New directions for the ambitwistor string

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1 Motivation and overview

The ambitwistor string was formulated by Mason and Skinner in [1]. Its tree amplitudes successfully reproduce the formulas for bosonic scattering amplitudes developed by Cachazo, He, and Yuan [2–4]. It is hoped that the ambitwistor string may also be useful for computing loop amplitudes and scattering amplitudes on non-trivial backgrounds. Since the ambitwistor string makes manifest the perturbative double copy (relating gauge and gravity amplitudes), it is hoped that the ambitwistor string on non-trivial backgrounds could lead to a formulation of a non-perturbative double copy. This report presents work towards both of these aims.

There has been some recent progress in the study of non-trivial backgrounds for the ambitwistor string. In 2014, Adamo, Casali, and Skinner [5] showed that the quantum consistency of the string imposes the supergravity equations for the background. More recently, a concrete ambitwistorial computation of a three-point amplitude on a non-trivial plane wave background was carried out in [6], following earlier work in [7]. This calculation uses the original formulation of the ambitwistor string. An alternative formulation is available in which the supergravity equations are realised classically, and not through quantum consistency. This is the pure spinor ambitwistor string first proposed by Berkovits [8]. It is related to Berkovits' formulation of the superstring [9]. In the same way that the ambitwistor string may be regarded as the infinite tension limit of the RNS superstring, the pure spinor version may be regarded as an infinite tension limit of the pure spinor superstring. The relation between the two formulations is the subject of some ongoing work by Berkovits [10, 11]. The pure spinor ambitwistor string has been shown to give the correct field theory amplitudes at tree level [12]. Adamo and Casali have offered evidence suggesting that it also gives the correct amplitudes at one loop [13]. Moreover, the pure spinor ambitwistor string gives rise to the supergravity equations at the classical level as consistency of the constraints [14]. The heterotic version has been shown to give the super Yang-Mills equations [15]. These results are based on the work by Berkovits and Howe in 2001 demonstrating that the supergravity and super Yang-Mills equations follow from the pure spinor superstring [16]. The emergence of the supergravity constraints at the classical level suggests that there is a classical ambitwistor construction of the supergravity equations making use of pure spinors. This would be analogous to the ambitwistor construction for super Yang-Mills equations in four dimensions [17, 18]. In 1986,

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Witten proposed that the ten dimensional super Yang-Mills and supergravity equations follow from an ambitwistor construction [19]. However, only the super Yang-Mills statement was fully proved (in refs. [19, 20]) and the supergravity construction remains incomplete. In Part 1 of this essay we show that there is an extended version of ambitwistor space which can be used to derive the super Yang-Mills and supergravity equations in ten dimensions.[1] The main interest of this work concerns the 'double copy' relating gauge theory and gravity amplitudes. This goes back to work by Bern, Cachazo, and Johansson [21] who formulated this relationship for tree amplitudes. Their work has a natural translation in the pure spinor superstring [22]. Some authors have suggested nonlinear examples of the double copy—which is to say, they find a dictionary between classical general relativity spacetimes and copies of classical Yang-Mills solutions. See ref. [23]. Since pure spinor ambitwistor space provides a classical construction of sueprgravity and super-Yang-Mills solutions in ten dimensions, it is possible to use our formulation of the classical equations to see how the double copy translates to a non-linear statement.[2]

