

- $(\mathcal{X}, g)$ pseudo–Riemannian manifold of dimension $4n$. A null–Kähler (NK) structure is $N : T\mathcal{X} \to T\mathcal{X}$ such that
  1. $N^2 = 0$, rank$(N) = 2n$,
  2. $g(NX, Y) + g(X, NY) = 0$,
  3. $\nabla N = 0$.

Null–Kähler structures

- $(\mathcal{X}, g)$ pseudo–Riemannian manifold of dimension $4n$. A null–Kähler (NK) structure is $N : T\mathcal{X} \to T\mathcal{X}$ such that
  1. $N^2 = 0$, $\mathrm{rank}(N) = 2n$,
  2. $g(NX, Y) + g(X, NY) = 0$,
  3. $\nabla N = 0$.
- Fundamental 2–form $\Omega(X, Y) = g(NX, Y)$

$$\Omega^{\wedge n} := \underbrace{\Omega \wedge \cdots \wedge \Omega}_{n} \neq 0, \quad \Omega^{\wedge (n+1)} = 0.$$

Null–Kähler structures

- $(\mathcal{X}, g)$ pseudo–Riemannian manifold of dimension $4n$. A null–Kähler (NK) structure is $N : T\mathcal{X} \to T\mathcal{X}$ such that
  1. $N^2 = 0$, $\text{rank}(N) = 2n$,
  2. $g(NX, Y) + g(X, NY) = 0$,
  3. $\nabla N = 0$.
- Fundamental 2–form $\Omega(X, Y) = g(NX, Y)$

$$\Omega^{\wedge n} := \underbrace{\Omega \wedge \cdots \wedge \Omega}_{n} \neq 0, \quad \Omega^{\wedge(n+1)} = 0.$$

- Motivation
  1. Signature of g is $(2n, 2n)$. Pseudo–Riemannian holonomy.
  2. Appearance in works of Bridgeland and Bridgeland and Strachan (in the complexified setting, and under additional curvature assumptions).
  3. Take $n = 1$, and impose anti–self–duality on Weyl. Dispersionless integrable system, and connections with isomonodromy.

Interlude. Dual numbers

- $a + \epsilon\, b \in \mathbb{D}$, $a, b \in \mathbb{R}$, and $\epsilon^2 = 0$.

$$(a_1 + \epsilon\, b_1)(a_2 + \epsilon\, b_2) = a_1 a_2 + \epsilon\, (a_1 b_2 + b_1 a_2).$$

- $a + \epsilon\, b \in \mathbb{D}$, $a, b \in \mathbb{R}$, and $\epsilon^2 = 0$.

$$(a_1 + \epsilon\, b_1)(a_2 + \epsilon\, b_2) = a_1 a_2 + \epsilon\, (a_1 b_2 + b_1 a_2).$$

- In geometry (Eduard Study 1903): a ruled surface in $\mathbb{R}^3$ is a curve in the space of oriented lines: a unit sphere in $\mathbb{D}^3$

$$(\mathbf{u} + \epsilon\mathbf{v}) \cdot (\mathbf{u} + \epsilon\mathbf{v}) = |\mathbf{u}|^2 + 2\epsilon\, \mathbf{u} \cdot \mathbf{v} = 1.$$

Interlude. Dual numbers

- $a + \epsilon\, b \in \mathbb{D}$, $a, b \in \mathbb{R}$, and $\epsilon^2 = 0$.

$$(a_1 + \epsilon\, b_1)(a_2 + \epsilon\, b_2) = a_1 a_2 + \epsilon\, (a_1 b_2 + b_1 a_2).$$

- In geometry (Eduard Study 1903): a ruled surface in $\mathbb{R}^3$ is a curve in the space of oriented lines: a unit sphere in $\mathbb{D}^3$

$$(\mathbf{u} + \epsilon \mathbf{v}) \cdot (\mathbf{u} + \epsilon \mathbf{v}) = |\mathbf{u}|^2 + 2\epsilon\, \mathbf{u} \cdot \mathbf{v} = 1.$$

- In nonstandard analysis: $1 \neq 0.999 \cdots$.

