Spinors, twistors and classical geometry

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FOUR DIMENSIONS

- spinor spaces S_+, S_- skew forms ϵ_+, ϵ_-

  $\Rightarrow$ inner product on $V = S_+ \otimes S_-$

- null vectors $\phi \otimes \psi$

- $\Lambda^2 V \cong S_+^2 \oplus S_-^2$ self-dual/anti-self-dual forms

- $S_+^2 \cong \mathrm{End}_0\, S_+ =$ trace zero endomorphisms

- Weyl tensor $S_+^4 \oplus S_-^4$

LINEAR TWISTOR THEORY

- P^3 complex projective space

- lines in $P^3 \sim$ points in $Q^4 \subset P^5$

  $=$ Klein quadric $=$ complexified compactified Minkowski space

- two points null separated if the lines intersect

- points in P^3

- $=$ null planes (α-planes) in Q^4

- planes in P^3 ($\sim$ points in dual projective space)

  $=$ null planes (β-planes) in Q^4

- null geodesic $=$ point contained in a plane

NONLINEAR TWISTOR THEORY

- complex 3-manifold Z

- rational curves $\mathbb{P}^1 \subset Z$

  normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$

- complex 3-manifold Z

- rational curves $\mathsf{P}^1 \subset Z$

  normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$

- $\Rightarrow$ complete 4-dimensional family M^4

  null separation = intersection of curves

  Weyl tensor self-dual

- point $z \in Z \Rightarrow$ null surface (one family)

INTEGRABLE SYSTEMS

- Riemann surface C, genus g, rank 2 vector bundle E over C is *stable* if for each subbundle $L \subset E$, $\deg L < \deg E/2$.

- fix $\Lambda^2 E$, moduli space $3g-3$-dimensional variety

- E defines P^1-bundle $\mathsf{P}(E)$

  (equivalence $E \sim E \otimes L$, L^2 trivial, finite group $\mathbf{Z}_2^{2g}$)

- E stable, cotangent space of $\mathcal{N}$ at $[E] \cong H^0(C, \mathrm{End}_0\, E \otimes K)$

  $\Phi \in H^0(C, \mathrm{End}_0\, E \otimes K)$, $\mathrm{tr}\, \Phi^2 \in H^0(C, K^2)$

- $\dim H^0(C, \mathrm{End}_0\, E \otimes K) = 3g - 3 = \dim H^0(C, K^2)$

- completely integrable system =

  geodesic flows for a $3g-3$-dimensional family of metrics

  ($\mathrm{End}_0\, E$ independent of $E \mapsto E \otimes L$)

- genus $g = 2$

- $\Lambda^2 E$ trivial, moduli space $\mathcal{N} = \mathrm{P}^3$

- $\Lambda^2 E$ odd degree, moduli space = intersection of two quadrics

M.S.Narasimhan & S.Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Annals of Maths. **89** 19-51 (1969)

P.Newstead, *Stable bundles of rank 2 and odd degree over a curve of genus 2*, Topology **7** 205-215 (1968)

- C genus 2 $\qquad y^2 = \prod_1^6 (x - x_i) = p(x)$

- sections of K^2 $\qquad \dfrac{a + bx + cx^2}{p(x)} dx^2$

B van Geemen and E Previato : *On the Hitchin system*, Duke Math. J. **85** (1996) 659–683.

K Gawędzki and P Tran-Ngoc-Bich: *Self-duality of the* SL_2 *Hitchin integrable system at genus* 2, Comm. Math. Phys. **196** (1998) 641–670.

$$T^*\mathsf{P}^3 = \{(p,q) : \langle p,q\rangle = 0, p \neq 0\}/\mathbf{C}^*$$

- **integrable system**

$$H(p,q) = -\frac{1}{128\pi^2}\sum_{i\neq j}\frac{\langle\sigma(ij)p,q\rangle^2}{(x-x_i)(x-x_j)}dx^2$$

- action of $H^1(\Sigma, \mathbf{Z}/2)$ on $\mathcal{M}$ by $E \mapsto L \otimes E$

