From non-linear gravitons to gravitational scattering

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Roger Penrose @ 90

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work with L. Mason & A. Sharma [2003.13501, 2103.16984, 2108.xxxxx]
also work with A. Ilderton [2005.05807]

Motivation

Non-linear graviton gives correspondence

self-dual 4-manifolds $\leftrightarrow$ twistor spaces

[Penrose, Ward, ...]

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self-dual 4-manifolds $\leftrightarrow$ twistor spaces

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This is a *non-linear, chiral* construction.

What can it tell us about perturbative gravity?

Scattering amplitudes $\rightarrow$ probes of full non-linear dynamics

Gravitational scattering amplitudes

Treat GR (possibly coupled to matter) as a perturbative QFT

- Tree-level amplitudes $\leftrightarrow$ multi-linear piece of action, evaluated on recursively constructed solutions
- UV-divergent beyond tree-level, but still a low-energy EFT for gravity [DeWitt, Donoghue, ...]
- Encode non-linear dynamics and observables (scattering angle, radiation reaction, memory effects,...)

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Real physics – these things are being measured!

cf., early inspiral BH mergers from gravitational amplitudes

[Bern et al., Bjerrum-Bohr et al., Cheung et al., ...]

State of play

Pert. expansion of EH action is a mess

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- Minkowski: Everything! Full tree-level S-matrix [Hodges, Cachazo-Geyer, Cachazo-Skinner, Cachazo-He-Yuan]

State of play

Pert. expansion of EH action is a mess

...what do we know (at tree-level)?

- Minkowski: Everything! Full tree-level S-matrix [Hodges, Cachazo-Geyer, Cachazo-Skinner, Cachazo-He-Yuan]
- Curved space-time: Almost nothing! State-of-art is 4-points; some special exceptions in AdS [Gonclaves-Pereira-Zhou, Green-Wen]

Some natural questions

Where do remarkable formulae in Minkowski space come from?

Do they generalise to curved space-times?

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Today: provide some answers using twistor theory!

Graviton scattering in 4d flat space

External states: *gravitons* = linear solutions on $\mathbb{M}$

labelled by:

- massless 4-momentum $k^{\alpha\dot{\alpha}} = \kappa^{\alpha}\,\tilde{\kappa}^{\dot{\alpha}}$
- helicity $= \pm 2 \leftrightarrow$ (anti-)self-duality of linear field

Spinor-helicity notation: $\langle i\, j\rangle := \kappa_i^{\alpha}\,\kappa_{j\,\alpha}$, $[i\, j] := \tilde{\kappa}_i^{\dot{\alpha}}\,\tilde{\kappa}_{j\,\dot{\alpha}}$

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Tree amplitudes arranged by *helicity configuration*

Integrability of SD sector $\Rightarrow$

$$\mathcal{M}_n^{(0)}(1^+,\ldots,n^+) = 0 = \mathcal{M}_n^{(0)}(1^-,2^+,\ldots,n^+)$$

MHV Scattering

Non-trivial *maximal helicity violating* amplitudes

$$\mathcal{M}_n^{(0)}(1^-, 2^-, 3^+, \ldots, n^+) := \mathcal{M}_{n,1}$$

Many formulae deduced through unitarity methods

[Berends-Giele-Kuijf, Bern-Dixon-Perelstein-Rozowsky, Nguyen-Spradlin-Volovich-Wen]

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Optimal formula by Hodges:

$$\mathcal{M}_{n,1} = \delta^4\left(\sum_{i=1}^{n} k_i\right) \frac{\langle 1\,2\rangle^6}{\langle 1\,j\rangle^2 \langle 2\,j\rangle^2} \left|\mathbb{H}_{12j}^{12j}\right| \, ,$$

$$\mathbb{H}_{ij} = \frac{[i\,j]}{\langle i\,j\rangle}\, , \qquad \mathbb{H}_{ii} = -\sum_{j\neq i} \frac{[i\,j]}{\langle i\,j\rangle} \frac{\langle 1\,j\rangle \langle 2\,j\rangle}{\langle 1\,i\rangle \langle 2\,i\rangle}$$

Hodges' formula is beautiful: compact & manifestly perm. symmetric

...but where did it come from?