- $a + \epsilon\, b \in \mathbb{D}$, $a, b \in \mathbb{R}$, and $\epsilon^2 = 0$.

$$(a_1 + \epsilon\, b_1)(a_2 + \epsilon\, b_2) = a_1 a_2 + \epsilon\, (a_1 b_2 + b_1 a_2).$$

- In geometry (Eduard Study 1903): a ruled surface in $\mathbb{R}^3$ is a curve in the space of oriented lines: a unit sphere in $\mathbb{D}^3$

$$(\mathbf{u} + \epsilon \mathbf{v}) \cdot (\mathbf{u} + \epsilon \mathbf{v}) = |\mathbf{u}|^2 + 2\epsilon\, \mathbf{u} \cdot \mathbf{v} = 1.$$

- In nonstandard analysis: $1 \neq 0.999 \cdots$.
- In algebra

$$a + \epsilon\, b \rightarrow \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = a\mathbf{1} + bN, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Null Kähler potential

- **Theorem B** (MD 2020) Let $(\mathcal{X}, g, N)$ be a $4n$–dimensional null–Kähler manifold. There exist a local coordinate system $(x^i, y^i), i = 1, \dots, 2n$ and a function $\Theta : \mathcal{X} \to \mathbb{R}$ such that

$$
\begin{aligned}
g &= \sum_{i,j} \omega_{ij} dy^i \odot dx^j + \frac{\partial^2 \Theta}{\partial y^i \partial y^j} dx^i \odot dx^j, \\
N &= \sum_i dx^i \otimes \frac{\partial}{\partial y^i}, \quad \text{where} \quad \omega_{ij} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.
\end{aligned}
$$

Conversely (g, N) is null–Kähler for any function $\Theta = \Theta(x^i, y^i)$.

- **Theorem B** (MD 2020) Let $(\mathcal{X}, g, N)$ be a $4n$–dimensional null–Kähler manifold. There exist a local coordinate system $(x^i, y^i), i = 1, \dots, 2n$ and a function $\Theta : \mathcal{X} \to \mathbb{R}$ such that

$$\begin{aligned} g &= \sum_{i,j} \omega_{ij} dy^i \odot dx^j + \frac{\partial^2 \Theta}{\partial y^i \partial y^j} dx^i \odot dx^j, \\ N &= \sum_i dx^i \otimes \frac{\partial}{\partial y^i}, \quad \text{where} \quad \omega_{ij} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}. \end{aligned}$$

  Conversely (g, N) is null–Kähler for any function $\Theta = \Theta(x^i, y^i)$.
- Proof
  - ker$(N) \subset T\mathcal{X}$ is a totally null integrable distribution.
  - $M = \mathcal{X}/\text{ker}(N)$ is a symplectic manifold, with Darboux coordinatex x^i.
  - Frobenius theorem: ker$(N) = \text{span}(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{2n}})$.
  - $\nabla N = 0$ give integrability conditions for the existence of Θ.

Curvature

- One arbitrary function Θ of $4n$ variables. Freedom: $(4n+2)$ functions of $2n$ variables.

Curvature

- One arbitrary function Θ of $4n$ variables. Freedom: $(4n+2)$ functions of $2n$ variables.
- Ricci scalar vanishes. Ricci tensor

$$
\begin{aligned}
r &= 2\sum_{i,j} \frac{\partial^2 f}{\partial y^i \partial y^j} dx^i \odot dx^j, \quad \text{where} \\
f &\equiv \sum_{i,j} \omega^{ij} \frac{\partial^2 \Theta}{\partial y^i \partial x^j} + \sum_{i,j,k,l} \frac{1}{2} \omega^{ik} \omega^{jl} \frac{\partial^2 \Theta}{\partial y^i \partial y^j} \frac{\partial^2 \Theta}{\partial y^k \partial y^l}.
\end{aligned}
$$