- (projective) action σ on $H^0(J^1, 2\Theta)$

- $\sigma(ij) = \sigma([x_i] - [x_j])$

INTERSECTION OF TWO QUADRICS

CLASSICAL GEOMETRY

Julius Plücker 1801–1868

Otto Hesse 1811 — 1874

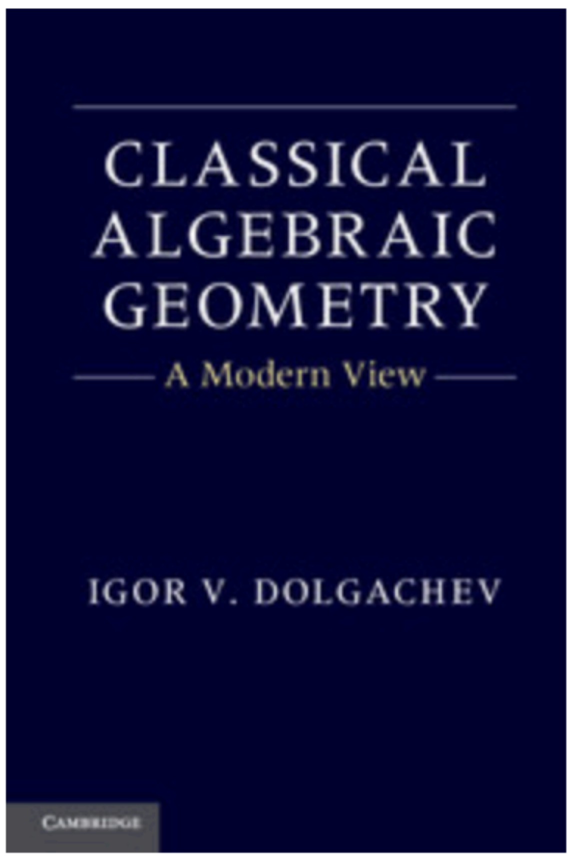
George Salmon 1819–1904

Arthur Cayley 1821 – 1895

Arthur Coble 1878–1966

Classical Algebraic Geometry

A Modern View



Igor V. Dolgachev, *University of Michigan, Ann Arbor*

Publisher:	Cambridge University Press
Online publication date:	September 2012
Print publication year:	2012
Online ISBN:	9781139084437
DOI:	https://doi.org/10.1017/CBO9781139084437

Subjects:	Mathematics, Geometry and Topology

- quadrics $Q_1, Q_2 \subset \mathsf{P}^5$

  $\sim$ quadratic forms q_1, q_2 on $\mathbf{C}^6$

- pencil – 1-parameter family $z_1 q_1 + z_2 q_2, [z_1, z_2] \in \mathsf{P}^1$

- singular quadrics: $\det(z_1 q_1 + z_2 q_2) = 0$: six points $x_i \in \mathsf{P}^1$

- $x \in Q_1 \cap Q_2$: point in each quadric of the pencil

- each point $z \in \mathsf{P}^1$, $z \neq x_i$ defines a quadric

  ... and a twistor space P^3 of α-planes

  .... α or β?

- well-defined if we take a double cover C of P^1 branched over $x_1, \ldots, x_6$ = curve of genus 2

- planes in singular quadric $\sim \mathsf{P}^3$

  (spin representations of $Spin(6)$: V^+, V^-

  $V^+ \cong V^-$ restricted to $Spin(5)$)

- $M^4 = \mathrm{P}^3$-bundle over C

- $x \in Q_1 \cap Q_2$ = point in each quadric of the pencil

  = line in each twistor space

  = P^1-bundle over C, contained in M^4

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- $x \in Q_1 \cap Q_2 =$ point in each quadric of the pencil

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  $= \mathrm{P}(E)$

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The Non-Linear Graviton

by Roger Penrose

Summary

A new approach to quantized gravitational theory is suggested. It is argued by analogy with Maxwell theory - and also from a principle that (physical) gravitons should carry space-time curvature - that a free graviton should be describable by a complex solution of Einstein's vacuum equations. For a left-handed

Ann. Global Anal. Geom.
Vol. 3, No. 2 (1985), 185-195

THE INTERSECTION OF TWO QUADRICS IN $\mathbb{P}_5(\mathbb{C})$ AS A TWISTOR SPACE

Jacques Hurtubise

A parametrization is constructed for the space Y^c of conics in the intersection of two quadrics in $\mathbb{P}_5(\mathbb{C})$ and the study is made of the conformal structure of Y^c.