The second aim of this essay is to discuss work that relates to the Ramond sector of the ambitwistor string and higher loop amplitudes. Immediately following the formulation of the ambitwistor string, Adamo, Casali and Skinner conjectured that the genus one amplitudes of the ambitwistor string are field theory loop amplitudes [5]. A partial proof of their conjecture in the case that all external states are bosonic was given in [24], with some further work on two loops in [25]. To complete the proof in [24], one needs to know the contribution that arises due to fermions in the loop. Some work towards this end was recently carried out in [26]. We make the connection in section 8.4, in Part 2 of this essay. Part 2 is concerned more generally with computing new correlators in the ambitwistor string involving the Ramond sector. We produce new formulas for correlators in dimension 4 and 6. We find that these do not give rise to formulas for amplitudes. It is hoped that more can be said in 10 dimensions, especially for the case of four fermions—which may lead us to a proof that the ambitwistor string gives correct 2-loop amplitudes. Many of our computations make use of earlier work by Schlotterer, Stieberger and Hartl in 2009 and 2010 [27–29]. Their work concerns superstring amplitudes and they embarked on a programme to compute higher point correlators involving Ramond insertions at arbitrary genus—with the motivation that these correlators are important for phenomenological particle physics. We adapt their work to our aims, giving complete proofs where references are otherwise unavailable. We also make use of some results in Lie polynomial theory to write our formulas in a manifestly Lie algebraic form. We conclude Part 2 with a discussion of how our formulas might be applied to higher loop and higher point amplitudes in the ambitwistor string.

Part One—pure spinors

The ambitwistor space of a complex manifold is the space of (complex) null geodesics. We begin, in section 2, with a review of key results about ambitwistor spaces and, in four dimensions, its relation to gauge theory. Section 2 concludes with a discussion of $D = 4$ super Yang-Mills from the ambitwistorial viewpoint. The main focus of Part 1 is super Yang-Mills and supergravity in ten dimensions. We give an ambitwistorial construction for the super Yang-Mills equations in section 3, and likewise for type IIB supergravity in section 4. These results are

[1] A number of homological results remain to be proved, as we explain in section 4.3. If these conjectures can be substantiated, we hope to show that the extended version of ambitwistor space can always be constructed, at least locally, for curved superspacetimes.

[2] It is dubious whether this is in any way interesting, and I have not burdened this report with any discussion of my calculations relating to this.

based on an extended version of ambitwistor space that we call pure spinor ambitwistor space. We conclude in section 4.3 by conjecturing that it is possible to give a local existence result for pure spinor ambitwistor space, by analogy with Le Brun's result for ambitwistor space.

2 Ambitwistor space

We can understand the local structure of ambitwistor space A by describing its tangent spaces. The tangent spaces to A can be identified with Jacobi fields in the following way. Let (M, g) be a complex manifold and consider some geodesic l. It has some tangent vector field K satisfying $\nabla_K K = 0$. We can obtain nearby geodesics by choosing a Jacobi field J, which satisfies $[K, J] = 0$ on l. At a particular point $p \in l$ we can specify the field J by choosing two vectors, $J(p)$ and $\dot{J}(p)$ (this is initial data for the Jacobi equation). The Jacobi fields on l can be regarded as the tangent vectors to the space of geodesics at l. However, the Jacobi fields corresponding to

$$J(p) = K,\ \dot{J}(p) = 0 \qquad \text{and} \qquad J(p) = 0,\ \dot{J}(p) = K$$

merely amount to reparametrisations of l. So the space of geodesics, when it exists, has expected dimension $2\dim M - 2$. We may represent the tangent vectors by choosing $J(p), \dot{J}(p)$ orthogonal and not parallel to K. There is a natural 1-form, ω, on the tangent space, which resembles the Wronskian. For any two Jacobi fields J_1, J_2, we can, at any point p on l, evaluate

$$\omega(J_1, J_2) = \dot{J}_1 \cdot J_2(p) - J_1 \cdot \dot{J}_2(p). \tag{1}$$

One easily verifies that the right hand side does not depend on our choice of $p \in l$ (by Jacobi's equation). If we constrain l to be null, $K \cdot K = 0$, we obtain a neighbouring null geodesic if $\dot{J}(p)^2 = 0$. So the space of null geodesics has expected dimension $2\dim M - 3$.

Theorem 1. *(Le Brun) Let M be a complex manifold and $p \in M$ a point. There exists some neighbourhood U of p such that the space of (null) geodesics in U is itself a Hausdorff complex manifold of the expected dimension.*[3]

The space of null geodesics is called ambitwistor space. In section 2.1 we give examples for flat space times. Our discussion of the relationship with gauge theory begins in section 2.3, and we conclude by discussing super Yang-Mills in section 2.6.