Can prove that it is correct, and generalizes to full tree-level S-matrix [Cachazo-Skinner] , yet

- no hint of this structure in GR perturbation theory
- twistor string theory computes formula [Skinner] , but does not clarify origins in classical GR

Idea: look for a geometric formulation of MHV scattering

MHV generating functional

$+$ helicity gravitons $\rightsquigarrow$ SD background $\mathscr{M}$

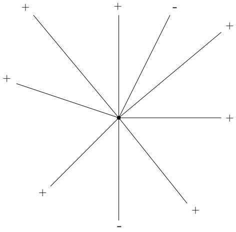
$-$ helicity gravitons $\rightsquigarrow$ ASD perturbations on $\mathscr{M}$

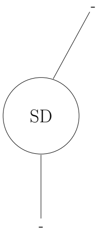
Generating functional [Mason-Skinner]

$$\mathcal{G}(1,2) = \frac{1}{\kappa^2} \int_{\mathscr{M}} \Sigma^{\alpha\beta} \wedge \gamma_{1\,\alpha}{}^{\gamma} \wedge \gamma_{2\,\beta\gamma}$$

- $\Sigma^{\alpha\beta}$ basis of ASD 2-forms on $\mathscr{M}$, $\Sigma^{(\alpha\beta} \wedge \Sigma^{\gamma\delta)} = 0 = \mathrm{d}\Sigma^{\alpha\beta}$
- $\gamma_{i\,\alpha\beta}$ linear perturbation to the ASD spin connection on $\mathscr{M}$; obeys $\mathrm{d}\gamma_{\alpha\beta} = \psi_{\alpha\beta\gamma\delta}\Sigma^{\gamma\delta}$

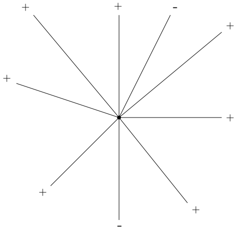
Expand $\mathscr{M}$ in positive helicity gravitons to obtain $\mathcal{M}_{n,1}$

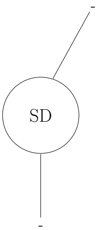




A nice geometric picture for MHV scattering, but some problems:

- not clear how to operationalize pert. expansion
- not manifestly gauge invariant





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Resolution: use ‘heavenly’/hyperkähler description of $\mathscr{M}$

As $\mathrm{d}\Sigma^{\alpha\beta} = 0$, $\exists$ coordinates $(z^{\dot\alpha}, \tilde{z}^{\dot{\tilde\alpha}})$ s.t.

$$\Sigma^{11} = \mathrm{d}z^{\dot\alpha} \wedge \mathrm{d}z_{\dot\alpha}\,, \quad \Sigma^{22} = \mathrm{d}\tilde{z}^{\dot{\tilde\alpha}} \wedge \mathrm{d}\tilde{z}_{\dot{\tilde\alpha}}\,, \quad \Sigma^{12} = \Omega_{\dot\alpha\dot{\tilde\beta}}\,\mathrm{d}z^{\dot\alpha} \wedge \mathrm{d}\tilde{z}^{\dot{\tilde\beta}}$$

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$$\Omega_{\dot\alpha\dot{\tilde\beta}} = \frac{\partial^2\Omega}{\partial z^{\dot\alpha}\partial\tilde z^{\dot{\tilde\beta}}}$$

$\Omega(z, \tilde z)$ the Kähler potential/first form:

$$\text{self-duality} \quad \Leftrightarrow \quad \Omega_{\dot\alpha\dot{\tilde\gamma}}\, \Omega^{\dot\alpha}{}_{\dot{\tilde\delta}} = \epsilon_{\dot{\tilde\gamma}\dot{\tilde\delta}}$$

the *first heavenly equation* [Plebanski]

Momentum eigenstates

neg. helicity gravitons $\leftrightarrow$ ASD perts on $\mathscr{M}$

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Choose ASD spin basis κ_1^{α}, κ_2^{α}. Then

$$\gamma_1^{\alpha\beta} = \kappa_1^{\alpha}\,\kappa_1^{\beta}\,\mathrm{e}^{\mathrm{i}\,[z\,1]}\,[b_1\,\mathrm{d}z]\,, \qquad \gamma_2^{\alpha\beta} = \kappa_2^{\alpha}\,\kappa_2^{\beta}\,\mathrm{e}^{\mathrm{i}\,[\tilde{z}\,2]}\,[b_2\,\mathrm{d}\tilde{z}]$$

$b_i^{\dot{\alpha}}$ encode gauge freedom, normalised so that $[b_i\,i] = 2$.