Introduction

In [3], Hitchin, classifying Kählerian twistor spaces, reduces the possibilities to four: $\mathbb{P}_3(\mathbb{C})$, $F_3(\mathbb{C})$ (the flags in $\mathbb{C}^3$), the intersection of two quadrics in $\mathbb{P}_5(\mathbb{C})$ and the double covering of $\mathbb{P}_3(\mathbb{C})$ branched over a non singular quartic surface. In the last two cases, however, there metric

$$
\begin{pmatrix}
\begin{matrix} -\nu_1 q_{23}(t) \\ -\nu_3 q_{12}(t)\tau_2^2 \\ -\nu_2 q_{13}(t)\tau_3^2 \end{matrix} & \cdot & \cdot & \cdot \\
\begin{matrix} \nu_3 q_{12}(t)\tau_1\tau_2 \\ + \begin{bmatrix} -(\mu_2 + 2\rho_2 t)q_{13}(t) \\ +(\mu_1 + 2\rho_1 t)q_{23}(t) \end{bmatrix}\tau_3 \end{matrix} & \begin{matrix} -\nu_2 q_{13}(t) \\ -\nu_3 q_{12}(t)\tau_1^2 \\ -\nu_1 q_{23}(t)\tau_3^2 \end{matrix} & \cdot & \cdot \\
\begin{matrix} \nu_2 q_{13}(t)\tau_1\tau_3 \\ + \begin{bmatrix} -(\mu_1 + 2\rho_1 t)q_{23}(t) \\ +(\mu_3 + 2\rho_3 t)q_{12}(t) \end{bmatrix}\tau_2 \end{matrix} & \begin{matrix} \nu_1 q_{23}(t)\tau_2\tau_3 \\ + \begin{bmatrix} -(\mu_3 + 2\rho_3 t)q_{12}(t) \\ +(\mu_2 + 2\rho_2 t)q_{13}(t) \end{bmatrix}\tau_1 \end{matrix} & \begin{matrix} -\nu_3 q_{12}(t) \\ -\nu_2 q_{13}(t)\tau_1^2 \\ -\nu_1 q_{23}(t)\tau_2^2 \end{matrix} & \cdot \\
0 & 0 & 0 & \frac{P(\tau)}{16}
\end{pmatrix}
$$

curvature

$R_{ijkl} = 0$ unless i,j,k,l all different or pairwise equal,

$$R_{abab} = \frac{1}{\nu_1\nu_2\nu_3}\left[3\nu_c^2 - (\mu_a - \mu_b)^2 + 4(\mu_a + \mu_c)(\mu_b + \mu_c) \right]$$

$$R_{abc4} = \frac{1}{\nu_1\nu_2\nu_3}\left[\begin{matrix} 4\mu_c^2 - 8\rho_c + 4\mu_a\mu_b \\ -2\mu_a^2 + 4\rho_a - 2\mu_b\mu_c - 2\mu_b^2 + 4\rho_b - 2\mu_a\mu_c \end{matrix} \right]$$

$$R_{a4a4} = \frac{1}{\nu_1\nu_2\nu_3}\left[12(\rho_b + \rho_c) + 18\mu_b\mu_c - 4(\mu_b + \mu_c)^2 \right]$$

- twistor space = intersection of quadrics $Q_1 \cap Q_2$ in P^5

  twistor lines = conics in $Q_1 \cap Q_2$

  conic = intersection of a plane and a quadric

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- pencil of quadrics Q_z, $z \in \mathsf{P}^1$

  take an α-plane in Q_a for some a, intersect with Q_1

  $Q_1 \cap Q_2 = Q_1 \cap Q_a \Rightarrow$ conic in $Q_1 \cap Q_2$

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- space-time = space of conics = pairs $a \in C$, α-plane in Q_a

  = P^3 bundle over $C = M^4$

- a point in twistor space $\Rightarrow$ null surface in complex space-time

- a point $x \in Q_1 \cap Q_2 \Rightarrow$ a null surface in M^4

  ... but x defines the ruled surface $\mathsf{P}(E) \to C$ in M^4

- **Prop**: These surfaces are null

- conformal structure is defined on an open set in M^4

  – where the intersection of a plane and a quadric is a non-singular conic

- complement is where the intersection is a pair of lines

  = singular quartic surface in each P^3 fibre ...