2.1 Examples

In principle, A could be given as the symplectic quotient of T^*M by the constraint $P^2 = 0$. However, for flat space we can give instructive global presentations of A using spinors and flag varieties. Let M_n be n-dimensional affine space over $\mathbb{C}$ with the obvious quadratic form. Write A_n for the associated ambitwistor space. We begin with some low dimensional examples.

2.1.1 Three dimensions

In three dimensions, a null vector k satisfies

$$k_1^2 + k_2^2 + k_3^2 = 0,$$

[3] Le Brun first proved this in his thesis [30] and later elaborated his work in ref. [31].

which is best solved using spinors. Indeed, all null vectors can be written as $k^a = \sigma^a{}_{\alpha\beta}\lambda^\alpha\lambda^\beta$ for some λ^α, unique up to a sign. Here λ^α belongs to the two dimensional spin representation of $so(3)$ and $\sigma^a_{\alpha\beta}$ are the Pauli matrices. A null geodesic is then of the form

$$l = \left\{ x_0^{\alpha\beta} + t\lambda^\alpha\lambda^\beta | t \in \mathbb{C} \right\} \subset M.$$

Notice that l is the solution to the equation

$$\mu^\alpha = x^{\alpha\beta}\lambda_\beta,$$

if we choose $\mu^\alpha = x_0^{\alpha\beta}\lambda_\beta$. The spinor μ^α does not depend on our choice of a base point x_0 on l. It follows that A_3 is given by the pairs $(\mu^\alpha, \lambda_\beta)$, up to a projective scaling. In other words, the ambitwistor space is $A_3 = \mathbb{CP}^3$, which has dimension $2 \times 3 - 3 = 3$, as anticipated.[4] We can equip this with a symplectic form

$$\omega = \mathrm{d}\lambda_\alpha \wedge \mathrm{d}\mu^\alpha. \tag{2}$$

The reason we choose this symplectic form—and not, say, the Kähler form on $\mathbb{CP}^3$—is that it is equivalent to equation (1).[5]

2.1.2 Four dimensions

In four dimensions, a null vector can be decomposed as $k^a\gamma_a{}^{\alpha\dot\alpha} = \tilde\lambda^\alpha\lambda^{\dot\alpha}$, unique up to a scaling. To parameterise the null rays, we can repeat the trick we used in three dimensions. The chiral structure of the spin representations proves to be only a minor complication. The solutions to $\mu^\alpha = x^{\alpha\dot\beta}\lambda_{\dot\beta}$ form a null 2-plane, $\{x_0^{\alpha\dot\beta} + \pi^\alpha\lambda^{\dot\beta} | \forall\pi^\alpha\}$. Likewise, the solutions to $\tilde\mu^{\dot\beta} = x^{\alpha\dot\beta}\tilde\lambda_\alpha$ form a null 2-plane, $\{x_1^{\alpha\dot\beta} + \tilde\lambda^\alpha\tilde\pi^{\dot\beta} | \forall\tilde\pi^{\dot\beta}\}$.[6] It is clear that if these two planes intersect for some choice of $\pi, \tilde\pi$, then they intersect in a null line with tangent vector $\tilde\lambda^\alpha\lambda^{\dot\alpha}$. The two planes intersect if there exist $\pi, \tilde\pi$ such that

$$x_0^{\alpha\dot\beta} - x_1^{\alpha\dot\beta} + \pi^\alpha\lambda^{\dot\beta} - \tilde\lambda^\alpha\tilde\pi^{\dot\beta} = 0.$$

Since the chiral spin representations are only two dimensional, such a π and $\tilde\pi$ exist iff

$$Q = \mu^\alpha\tilde\lambda_\alpha - \lambda_{\dot\alpha}\tilde\mu^{\dot\alpha} = 0. \tag{3}$$