With this choice, MHV generating functional becomes:

$$\mathcal{G}(1,2) = \frac{1}{4\,\kappa^2} \int_{\mathscr{M}} \mathrm{d}^2 z\, \mathrm{d}^2 \tilde{z}\, \Omega_{\dot\alpha \dot{\tilde\beta}}\, b_1^{\dot\alpha}\, b_2^{\dot{\tilde\beta}}\, \mathrm{e}^{\mathrm{i}\,[z\,1] + \mathrm{i}\,[\tilde z\,2]}$$
$$= -\frac{\langle 1\,2\rangle^4}{\kappa^2} \int_{\mathscr{M}} \mathrm{d}^2 z\, \mathrm{d}^2 \tilde{z}\, \Omega\, \mathrm{e}^{\mathrm{i}\,[z\,1] + \mathrm{i}\,[\tilde z\,2]}$$

Manifestly gauge-invariant, but how to perturbatively expand?

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Manifestly gauge-invariant, but how to perturbatively expand?

Twistor theory to the rescue!

Twistor theory recap

What is a twistor space, $\mathbb{P}\mathscr{T}$?

A 3d complex manifold, which:

- is a $\mathbb{C}$-deformation of an open neighbourhood in $\mathbb{CP}^3$
- contains a 4-parameter family of holomorphic rational curves $X \cong \mathbb{CP}^1$, with $N_X \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$

Twistor theory recap

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Ex. Minkowski space: $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ homog. coords on $\mathbb{CP}^3$

$$\mathbb{PT} = \left\{ Z \in \mathbb{CP}^3 \,|\, \lambda_\alpha \neq 0 \right\} \,, \text{ twistor lines } \mu^{\dot\alpha} = \mathrm{i}\, x^{\alpha\dot\alpha} \lambda_\alpha$$

Non-linear graviton construction

Theorem (Penrose)

There is a 1:1 correspondence between:

- *Vacuum, half-flat SD 4-manifolds* $\mathscr{M}$, *and*
- $\mathbb{P}\mathscr{T}$ *which admit a holomorphic fibration* $\mathbb{P}\mathscr{T} \to \mathbb{CP}^1$ *with degenerate, holomorphic weighted Poisson structure on the fibres*

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In real money: points $x \in \mathscr{M} \leftrightarrow X \subset \mathbb{P}\mathscr{T}$ holomorphic wrt

$$\bar{\nabla} = \bar{\partial} + V\,, \qquad V \in \Omega^{0,1}(\mathbb{PT}, T)\,, \qquad \text{obeying } \bar{\nabla}^2 = 0$$

Let λ_α be coordinates on base of $\mathbb{P}\mathscr{T} \to \mathbb{CP}^1$,
$\mu^{\dot\alpha}$ coordinates on fibres s.t.

$$\{\cdot, \cdot\} = \frac{\partial}{\partial \mu^{\dot\alpha}} \wedge \frac{\partial}{\partial \mu_{\dot\alpha}}$$

is weighted Poisson structure.

Holomorphicity of λ_α + Poisson structure $\Rightarrow$

$$V = \frac{\partial h}{\partial \mu^{\dot\alpha}} \frac{\partial}{\partial \mu_{\dot\alpha}}\,, \qquad h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$$

Integrability condition now

$$\bar\nabla^2 = 0 \qquad \Leftrightarrow \qquad \bar\partial h + \frac{1}{2}\,\{h,\ h\} = 0$$

Twistor curves

Points $x \in \mathscr{M} \leftrightarrow$ holomorphic (wrt $\bar{\nabla}$) sections

$$(\mu^{\dot{\alpha}} = F^{\dot{\alpha}}(x, \lambda),\ \lambda_{\alpha}) : \mathbb{CP}^1 \to \mathbb{P}\mathscr{T}$$

Holomorphicity $\Rightarrow$

$$\bar{\partial}|_X F^{\dot{\alpha}}(x, \lambda) = \left.\frac{\partial h}{\partial \mu_{\dot{\alpha}}}\right|_X$$

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Look for solutions compatible w/ coordinates $(z^{\dot{\alpha}}, \tilde{z}^{\dot{\tilde{\alpha}}})$ on $\mathscr{M}$:

$$F^{\dot{\alpha}}(x, \lambda) = \frac{\langle \lambda\, 2\rangle}{\langle 1\, 2\rangle}\, z^{\dot{\alpha}} + \frac{\langle \lambda\, 1\rangle}{\langle 2\, 1\rangle}\, \tilde{z}^{\dot{\tilde{\alpha}}} + m^{\dot{\alpha}}(x, \lambda)$$

where $m^{\dot{\alpha}}$ has zeros at $\lambda_{\alpha} = \kappa_{1\,\alpha},\ \kappa_{2\,\alpha}$