- ... Kummer surface, covering = space of lines in $Q_1 \cap Q_2$

- there are 4 lines through a generic point x in $Q_1 \cap Q_2$
- a line $\ell \subset Q_a$ = null geodesic $\sim$ point in a plane in P^3_a

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- there are 4 lines through a generic point x in $Q_1 \cap Q_2$

- a line $\ell \subset Q_a$ = null geodesic $\sim$ point in a plane in P_a^3

- $x \in Q_1 \cap Q_2, x \in \ell \sim$ point in a line in a plane in P_a^3

- $x \in Q_1 \cap Q_2 \Rightarrow \mathrm{P}(E) \subset M^4$

  a point in a line $\Rightarrow$ section $s : C \to \mathrm{P}(E) \sim$ line bundle in E.

  $\Rightarrow$ 4 distinguished subbundles in each stable bundle E.

COMPLEX FIBRE BUNDLES AND RULED SURFACES

By M. F. ATIYAH

[Received 5 August 1954.—Read 25 November 1954]

Introduction

ALTHOUGH much work has been done in the topological theory of fibre bundles, very little appears to be known on the complex analytic side. In this paper we propose to study certain types of complex fibre bundle, showing how they can be classified. The methods we shall employ will be

- represent $\mathsf{P}(E)$ by an extension $\mathcal{O} \to E \to L$, $\deg L = 1$

  extension class in $H^1(C, L^{-1})$

- $\dim H^1(C, L^{-1}) = 2 = \dim H^0(C, LK)$ (Serre dual)

  projective line $\mathsf{P}(H^1(C, L^{-1}))$ isomorphic to dual $\mathsf{P}(H^0(C, LK))$

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  $\Rightarrow$ extension class $=$ annihilator of s

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- 4 subbundles of E: $\tau : C \to C$ involution

  $(p, q, r) \sim (\tau(p), \tau(q), r) \sim (p, \tau(q), \tau(r)) \sim (\tau(p), q, \tau(r))$

- $\pi : C \to \mathsf{P}^1$ unordered triple $(\pi(p), \pi(q), \pi(r)) \in \mathsf{P}^1$

  $\Rightarrow$ symmetric product $S^3\mathsf{P}^1 = \mathsf{P}^3$

- $Q_1 \cap Q_2$ modulo $H^1(C, \mathbf{Z}_2)$ = double covering of P^3

  branched over six planes (x_i, q, r)

- $\mathsf{P}^3 \sim$ space of cubic polynomials

NETS OF QUADRICS

- genus 2, fix E, $\operatorname{tr}\Phi^2 \in H^0(C, K^2)$:
- 3 quadratic forms on the 3-dimensional space $H^0(C, \operatorname{End}_0 E \otimes K)$
- $=$ net of conics

Problem: Given E

1. Find all $\Phi \in H^0(C, \mathrm{End}_o\, E \otimes K)$
2. Calculate $\mathrm{tr}\, \Phi^2$
3. Classify family of quadrics

- quadratic forms q_1, q_2, q_3 on $\mathbf{C}^3 = H^0(C, \mathrm{End}_0\, E \otimes K)$

- discriminant $\det(z_1 q_1 + z_2 q_2 + z_3 q_3) = 0$

  cubic curve in P^2

- invariant = modulus of elliptic curve

- genus 2, $\mathcal{O} \to E \to L$

  3-dimensional space $H^0(C, \mathrm{End}_0\, E \otimes K)$

- 2-dimensional space preserving subbundle $\cong H^0(C, K)$

- 1-dimensional space

  $\Phi : \mathcal{O} \to (E/\mathcal{O}) \otimes K = LK$ defined by divisor $p + q + r$

  $\Rightarrow$ basis for $H^0(C, \mathrm{End}_0\, E \otimes K)$

- Atiyah's description of moduli space : cover of $S^3\mathrm{P}^1 = \mathrm{P}^3$

- generic (p,q,r): $\{x^2, y^2, z^2\}$

net of conics

big diagonal (p,p,q): $\{x^2, y^2, yz\}$

small diagonal (p,p,p): $\{x^2, 2xy, y^2 + 2xz\}$

branch locus (x_i, q, r): $\{y^2 + xz, z^2, 0\}$

- discriminant cubic = lines

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- genus 3?