This defines a quadric in $\mathbb{CP}^3 \times \mathbb{CP}^3$. Conversely, every null line can be obtained as the intersection of two such planes. The null ray through $x_0^{\alpha\dot\beta}$ with momentum $\tilde\lambda^\alpha\lambda^{\dot\beta}$ corresponds to the intersection of the planes $(x_0^{\alpha\dot\beta}\lambda_{\dot\beta}, \lambda_{\dot\beta})$ and $(x_0^{\alpha\dot\beta}\tilde\lambda_\alpha, \tilde\lambda_\alpha)$. It follows that A_4 is the quadric $Q = 0$ in $\mathbb{CP}^3 \times \mathbb{CP}^3$, which has dimension $2 \times 4 - 3 = 5$. It has a symplectic form

$$\omega = \mathrm{d}\tilde\lambda_\alpha \wedge \mathrm{d}\mu^\alpha + \mathrm{d}\tilde\mu^{\dot\alpha} \wedge \mathrm{d}\lambda_{\dot\alpha}.$$

This is again identical with equation (1).

[4] This example is discussed by Le Brun in ref. [32].

[5] The translation is straightforward. We identify

$$k^a = \sigma^{a\alpha\beta}\lambda_\alpha\lambda_\beta \qquad \text{and} \qquad \mu^\alpha = x_0^a\sigma^{a\alpha\beta}\lambda_\beta.$$

A variation in the geodesic is specified by $\delta x_0^a = J^a$ and $\delta k^a = \dot J^a$. This is related to the spinors μ^α and λ_α by

$$\delta\mu^a = J^a\sigma^{a\alpha\beta}\lambda_\beta, \qquad \dot J^a = 2\sigma^{a\alpha\beta}\lambda_{(\alpha}\delta\lambda_{\beta)}.$$

It follows that

$$\delta_1\lambda_\alpha\,\delta_2\mu^\alpha = \dot J_1 \cdot J_2,$$

and this shows that equation (2) is the same as equation (1).

[6] Penrose called these α and β planes. They are distinguished in the Grassmannian of null planes by belonging to distinct $SO(4)$ orbits—see below.

2.1.3 Six dimensions

In six dimensions, the two chiral spin representations are isomorphic and the chiral gamma matrix $\gamma_a^{\alpha\beta}$ is antisymmetric. A null vector has form $\epsilon^{\alpha\beta\gamma\delta}\lambda_\gamma\tilde{\lambda}_\delta$, though this is not unique. To parameterise the null rays we can again consider $\mu^\alpha = x^{\alpha\beta}\lambda_\beta$, as in three dimensions. However, this has a solution iff $\mu^\alpha\lambda_\alpha = 0$, since $x^{\alpha\beta}$ is skew. The solution is a null 3-plane with tangents of the form $\epsilon^{\alpha\beta\gamma\delta}\pi_\gamma\lambda_\delta$. Likewise, consider a second null 3-plane defined by $\tilde{\mu}^\alpha = x^{\alpha\beta}\tilde{\lambda}_\beta$. Again, we must impose $\tilde{\mu}^\alpha\tilde{\lambda}_\alpha = 0$. If the two planes intersect, they do so in a null ray with tangent $\epsilon^{\alpha\beta\gamma\delta}\lambda_\gamma\tilde{\lambda}_\delta$. Following the same argument as in four dimensions, they intersect if

$$Q = \mu^\alpha\tilde{\lambda}_\alpha + \lambda_\alpha\tilde{\mu}^\alpha = 0.$$

So, if $\mu^\alpha\lambda_\alpha = 0$, $\tilde{\mu}^\alpha\tilde{\lambda}_\alpha = 0$, and $Q = 0$, the two planes intersect in a null ray. But the null ray is not uniquely represented by the two null planes. If a null ray through x_0 with tangent $\epsilon^{\alpha\beta\gamma\delta}\lambda_\gamma\tilde{\lambda}_\delta$ is contained in the α-plane $(\mu^\alpha, \lambda_\alpha)$, it is also contained in the plane

$$(\mu^\alpha + tx_0^{\alpha\beta}\tilde{\lambda}_\beta, \lambda_\alpha + t\tilde{\lambda}_\alpha),$$