Twistor sigma model

There is an action principle for these holomorphic curves

Sigma model governing holomorphic maps $\mathbb{CP}^1 \to \mathbb{P}\mathscr{T}$ w/ appropriate boundary conditions

$$S[m] = \int_X \frac{\mathrm{D}\lambda}{\langle 1\,\lambda\rangle^2\,\langle 2\,\lambda\rangle^2}\,\left([m\,\bar{\partial}|_X\,m] + 2\,h|_X\right)$$

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So what?

Back to amplitudes...

Lemma (TA-Mason-Sharma)

The first form Ω *on* $\mathscr{M}$ *is given by*

$$\Omega(z,\tilde{z}) = [z\,\tilde{z}] - \frac{1}{4\,\pi\,\mathrm{i}}\, S[m]|_{\textit{on-shell}}$$

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Upshot: MHV generating functional now

$$\mathcal{G}(1,2) = \frac{\langle 1\,2\rangle^4}{4\pi\mathrm{i}\,\kappa^2} \int_{\mathscr{M}} \mathrm{d}^2 z\,\mathrm{d}^2\tilde{z}\,\mathrm{e}^{\mathrm{i}\,[z\,1]+\mathrm{i}\,[\tilde{z}\,2]}\, S[m]|_{\text{on-shell}}$$

Recap

Let's recall what we've got so far:

- MHV generating functional $= (--)$ 2-point function on SD background $\mathscr{M}$
- Gauge-invariant form using Kähler potential Ω
- Ω computed by sigma model for non-linear graviton of $\mathscr{M}$

Recap

Let's recall what we've got so far:

- MHV generating functional = $(--)$ 2-point function on SD background $\mathscr{M}$
- Gauge-invariant form using Kähler potential Ω
- Ω computed by sigma model for non-linear graviton of $\mathscr{M}$

Next: write $\mathscr{M}$ as sum of $+$ helicity gravitons on $\mathbb{M}$

perturbative expansion $\rightarrow$ tree-level correlation function in $\mathbb{CP}^1$ theory

Setting up expansion

Penrose transform $\Rightarrow h = \sum_i \epsilon_i h_i$, for $h_i \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2))$ momentum eigenstates:

$$h_i(Z) = \int_{\mathbb{C}^*} \frac{\mathrm{d}s_i}{s_i^3}\, \bar{\delta}^2(\kappa_i - s_i\,\lambda)\, \mathrm{e}^{\mathrm{i}\, s_i\, [\mu\, i]}$$

Want to compute:

$$\left(\prod_i \frac{\partial}{\partial \epsilon_i}\right) S[m]|_{\text{on-shell}}\Bigg|_{\epsilon_i = 0}$$

i.e., extract multi-linear piece of on-shell classical action

This multi-linear piece is computed by correlator

$$\left\langle \prod_i V_i \right\rangle_{\text{conn., tree}}$$

in theory

$$\int_{\mathbb{CP}^1} \frac{\mathrm{D}\lambda}{\langle 1\,\lambda\rangle^2\,\langle 2\,\lambda\rangle^2}\,[m\,\bar{\partial}|_X\,m]$$

with vertex operators

$$V_i = 2\int_{\mathbb{CP}^1} \frac{\mathrm{D}\lambda_i}{\langle 1\,\lambda_i\rangle^2\,\langle 2\,\lambda_i\rangle^2}\,h_i(x,\lambda_i)$$

Deriving Hodges' formula

Need to sum over weighted spanning trees on $n-2$ vertices, edges given by OPE

$$m^{\dot\alpha}(\lambda_i)\, m^{\dot\beta}(\lambda_j) \sim \frac{\epsilon^{\dot\alpha\dot\beta}}{\langle \lambda_i\, \lambda_j\rangle} \langle 1\, \lambda_i\rangle \langle 2\, \lambda_i\rangle \langle 1\, \lambda_j\rangle \langle 2\, \lambda_j\rangle$$

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Easy using weighted matrix tree theorem!