GENUS 3

- C curve of genus 3, quartic in P^2

  moduli space of rank 2 stable bundles, $\Lambda^2 E$ trivial

  $=$ quartic hypersurface Q in P^7, the Coble quartic

  M.S.Narasimhan & S.Ramanan, 2Θ *linear systems on abelian varieties*, Vector bundles and algebraic varieties (Bombay, 1984), 415- 427, Oxford University Press, (1987)

- $E = L \oplus L^* \Rightarrow \mathrm{Jac}(C)/\mathbf{Z}_2 =$ Kummer variety $\subset Q$

- $E \mapsto E \otimes L$, $L \in H^1(C, \mathbf{Z}_2)$, Q invariant under $\mathbf{Z}_2^6$-action

- C genus 3, non hyperelliptic $\Rightarrow$ quartic in P^2

- sections of K = restriction of sections of $\mathcal{O}(1)$ on P^2

  = linear forms on $\mathbf{C}^3$ $a_1 z_1 + a_2 z_2 + a_3 z_3$

- sections of K^2 = quadratic forms $\displaystyle\sum_{i,j=1}^{3} a_{ij} z_i z_j$

- $\mathrm{tr}\,\Phi^2 : H^0(C, \mathrm{End}_0\, E \otimes K) \to H^0(C, K^2)$ .... six quadrics in P^5

- C.Pauly, *Self-duality of Coble's quartic hypersurface and applications*, Michigan Math. J. **50** (2002) 551-574.

  dual variety is also a Coble quartic: moduli space of bundles F with $\Lambda^2 F \cong K$

- associate to very stable E a bundle F

  with $\dim H^0(C, E \otimes F) = 4$

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  $\Rightarrow$ inner product on $V = S_+ \otimes S_-$

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- $\Lambda^2 H^0(C, E \otimes F) \to H^0(C, \Lambda^2(E \otimes F)) \to H^0(C, S^2 E \otimes K)$

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- $\Lambda^2 H^0(C, E \otimes F) \to H^0(C, \Lambda^2(E \otimes F)) \to H^0(C, S^2 E \otimes K)$

  $S^2 E \cong \mathrm{End}_0\, E$

- $\Phi \in H^0(C, \mathrm{End}_0\, E \otimes K)$ is the self-dual part of

  $\alpha \in \Lambda^2 H^0(C, E \otimes F)$

- $H^0(C, E \otimes F)$, $\Lambda^2 E \cong \mathcal{O}, \Lambda^2 F \cong K \Rightarrow$ skew forms

  inner product on $H^0(C, E \otimes F) \cong \mathbf{C}^4$ with values in $H^0(C, K)$

- $\Phi \in \Lambda^2 H^0(C, E \otimes F)$

  $\operatorname{tr} \Phi^2 =$ induced inner product on self-dual component

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- basis $v_1, \dots v_4$ of $H^0(C, E \otimes F)$, $(v_i, v_j) \in H^0(C, K)$

- $((v_1 \wedge v_2)_+, (v_3 \wedge v_4)_+) =$

  $$\frac{1}{2}((v_1, v_3)(v_2, v_4) - (v_1, v_4)(v_2, v_3) + \sqrt{\det(v_i, v_j)})$$

$$((v_1 \wedge v_2)_+, (v_3 \wedge v_4)_+) =$$

$$\frac{1}{2}((v_1, v_3)(v_2, v_4) - (v_1, v_4)(v_2, v_3) + \sqrt{\det(v_i, v_j)})$$

?

square root of a quartic polynomial?

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$$\frac{1}{2}((v_1, v_3)(v_2, v_4) - (v_1, v_4)(v_2, v_3) + \sqrt{\det(v_i, v_j)})$$

?

square root of a quartic polynomial?

- Answer: $\det(v_i, v_j) = p^2$ modulo the quartic equation of C
- $\det(v_i, v_j) = 0$ quartic curve X, $\det(v_i, v_j) - p^2 = 0$ curve C

  $\Rightarrow X$ meets C tangentially in 8 points

- moduli space $\mathcal{N}/H^1(C, \mathbf{Z}_2) \cong$ 72-fold cover of space of tangential quartics (C.Pauly)

- $H^0(C, E \otimes F)$ quadratic form values in $H^0(C, K) \cong \mathbf{C}^3$ discriminant $\det Q = \det(v_i, v_j) = 0 =$ quartic curve $X \subset \mathsf{P}^2$

- moduli space $\mathcal{N}/H^1(C,\mathbf{Z}_2) \cong$ 72-fold cover

  of space of tangential quartics (C.Pauly)