for all t. With respect to the symplectic form

$$\omega = \mathrm{d}\mu^\alpha \wedge \mathrm{d}\tilde{\lambda}_\alpha + \mathrm{d}\tilde{\mu}^\alpha \wedge \mathrm{d}\lambda_\alpha,$$

this shift in μ^α and λ_α is generated by the Hamiltonian $\tilde{\mu}^\alpha\tilde{\lambda}_\alpha$. Likewise for $\mu^\alpha\lambda_\alpha$. So A_6 is obtained as a symplectic reduction of $\mathbb{CP}^7 \times \mathbb{CP}^7$ by the Hamiltonians $\tilde{\mu}^\alpha\tilde{\lambda}_\alpha$, $\mu^\alpha\lambda_\alpha$, and Q. The first two Hamiltonians reduce the dimension from 14 by two each. Q only reduces the dimension by one since the Hamiltonian vector field associated to Q is just the difference of the two Eulerian vector fields which we have already reduced by when passing to the projective spaces $\mathbb{CP}^7$. The symplectic reduction by these Hamiltonians thus has dimension $14 - 5 = 9$, as expected.

2.1.4 Eight dimensions

In eight dimensions, the two chiral spin representations have dimension eight, and there are symmetric inner products $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ on each chiral representation. The gamma matrices satisfy

$$\gamma_{(a}{}^{\alpha\dot{\alpha}}\gamma_{b)\alpha}{}^{\dot{\beta}} = g_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}, \tag{4}$$

where g is the Euclidean metric on M. In fact, analogous identities can be obtained by permuting the roles of the indices. As in lower dimensions, a spinor $\lambda^{\dot{\alpha}}$ defines a plane with tangents $k^a = \gamma^a{}_{\alpha\dot{\alpha}}\pi^\alpha\lambda^{\dot{\alpha}}$. However, this plane is totally null only if $\lambda^{\dot{\alpha}}\lambda_{\dot{\alpha}} = 0$, by equation (4). In this case, $\lambda^{\dot{\alpha}}$ defines a null 4-plane. Such a $\lambda^{\dot{\alpha}}$ is called a pure spinor, and the space of projective pure spinors can be identified with null planes of maximal dimension. (See appendix A.2.) The incidence relation is now

$$\mu_\alpha = x_{\alpha\dot{\alpha}}\lambda^{\dot{\alpha}},$$

and μ_α is a pure spinor, as follows from equation (4). If x_0 is a solution, the space of solutions is a null 4-plane of the form $x_0^a + \gamma^a{}_{\alpha\dot{\alpha}}\pi^\alpha\lambda^{\dot{\alpha}}$, for all π^α. However, the incidence relation has a solution only if

$$\mu_\alpha\lambda_{\dot{\alpha}}\gamma_a{}^{\alpha\dot{\alpha}} = 0. \tag{5}$$

Together with the purity of $\lambda^{\dot{\alpha}}$ and μ_α, this condition means that $(\mu_\alpha, \lambda^{\dot{\alpha}})$ is a pure $SO(10)$ spinor. Likewise, the dual relation $\tilde{w}^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\tilde{\lambda}_\alpha$ gives rise to a null 4-plane, provided that

$$\tilde{\lambda}_\alpha\tilde{w}_{\dot{\alpha}}\gamma_a{}^{\alpha\dot{\alpha}} = 0. \tag{6}$$

If the two null planes intersect, they do so in a null ray. The intersection condition is again the quadric analogous to equation (3). The space of two projective $SO(10)$ pure spinors satisfying $Q = 0$ has dimension 19, as compared with $\dim A_8 = 13$. The reason for the discrepancy is that a null vector k^a is not uniquely written as $\gamma^a_{\alpha\dot{\alpha}}\tilde{\lambda}^\alpha\lambda^{\dot{\alpha}}$. We may add to $\lambda^{\dot{\alpha}}$ any spinor of the form $\gamma_a^{\alpha\dot{\alpha}}l^a\tilde{\lambda}_\alpha$. The space of spinors with this form has dimension 4 or, projectively, 3. With respect to the symplectic form, these translations are generated by equation (5). Likewise, equation (6) generates the analogous translations in $\tilde{\lambda}^\alpha$. The symplectic reduction by these constraints then gives A_8, with dimension 13.