Result...Hodges formula!

$$\mathcal{M}_{n,1} = \delta^4\left(\sum_{i=1}^{n} k_i\right) \frac{\langle 1\, 2\rangle^6}{\langle 1\, j\rangle^2 \langle 2\, j\rangle^2} \left|\mathbb{H}^{12j}_{12j}\right|$$

What next?

Twistor theory enables a first principles derivation of well-known formula!

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YES! Generating functional & sigma model extend to

any self-dual space-time!

SD radiative space-times

A *radiative* space-time is: [Sachs, Bondi-Metzner-van der Burg, Newman-Penrose, Friedrich]

- (almost everywhere) asymptotically flat
- vacuum/source-free
- completely characterized by free data σ^0 at $\mathscr{I}^+$ (spin-weight -2, conformal weight -1)

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A *SD* radiative space-time is a complex radiative space-time with complexified data: [Newman, Ludvigsen-Newman-Tod]

$$\sigma^0 = 0\,, \qquad \tilde{\sigma}^0 \neq 0$$

Scattering amplitudes on a SD rad. space-time look impossible to compute:

- Background field Einstein-Hilbert Lagrangian a nightmare
- *Functional* degrees of freedom in the background
- No momentum conservation
- No Huygens' principle $\Rightarrow$ tails [Friedlander, Harte, TA-Casali-Mason-Nekovar]
- Memory effect [Christodoulou, Bieri-Garfinkle-Yau,...]

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But twistor theory covers these space-times with the non-linear graviton!

Twistor theory

Let $\mathcal{M}$ be SD radiative w/ data $\tilde{\sigma}^0$

$\mathbb{C}$-structure on associated $\mathbb{P}\mathscr{T}$: [Sparling, Newman, Eastwood-Tod]

$$\bar{\nabla} = \bar{\partial} + \tilde{\sigma}^0([\mu\,\bar{\lambda}], \lambda, \bar{\lambda})\,\mathrm{D}\bar{\lambda}\,\bar{\lambda}^{\dot{\alpha}}\,\frac{\partial}{\partial \mu^{\dot{\alpha}}} := \bar{\partial} + \frac{\partial \mathsf{h}}{\partial \mu_{\dot{\alpha}}}\,\frac{\partial}{\partial \mu^{\dot{\alpha}}}$$

holomorphic curves described by $\mu^{\dot{\alpha}} = \mathsf{F}^{\dot{\alpha}}(x, \lambda)$ obeying $\bar{\partial}|_X \mathsf{F}^{\dot{\alpha}} = \frac{\partial \mathsf{h}}{\partial \mu_{\dot{\alpha}}}|_X$

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Gravitons on $\mathcal{M}$ represented on $\mathbb{P}\mathscr{T}$ by $H^{0,1}_{\bar{\nabla}}(\mathbb{P}\mathscr{T}, \mathcal{O}(2))$

[Hitchin, Ward-Wells, Baston-Eastwood]

We can still use 'flat' momentum eigenstate reps!

$$h_i(Z) = \int_{\mathbb{C}^*} \frac{\mathrm{d}s_i}{s_i^3}\,\bar{\delta}^2(\kappa_i - s_i\,\lambda)\,\mathrm{e}^{\mathrm{i}\,s_i\,[\mu\,i]} \in H^{0,1}_{\bar{\nabla}}(\mathbb{P}\mathscr{T}, \mathcal{O}(2))$$

On-shell kinematics in the background

Here $\kappa^{\alpha}\,\tilde{\kappa}^{\dot{\alpha}}$ is incoming momentum (i.e., at $\mathscr{I}^{-}$)
This gets *dressed* as perturbation moves through $\mathcal{M}$ [TA-Ilderton]

$$\kappa_{\alpha} \to \kappa_{\alpha}\,, \qquad \tilde{\kappa}_{\dot{\alpha}} \to \tilde{K}_{\dot{\alpha}}(x) := H^{\dot{\beta}}{}_{\dot{\alpha}}(x,\kappa)\,\tilde{\kappa}_{\dot{\beta}}$$

where $H^{\dot{\beta}}{}_{\dot{\alpha}}(x,\lambda)$ trivializes SD spinor bundle

Use a 'dressed' spinor-helicity notation

$$[\![i\,j]\!] := \tilde{K}_i^{\dot{\alpha}}(x)\,\tilde{K}_{j\,\dot{\alpha}}(x) = \epsilon_{\dot{\beta}\dot{\alpha}}\,H^{\dot{\gamma}\dot{\alpha}}(x,\kappa_i)\,H^{\dot{\delta}\dot{\beta}}(x,\kappa_j)\,\tilde{\kappa}_{i\,\dot{\gamma}}\,\tilde{\kappa}_{j\,\dot{\delta}}$$