Curvature

- One arbitrary function Θ of $4n$ variables. Freedom: $(4n+2)$ functions of $2n$ variables.
- Ricci scalar vanishes. Ricci tensor

$$r = 2\sum_{i,j} \frac{\partial^2 f}{\partial y^i \partial y^j} dx^i \odot dx^j, \quad \text{where}$$

$$f \equiv \sum_{i,j} \omega^{ij} \frac{\partial^2 \Theta}{\partial y^i \partial x^j} + \sum_{i,j,k,l} \frac{1}{2} \omega^{ik} \omega^{jl} \frac{\partial^2 \Theta}{\partial y^i \partial y^j} \frac{\partial^2 \Theta}{\partial y^k \partial y^l}.$$

- Ricci flat NK: non–integrable 2nd order PDE on Θ: Cauchy–Kowalewskaya: 2 functions of $4n-1$ variables. Example

$$\Theta = \frac{c}{\rho^{2n-1}} \quad \text{where} \quad \rho = \sum_{i,j} \omega_{ij} y^i x^j, \quad c = \text{const}.$$

- Complexified hyper–Kähler $(\mathcal{X}_{\mathbb{C}}, I, J, K)$. $\mathcal{X}_{\mathbb{C}} = TM_{\mathbb{C}}$, where $(M_{\mathbb{C}}, \omega)$ complex symplectic mfd of dimension $2n$.

Bridgeland–Strachan arXiv:2006.13059

- Complexified hyper–Kähler $(\mathcal{X}_{\mathbb{C}}, I, J, K)$. $\mathcal{X}_{\mathbb{C}} = TM_{\mathbb{C}}$, where $(M_{\mathbb{C}}, \omega)$ complex symplectic mfd of dimension $2n$.
- Null structure $N = I + iJ$, $\mathcal{X}_{\mathbb{C}} = TM_{\mathbb{C}}, \Omega = \pi^*(\omega)$. Equations on Θ

$$[l_i, l_j] = 0, \quad l_i \equiv \frac{\partial}{\partial y^i} + \lambda\Big(\frac{\partial}{\partial x^i} + \sum_{j,k} \omega^{jk} \frac{\partial^2 \Theta}{\partial y^i \partial y^j} \frac{\partial}{\partial y^k}\Big), \quad i = 1, \dots, 2n.$$

Bridgeland–Strachan arXiv:2006.13059

- Complexified hyper–Kähler $(\mathcal{X}_{\mathbb{C}}, I, J, K)$. $\mathcal{X}_{\mathbb{C}} = TM_{\mathbb{C}}$, where $(M_{\mathbb{C}}, \omega)$ complex symplectic mfd of dimension $2n$.
- Null structure $N = I + iJ$, $\mathcal{X}_{\mathbb{C}} = TM_{\mathbb{C}}, \Omega = \pi^*(\omega)$. Equations on Θ

$$[l_i, l_j] = 0, \quad l_i \equiv \frac{\partial}{\partial y^i} + \lambda\Big(\frac{\partial}{\partial x^i} + \sum_{j,k} \omega^{jk} \frac{\partial^2 \Theta}{\partial y^i \partial y^j} \frac{\partial}{\partial y^k}\Big), \quad i = 1, \dots, 2n.$$