2.2 Grassmannians

As we saw in the previous section, the spinorial presentation of A_n becomes unwieldy in higher dimensions for two reasons. The first reason is purity. In dimension $2n$, we have been imposing that the pairs (μ, λ) are pure spinors for $SO(2n+2)$. (A pure spinor for $SO(2n+2)$ is called a 'twistor' for M_{2n}.) However, in the spinorial representation, a pure spinor must satisfy many constraints. The second reason is the little group. The representation of a null vector k as $\lambda\tilde{\lambda}$ is not unique in dimensions greater than 2. This is manageable in low dimensions, where the orbits of the little group have low dimensions, but not in higher dimensions. We can solve both problems at once by working directly with the null planes represented by pure spinors. In this section, we will explain this approach in four dimensions, before giving its generalising to all even dimensions. The Grassmannian approach to four dimensions will be used again in section 2.6 in our discussion of super Yang-Mills. The generalisation to all even dimensions is somewhat technical, and may be skipped.

Four dimensions

Earlier, we constructed A_4 as a quadric in $\mathbb{CP}^3 \times \mathbb{CP}^3$. We can identify A_4 with the flag variety $F(1,3;4)$ of (1,3)-flags in a four dimensional vector space. To make this clear, introduce homogeneous coordinates v^i, w^i for $\mathbb{CP}^3$. Then A_4 is the quadric $v \cdot w = 0$ in $\mathbb{CP}^3 \times \mathbb{CP}^3$. But $v \cdot w$ is the Plücker relation for the embedding the flag variety $F(1,3;4)$ into the Grassmannians $\mathrm{Gr}(1;4) \times \mathrm{Gr}(3;4)$. This is because $\mathrm{Gr}(1;4)$ is isomorphic to $\mathbb{CP}^3$ where a ray with tangent v^i is sent to the point $[v^i]$. Similarly, $\mathrm{Gr}(3;4)$ is isomorphic to $\mathbb{CP}^3$ where a plane with normal w^i is sent to the point $[w^i]$. Then the ray is contained in the plane iff $v \cdot w = 0$. In fact, we can never obtain the lines $v^i = (\mu^\alpha, 0)$ for finite $x^{\alpha\dot{\alpha}}$ from the incidence relation. So A is only an open subset of $F(1,3;4)$. Alternatively, $F(1,3;4)$ is a natural compactification of A. On the other hand, the Euclidean space M_4 can be identified with an open subset of $Gr(2,4)$, which we might call compactified Euclidean space, M_4^c. This identification leads to the fibrations

$$A_4 \subset F(1,3;4) \leftarrow F(1,2,3;4) \rightarrow Gr(2;4) \supset M_4, \tag{7}$$

which we make use of in sections 2.4 and 2.5. Let us now explain the identification of M_4 with a subset of $Gr(2,4)$. Generally, the Grassmannian $Gr(r,d)$, which has dimension $r(d-r)$, may be covered by patches associated to dimension $d-r$ subspaces. This is completely analogous to the covering of $\mathbb{CP}^d$ by inhomogeneous coordinate patches. Given a subspace S of dimension $d-r$, choose a basis $e^1, ..., e^d$ so that S is the span of $e^1, ..., e^{d-r}$. Then we may consider the open subset of planes in $Gr(2,4)$ which are transversal to S. These take the form

$$\begin{bmatrix} \mathbb{1}_{d-r} \\ X \end{bmatrix} \in Gr(r,d),$$

where X is an $r \times d-r$ matrix. The corresponding subspace is the column span of this matrix. In our particular case, take $r = 2$ and $d = 4$. Then one patch of $Gr(2,4)$ may be written as