Upshot

Generating functional for MHV amplitudes on $\mathcal{M}$:

$$\mathcal{G}(1,2) = \frac{\langle 1\,2\rangle^4}{4\pi \mathrm{i}\,\kappa^2} \int_{\mathscr{M}} \mathrm{d}^2 z\, \mathrm{d}^2 \tilde{z}\, \mathrm{e}^{\mathrm{i}\,[z\,1] + \mathrm{i}\,[\tilde{z}\,2]}\, S[m]\big|_{\text{on-shell}}$$

where $\mathscr{M} \simeq \mathcal{M} \oplus \sum(+$ helicity gravitons), or

$$\bar{\nabla} = \bar{\partial} + \frac{\partial \mathrm{h}}{\partial \mu_{\dot{\alpha}}} \frac{\partial}{\partial \mu^{\dot{\alpha}}} + \sum_i \frac{\partial h_i}{\partial \mu_{\dot{\alpha}}} \frac{\partial}{\partial \mu^{\dot{\alpha}}}$$

on twistor space

Background-coupled sigma model

Now $S[m]$ looks like:

$$S[m] = \int_{\mathbb{CP}^1} \frac{\mathrm{D}\lambda}{\langle 1\,\lambda\rangle^2\,\langle 2\,\lambda\rangle^2}\left(\left[m\,\bar\partial m\right] + 2\sum_i h_i(\mathsf{F}+m,\lambda) + \sum_{p=2}^{\infty}\frac{2}{p!}\,\frac{\partial^p \mathsf{h}}{\partial\mu^{\dot\alpha_1}\cdots\partial\mu^{\dot\alpha_p}}(\mathsf{F},\lambda)\,m^{\dot\alpha_1}\cdots m^{\dot\alpha_n}\right)$$

where h encodes the SD rad. background:

$$\mathsf{h} = \mathrm{D}\bar\lambda\int^{u}\mathrm{d}s\,\tilde\sigma^0(s,\lambda,\bar\lambda)$$

Can now translate into $\mathbb{CP}^1$ correlation function as before...

...but now with 'background' vertex operators

$$\sum_{t=0}^{\infty} \sum_{p_1,\ldots,p_t} \left\langle \prod_{i=1} V_i \prod_{\mathrm{m}=1}^{t} U^{(p_\mathrm{m})} \right\rangle_{\text{conn., tree}}$$

with

$$U^{(p)} = \frac{2}{p!} \int_{\mathbb{CP}^1} \frac{\mathrm{D}\lambda \wedge \mathrm{D}\bar{\lambda}}{\langle \lambda\, 1 \rangle^2 \langle \lambda\, 2 \rangle^2} N^{(p-2)}(u, \lambda, \bar{\lambda}) \, [m\, \bar{\lambda}]^p$$

$N^{(k)} = -\partial_u^{k+1} \tilde{\sigma}^0$ the k^{th} derivative of news function on $\mathcal{M}$, $u = [\mathsf{F}\, \bar{\lambda}]$

Now need to sum over spanning trees w/ some vertices of fixed valence.

Matrix-tree theorem to the rescue once again!

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Matrix-tree theorem to the rescue once again!

Result:

$$\sum_{t=0}^{\infty} \sum_{p_1,\ldots,p_t} \int_{\mathcal{M}\times(\mathbb{CP}^1)^t} \mathrm{d}^4 x \sqrt{g} \, \frac{\langle 1\, 2\rangle^6}{\langle 1\, i\rangle^2 \langle 2\, i\rangle^2} \left(\prod_{\mathrm{m}=1}^{t} \frac{\partial^{p_\mathrm{m}}}{\partial \varepsilon_\mathrm{m}^{p_\mathrm{m}}} \right) |\mathcal{H}^i_i| \Bigg|_{\varepsilon=0}$$

$$\exp\left(\mathrm{i} \sum_{i=1}^{n} \mathsf{F}^{\dot\alpha}(x, \kappa_i)\, \tilde\kappa_{i\,\dot\alpha} \right) \prod_{\mathrm{m}=1}^{t} \mathrm{D}\lambda_\mathrm{m} \wedge \mathrm{D}\bar\lambda_\mathrm{m} \, N_\mathrm{m}^{(p-2)}$$