- $H^0(C, E\otimes F)$ quadratic form values in $H^0(C,K) \cong \mathbf{C}^3$

  discriminant $\det Q = \det(v_i, v_j) = 0 =$ quartic curve $X \subset \mathsf{P}^2$

- e.g. $X =$ four lines $\Rightarrow$ four bitangents to C

  for this E, family of quadrics:

  $x_i^2 - a_i(x_1y_1 + x_2y_2 + x_3y_3), \quad y_i^2 - b_i(x_1y_1 + x_2y_2 + x_3y_3)$

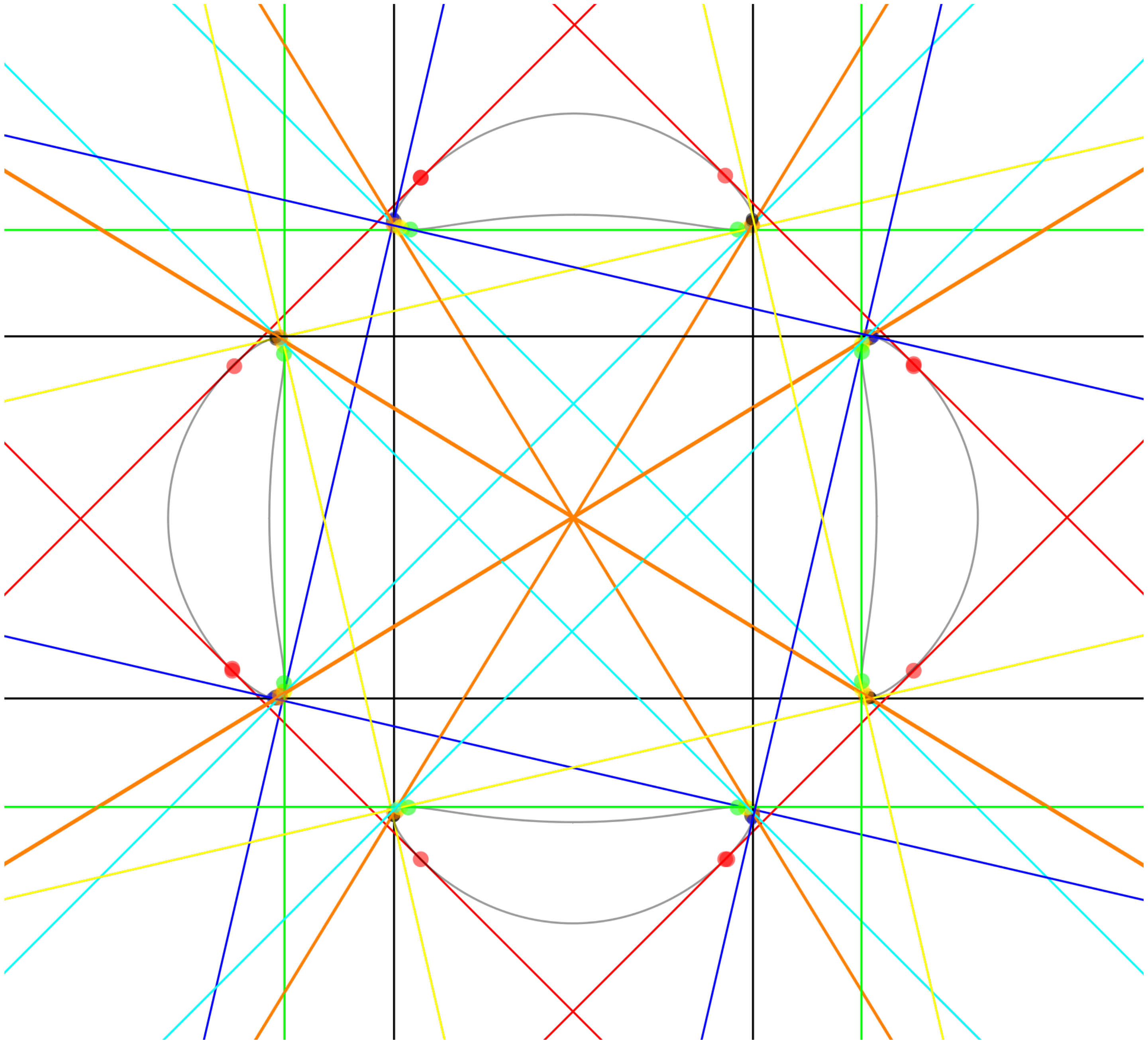
- four linear forms $\ell_1, \ell_2, \ell_3, \ell_4$