$$\begin{bmatrix} \mathbb{1}_2 \\ x^{\alpha\dot{\alpha}} \end{bmatrix},$$

and the $x^{\alpha\dot{\alpha}}$ can be identified here with coordinates on M_4. Indeed, in these coordinates, the construction of the tautological bundle over $Gr(2,4)$ shows that we may identify it with the unprimed chiral spinor bundle on M_4. The Grassmannian $Gr(2,4)$ admits an embedding into $\mathbb{P}\left(\wedge^2\mathbb{C}^4\right) = \mathbb{CP}^5$. If z^i are homogeneous coordinates for $\mathbb{CP}^5$, the Plücker relation is $q(z) = z^1z^2 + z^3z^4 + z^5z^6 = 0$. On the open set $z^6 \neq 0$, we recover z^i/z^6, with $1 \leq i \leq 4$, as coordinates on M_4.[7] So we may view M_4^c as the null rays in $\mathbb{CP}^5$ with respect to q, which we denote as $Gr^0(1,6)$. Likewise, we can consider null planes $Gr^0(2,6)$, and so on. In fact, suppose $Z_1, Z_2 \in Gr^0(1,6)$ are two points in a null plane in $Gr^0(2,6)$. If Z_i corresponds to the point x_i in M_4, then one finds that $q(x_1 - x_2) = 0$.[8] So x_1 and x_2 are null separated in M_4 and $x_1 + t(x_1 - x_2)$ is a null geodesic in M_4. Conversely, given a null geodesic, we obtain such a plane in $Gr^0(2,6)$ by taking two points, X_1 and X_2, on the null ray and taking the null plane associated to $X_1 \wedge X_2$. So $A_4^c = Gr^0(2,6)$. We can present an alternative double fibration

$$A_4 \subset Gr^0(2,6) \leftarrow F^0(1,2;6) \rightarrow Gr^0(1,6) \supset M_4.$$

This looks markedly different from equation (7). Earlier, we identified A_4 with $F(1,3;4)$, whereas now we are identifying it with $Gr^0(2,6)$. The translation between $Gr^0(2,6)$ and $F(1,3;4)$ is spinorial. A plane in $Gr^0(2,6)$ is the same as a simple 2-form $X \wedge Y$ on $V = \mathbb{C}^6$. The even forms in $\bigwedge V$ can be identified with $Spin(6)$. Choosing a decomposition $V = W \oplus W^*$, the irreducible representation of the Clifford algebra is $\bigwedge W$, which has dimension 8. The irreducible chiral spin representations are the even and odd forms in $\bigwedge W$, which we denote by $S_6^{\pm}$. These are each four dimensional. The associated representations of $Spin(6)$ are denoted γ and $\tilde{\gamma}$. Then $\gamma(X \wedge Y)$ is an endomorphism of S_6^+. Since X and Y are both null, the image of $\gamma(X \wedge Y)$ is a 1-dimensional ray in $S_6^{\pm}$. Doing the same for $\tilde{\gamma}$, and identifying S_6^- as the dual of S_6^+, we recover the presentation $A_4 = F(1,3;4)$.

All even dimensions

The approach we have sketched to four dimensions was generalised to higher dimensions by Harnad and Schnider. [33] Since the construction is not quite captured by the example of A_4, we will briefly present their result.

Theorem 2. *(i) The ambitwistor space A_{2n} for M_{2n} is an open subset of the isotropic Grassmannian, $Gr^0(2, 2n+2)$. We may present A_{2n} and M_{2n} as fibrations of an isotropic flag variety,*

$$A_{2n} \subset Gr^0(2, 2n+2) \leftarrow F^0(1,2; 2n+2) \rightarrow Gr^0(1, 2n+2) \supset M_{2n}.$$

(ii) For $n > 2$ we have the further identification

$$A_{2n} \subset Gr^0(2, 2n+2) = Gr^0(S_{2n-2}, S_{2n+2}^+),$$

given by (a generalisation of) the Cartan map.[9]

[7] These are related to the usual ones by $z^1/z^6 = x_1 + ix_2$, and so on, such that $q(x) = (x^1)^2 + (x^2)^2 + ...$

[8] Since $Z_1 + tZ_2$ is null for all t, we find that, writing $Z_i = (z_i^a, z_i^5, z_i^6)$,

$$q(z_i) + z_i^5 z_i^6 = 0 \qquad \text{and} \qquad 2q(z_1, z_2) + z_1^5 z_2^6 + z_1^6 z_2^5 = 0.$$

Combining these gives $x_1^2 + x_2^2 - 2x_1 \cdot x_2 = 0$, where $x_i^a = z_i^a / z_i^6$.