$t > 0$ terms are *tail* contributions

Some unpacking:

$\mathcal{H}$ is a $(n+t-2)\times(n+t-2)$ matrix:

$$\mathcal{H} = \begin{pmatrix} \mathbb{H} & \mathfrak{h} \\ \mathfrak{h}^{\mathrm{T}} & \mathbb{T} \end{pmatrix}$$

$\mathbb{H}$ is 'dressed' Hodges' matrix

$$\mathbb{H}_{ij} = \frac{[\![i\,j]\!]}{\langle i\,j\rangle}\,,$$

$$\mathbb{H}_{ii} = -\sum_{j\neq i}\frac{[\![i\,j]\!]}{\langle i\,j\rangle}\,\frac{\langle 1\,j\rangle\,\langle 2\,j\rangle}{\langle 1\,i\rangle\,\langle 2\,i\rangle} - \sum_{\mathrm{m}=1}^{t}\varepsilon_{\mathrm{m}}\,\frac{[\![i\,\bar{\lambda}_{\mathrm{m}}]\!]}{\langle i\,\lambda_{\mathrm{m}}\rangle}\,\frac{\langle 1\,\lambda_{\mathrm{m}}\rangle\,\langle 2\,\lambda_{\mathrm{m}}\rangle}{\langle 1\,i\rangle\,\langle 2\,i\rangle}$$

$\mathfrak{h}$, $\mathbb{T}$ encode edges involving background vertices:

$$\mathfrak{h}_{i\mathrm{m}} = \varepsilon_{\mathrm{m}} \frac{[\![i\, \bar\lambda_{\mathrm{m}}]\!]}{\langle i\, \lambda_{\mathrm{m}} \rangle}\,, \qquad \mathbb{T}_{\mathrm{mn}} = \varepsilon_{\mathrm{m}}\, \varepsilon_{\mathrm{n}} \frac{[\![\bar\lambda_{\mathrm{m}}\, \bar\lambda_{\mathrm{n}}]\!]}{\langle \lambda_{\mathrm{m}}\, \lambda_{\mathrm{n}} \rangle}\,,$$

$$\mathbb{T}_{\mathrm{mm}} = -\varepsilon_{\mathrm{m}} \sum_{\mathrm{n} \neq m} \varepsilon_{\mathrm{n}} \frac{[\![\bar\lambda_{\mathrm{m}}\, \bar\lambda_{\mathrm{n}}]\!]}{\langle \lambda_{\mathrm{m}}\, \lambda_{\mathrm{n}} \rangle} \frac{\langle 1\, \lambda_{\mathrm{n}} \rangle \langle 2\, \lambda_{\mathrm{n}} \rangle}{\langle 1\, \lambda_{\mathrm{m}} \rangle \langle 2\, \lambda_{n} \rangle} - \varepsilon_{\mathrm{m}} \sum_{i} \frac{[\![\bar\lambda_{\mathrm{m}}\, i]\!]}{\langle \lambda_{\mathrm{m}}\, i \rangle} \frac{\langle 1\, i \rangle \langle 2\, i \rangle}{\langle 1\, \lambda_{\mathrm{m}} \rangle \langle 2\, \lambda_{\mathrm{m}} \rangle}$$

So what?

This formula may look complicated, but...

- amazing that *any* all-multiplicity formula is possible
- wildly simpler vs. naïve expectations (e.g., 4 vs. $4(n-2)$ space-time integrals)
- further simplifications for specific examples (e.g., SD plane waves, impulsive limit)
- can be generalised to *full* tree-level S-matrix (i.e., any N^kMHV amplitude) – conjectural [TA-Mason-Sharma]

Roundup

Non-linear graviton provides novel understanding of some remarkable amplitude formulae

Enables computation of new observables in curved space-time, impossible with other techniques!

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Lots to think about...

- non-radiative SD backgrounds (e.g., SD Schwarzschild)
- 'double copy' on curved backgrounds
- (A)dS backgrounds [TA-Mason, TA, Röhrig-Skinner, Eberhardt-Komatsu-Mizera]
- non-chiral (Lorentzian-real) backgrounds $\to$ ambitwistor (string) theory
- relation to GW observables [Bautista-Guevara-Kavanagh-Vines, Cristofoli-Gonzo-Kosower-O'Connell]

2008

Happy birthday, Roger!