  quartic curve C: $\ell_1\ell_2\ell_3\ell_4 + p^2 = 0$

- syzygetic tetrad of bitangents

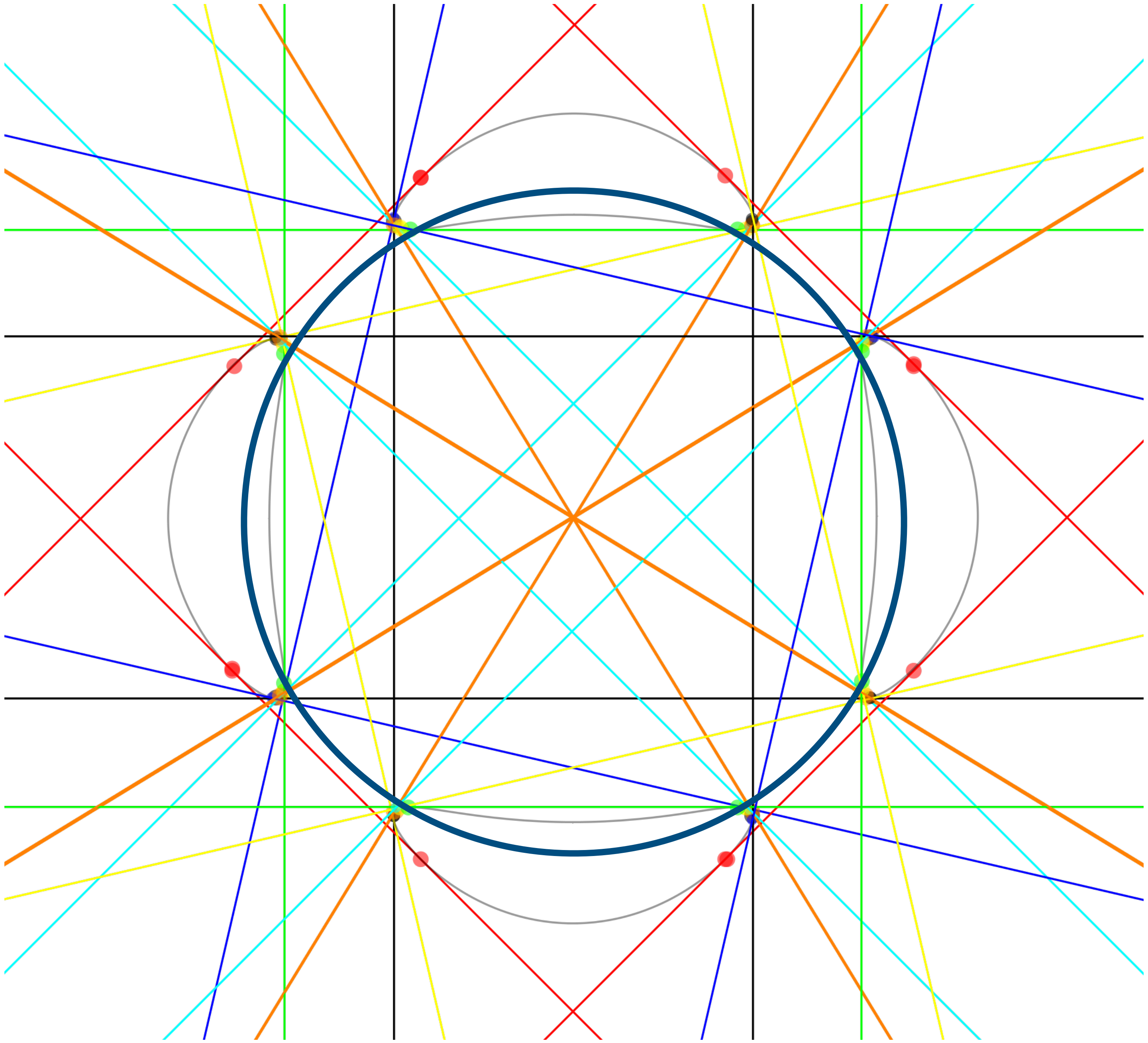
  ($4 \times 2 = 8$ points on a conic $p = 0$)

- **Thm** (Plücker 1839) Each quartic can be written in the form $\ell_1\ell_2\ell_3\ell_4 + p^2 = 0$ and in 315 ways.