- Additional conditions (aka 'A strong Joyce' structure)
  1. Θ is odd in the variables y^i.
  2. $Z \equiv \sum_i x^i \frac{\partial}{\partial x^i}$ is a homothetic Killing vector field such that

$$\mathcal{L}_Z g = g, \quad \mathcal{L}_Z \Theta = -\Theta.$$

  3. The metric is invariant under the lattice transformations

$$y^i \to y^i + 2\pi\sqrt{-1}, \quad i = 1, \dots, 2n.$$

Bridgeland–Strachan arXiv:2006.13059

- Complexified hyper–Kähler $(\mathcal{X}_\mathbb{C}, I, J, K)$. $\mathcal{X}_\mathbb{C} = TM_\mathbb{C}$, where $(M_\mathbb{C}, \omega)$ complex symplectic mfd of dimension $2n$.
- Null structure $N = I + iJ$, $\mathcal{X}_\mathbb{C} = TM_\mathbb{C}, \Omega = \pi^*(\omega)$. Equations on Θ

$$[l_i, l_j] = 0, \quad l_i \equiv \frac{\partial}{\partial y^i} + \lambda\Big(\frac{\partial}{\partial x^i} + \sum_{j,k} \omega^{jk} \frac{\partial^2 \Theta}{\partial y^i \partial y^j} \frac{\partial}{\partial y^k}\Big), \quad i = 1, \dots, 2n.$$

- Additional conditions (aka 'A strong Joyce' structure)
  1. Θ is odd in the variables y^i.
  2. $Z \equiv \sum_i x^i \frac{\partial}{\partial x^i}$ is a homothetic Killing vector field such that

$$\mathcal{L}_Z g = g, \quad \mathcal{L}_Z \Theta = -\Theta.$$

  3. The metric is invariant under the lattice transformations

$$y^i \to y^i + 2\pi\sqrt{-1}, \quad i = 1, \dots, 2n.$$

- Tom Bridgeland+MD (in progress). Lots of hyper-Lagrangian examples: $\mathcal{X}_\mathbb{C}$ is foliated by $2n$ dimensional manifolds which are Lagrangian w.r.t. I, J, K.

Four Dimensions

- $* : \Lambda^2 \to \Lambda^2$. Chose an orientation s.t. $*\Omega = \Omega$.

Four Dimensions

- $* : \Lambda^2 \to \Lambda^2$. Chose an orientation s.t. $*\Omega = \Omega$.
- Three 'remarkable' isomorphisms
  1. $\Lambda^2(T^*\mathcal{X}) = \Lambda^2_+(T^*\mathcal{X}) \oplus \Lambda^2_-(T^*\mathcal{X})$.
  2. $T\mathcal{X} \cong \mathbb{S}_+ \otimes \mathbb{S}_-$
  3. $\Lambda^2_+ \cong \text{Sym}^2(\mathbb{S}_+{}^*)$

Four Dimensions

- $* : \Lambda^2 \to \Lambda^2$. Chose an orientation s.t. $*\Omega = \Omega$.
- Three 'remarkable' isomorphisms
  1. $\Lambda^2(T^*\mathcal{X}) = \Lambda^2_+(T^*\mathcal{X}) \oplus \Lambda^2_-(T^*\mathcal{X})$.
  2. $T\mathcal{X} \cong \mathbb{S}_+ \otimes \mathbb{S}_-$
  3. $\Lambda^2_+ \cong \mathsf{Sym}^2(\mathbb{S}_+{}^*)$
- Null–Kähler $\iff$ $\exists$ parallel section of $\mathbb{S}_+$.

Four Dimensions

- $* : \Lambda^2 \to \Lambda^2$. Chose an orientation s.t. $*\Omega = \Omega$.
- Three 'remarkable' isomorphisms
  1. $\Lambda^2(T^*\mathcal{X}) = \Lambda^2_+(T^*\mathcal{X}) \oplus \Lambda^2_-(T^*\mathcal{X})$.
  2. $T\mathcal{X} \cong \mathbb{S}_+ \otimes \mathbb{S}_-$
  3. $\Lambda^2_+ \cong \mathsf{Sym}^2({\mathbb{S}_+}^*)$
- Null–Kähler $\iff$ $\exists$ parallel section of $\mathbb{S}_+$.
- **Theorem C** (Bridgeland + MD 2021) If $\mathcal{X}_\mathbb{C}$ is complex HK, and foliaded by hyper-Lagrangian surfaces, then
  1. Θ is at most quadratic in one of x^1 or x^2, and the heavenly equation linearise.
  2. $\mathcal{X}_\mathbb{C}$ admits a two–paramter family of β–surfaces.

Cohomogeneity one

- $\mathcal{X} = \mathbb{R} \times SL(2,\mathbb{R})$, or $\mathcal{X}_{\mathbb{C}} = \mathbb{C} \times SL(2,\mathbb{C})$, and $SL(2)$ acts isometrically and preserves N.

Cohomogeneity one

- $\mathcal{X} = \mathbb{R} \times SL(2, \mathbb{R})$, or $\mathcal{X}_{\mathbb{C}} = \mathbb{C} \times SL(2, \mathbb{C})$, and $SL(2)$ acts isometrically and preserves N.
- Left–invariant one–forms $\sigma^1, \sigma^2, \sigma^3$ on $SL(2)$

$$d\sigma^1 = 2\sigma^3 \wedge \sigma^2, \quad d\sigma^2 = \sigma^2 \wedge \sigma^1, \quad d\sigma^3 = \sigma^1 \wedge \sigma^3$$

and $g = \sum_{\alpha,\beta=1}^{3} \gamma_{\alpha\beta}(t)\sigma^\alpha \odot \sigma^\beta + 2\sum_{\alpha=1}^{3} n_\alpha(t)\sigma^\alpha \odot dt$.

Cohomogeneity one

- $\mathcal{X} = \mathbb{R} \times SL(2, \mathbb{R})$, or $\mathcal{X}_{\mathbb{C}} = \mathbb{C} \times SL(2, \mathbb{C})$, and $SL(2)$ acts isometrically and preserves N.
- Left–invariant one–forms $\sigma^1, \sigma^2, \sigma^3$ on $SL(2)$

$$d\sigma^1 = 2\sigma^3 \wedge \sigma^2, \quad d\sigma^2 = \sigma^2 \wedge \sigma^1, \quad d\sigma^3 = \sigma^1 \wedge \sigma^3$$

  and $g = \sum_{\alpha,\beta=1}^{3} \gamma_{\alpha\beta}(t)\sigma^\alpha \odot \sigma^\beta + 2\sum_{\alpha=1}^{3} n_\alpha(t)\sigma^\alpha \odot dt$.
- **Theorem D** (MD 2020) $SL(2)$–invariant ASD, NK structure:
  1. Either g conformal to Ricci–flat (all known explicitly: MD+Tod 2017).
  2. Or $\gamma(t), n(t)$ given by solutions to Painlevé I, II, or completely solvable.

Cohomogeneity one

- $\mathcal{X} = \mathbb{R} \times SL(2, \mathbb{R})$, or $\mathcal{X}_{\mathbb{C}} = \mathbb{C} \times SL(2, \mathbb{C})$, and $SL(2)$ acts isometrically and preserves N.
- Left–invariant one–forms $\sigma^1, \sigma^2, \sigma^3$ on $SL(2)$

$$d\sigma^1 = 2\sigma^3 \wedge \sigma^2, \quad d\sigma^2 = \sigma^2 \wedge \sigma^1, \quad d\sigma^3 = \sigma^1 \wedge \sigma^3$$

  and $g = \sum_{\alpha,\beta=1}^3 \gamma_{\alpha\beta}(t)\sigma^\alpha \odot \sigma^\beta + 2\sum_{\alpha=1}^3 n_\alpha(t)\sigma^\alpha \odot dt.$
- **Theorem D** (MD 2020) $SL(2)$–invariant ASD, NK structure:
  1. Either g conformal to Ricci–flat (all known explicitly: MD+Tod 2017).
  2. Or $\gamma(t), n(t)$ given by solutions to Painlevé I, II, or completely solvable.
- Example

$$g = \sigma^1 \odot \Big(\frac{12y^2 + 2t}{z}\sigma^1 + 8\sigma^2 - 6\sigma^3\Big) + \sigma^3 \odot (z\sigma^3 + 2z dt),$$
$$\Omega = 2\sigma^3 \wedge \sigma^1.$$

  ASD Null–Kähler iff $\dot{y} = z, \dot{z} = 6y^2 + t$ (Painlevé I).