28 bitangents

$$144(x^4 + y^4) - 225(x^2 + y^2) + 350x^2y^2 + 81 = 0$$



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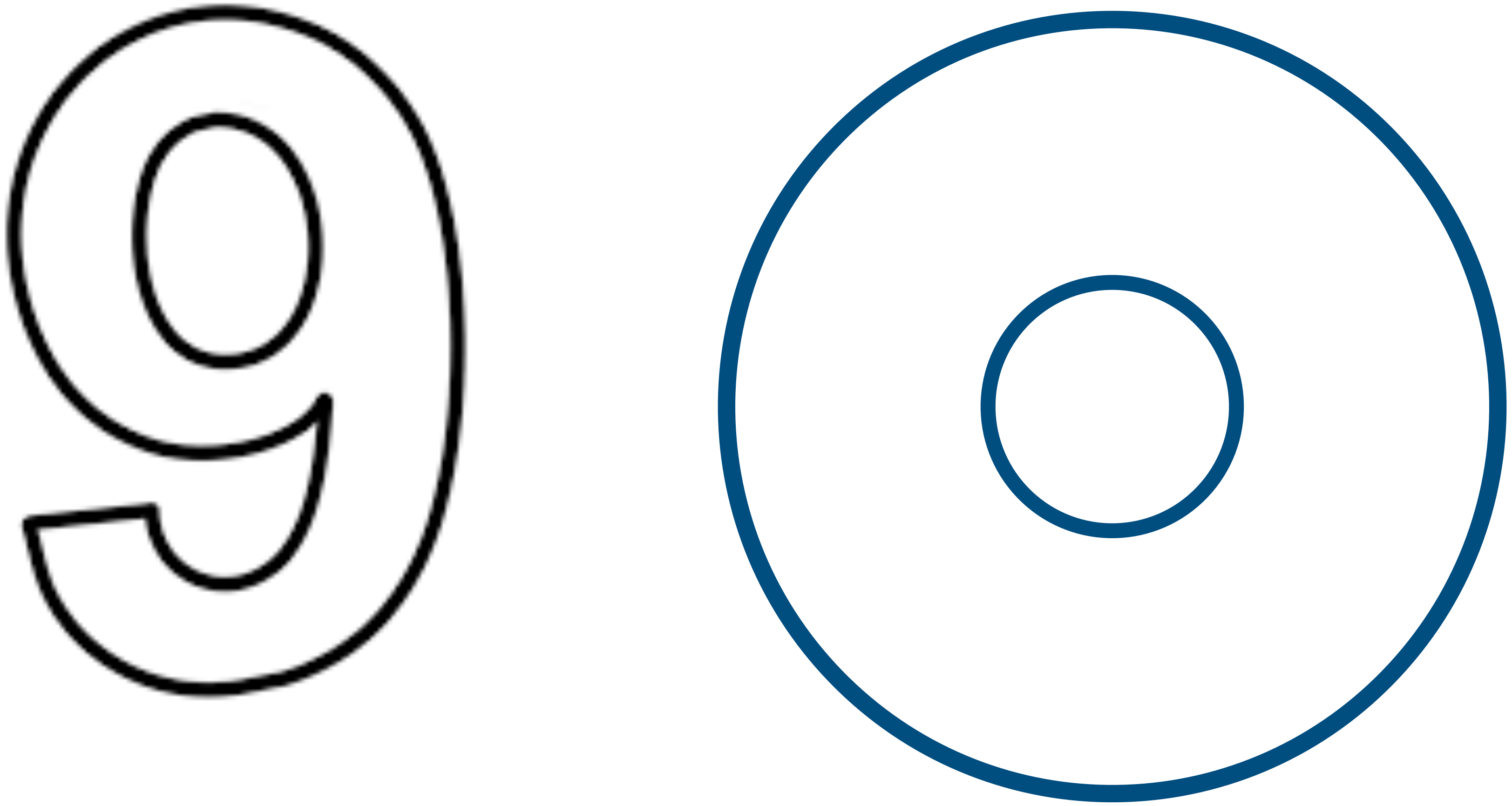

Happy birthday, Roger!