- $SL(2)$ action on the twistor space: vector bundle homomorphism

$$\phi : \mathfrak{sl}(2, \mathbb{C}) \times \mathcal{Y} \to T\mathcal{Y}.$$

(Buliding on Hitchin 1995, and Mason & Woodhouse 1993).

- $SL(2)$ action on the twistor space: vector bundle homomorphism

$$\phi : \mathfrak{sl}(2, \mathbb{C}) \times \mathcal{Y} \to T\mathcal{Y}.$$

(Buliding on Hitchin 1995, and Mason & Woodhouse 1993).

- $\Omega \leftrightarrow$ two–parameter family of α–surfaces in $\mathcal{X}_{\mathbb{C}} \leftrightarrow$ hypersurface $\mathcal{N} \subset \mathcal{Y}$. The divisor $\mathcal{N}$ meets each twistor line to order 4.

- $SL(2)$ action on the twistor space: vector bundle homomorphism

$$\phi : \mathfrak{sl}(2, \mathbb{C}) \times \mathcal{Y} \to T\mathcal{Y}.$$

  (Buliding on Hitchin 1995, and Mason & Woodhouse 1993).
- $\Omega \leftrightarrow$ two–parameter family of α–surfaces in $\mathcal{X}_{\mathbb{C}} \leftrightarrow$ hypersurface $\mathcal{N} \subset \mathcal{Y}$. The divisor $\mathcal{N}$ meets each twistor line to order 4.
- $\mathcal{N}$ preserved by the $SL(2)$ action, so $\exists \tau \in \mathfrak{sl}(2)$ s. t. $\phi(\tau) = 0$.

- $SL(2)$ action on the twistor space: vector bundle homomorphism

$$\phi : \mathfrak{sl}(2, \mathbb{C}) \times \mathcal{Y} \to T\mathcal{Y}.$$

(Buliding on Hitchin 1995, and Mason & Woodhouse 1993).

- $\Omega \leftrightarrow$ two–parameter family of α–surfaces in $\mathcal{X}_{\mathbb{C}} \leftrightarrow$ hypersurface $\mathcal{N} \subset \mathcal{Y}$. The divisor $\mathcal{N}$ meets each twistor line to order 4.
- $\mathcal{N}$ preserved by the $SL(2)$ action, so $\exists \tau \in \mathfrak{sl}(2)$ s. t. $\phi(\tau) = 0$.
- The element τ is nilpotent for PI, and semisimple for PII.

- $SL(2)$ action on the twistor space: vector bundle homomorphism

$$\phi : \mathfrak{sl}(2, \mathbb{C}) \times \mathcal{Y} \to T\mathcal{Y}.$$

  (Buliding on Hitchin 1995, and Mason & Woodhouse 1993).
- $\Omega \leftrightarrow$ two–parameter family of α–surfaces in $\mathcal{X}_{\mathbb{C}} \leftrightarrow$ hypersurface $\mathcal{N} \subset \mathcal{Y}$. The divisor $\mathcal{N}$ meets each twistor line to order 4.
- $\mathcal{N}$ preserved by the $SL(2)$ action, so $\exists \tau \in \mathfrak{sl}(2)$ s. t. $\phi(\tau) = 0$.
- The element τ is nilpotent for PI, and semisimple for PII.
- The inverse of ϕ is the $SL(2, \mathbb{C})$ connection with a pole of order 4 on the divisor, underlying the isomonodromy problem for Painlevé I, II.

Happy birthday Roger!