[9] The generalisation is due to Hanard and Schnider.

The first part of the theorem proceeds as in our discussion of A_4 with no changes. The second part requires some explanation. The Cartan map identifies maximal null planes with projective pure spinors. Take again V with dimension $2n+2$, and a decomposition $V = W \oplus W^*$ so that the quadratic form on V is

$$q : (W \oplus W^*) \times (W \oplus W^*) \to \mathbb{C} : (w_1, w_1'), (w_2, w_2') \mapsto w_1'(w_2) + w_2'(w_1).$$

The structure of $Gr^0(n+1, 2n+2)$ depends on $n+1$ mod 2. When $n+1$ is odd, $Gr^0(n+1, 2n+2)$ decomposes into two $SO(2n+2)$ orbits. When $n+1$ is even, $Gr^0(n+1, 2n+2)$ is itself an $SO(2n+2)$ orbit. To see this, consider the orbits of W and W^* under $SO(2n+2)$. The matrix J exchanging W and W^* has determinants $\det J = n+1$. Let us fix $n+1$ odd. We denote the disjoint orbits of W and W^* by $Gr^+(n+1, 2n+2)$ and $Gr^-(n+1, 2n+2)$. The Cartan map takes a null plane $X \in Gr^+(n+1, 2n+2)$ to the image of $\gamma(X)$, which is a 1-dimensional ray in Λ^+_{2n+2}. By choosing an ordering of the coordinates, we can obtain a canonical ray $S_2^+ \subset S^+_{2n+2}$ which is the spin representation of the subgroup $Spin(2) \subset Spin(2n+2)$ corresponding to the 'first two coordinates'. In fact, let us choose S_2^+ so that $S_2^+ = [\gamma(W)]$. Next, we use that γ is equivariant with respect to $Spin(2n+2)$. The ray $[\gamma(X)] \subset S^+_{2n+2}$ is thus connected to $S_2^+ \subset S^+_{2n+2}$ under $Spin(2n+2)$ since X is connected to W under $SO(2n+2)$. This might appear somewhat tautological! But this is Cartan's result. Cartan (via Chevalley) called the space of rays connected to S_2^+ the projective pure spinors,

$$Gr^0(S_2^+; S^+_{2n+2}).$$

Since Cartan's map identifies this space with the space of $n+1$-dimensional null planes, it has dimension $n(n+1)/2$. (See appendix A.2.) We can generalise Cartan's map to other isotropic Grassmannians besides $Gr^0(n+1, 2n+2)$. We define $Gr^0(S_{2n-2}, S^+_{2n+2})$ to be the 2^{n-1}-dimensional subspaces in S^+_{2n+2} connected to $S^\pm_{2n-2}$ by $Spin(2n+2)$. Then, in the same way as for the pure spinors, we can identify a null plane $X \wedge Y \in Gr^0(2, 2n+2)$ with the image of $\gamma(X \wedge Y)$ in S^+_{2n+2}.

2.3 Double fibrations

There is an intimate relationship between the Yang-Mills equations and the ambitwistor space in four dimensions. We return to this in section 2.4 after developing some prerequisites. As mentioned earlier, both A_4 and M_4 can be presented as fibrations of $F = F(1, 2, 3; 4)$. F can also be identified with the total space of the projective spinor bundles $(\mathbb{P}S \oplus \mathbb{P}\tilde{S})M$ or, equivalently, with the bundle of null quadrics in the projective tangent space $\mathbb{P}T_0 M$. For a general complex manifold M, there is no reason to expect that the null geodesics will define a fibration of $\mathbb{P}T_0 M$. However, by Le Brun's theorem, this is always possible locally. So consider any double fibration (of complex analytic spaces)

$$A \xleftarrow{p} F \xrightarrow{q} M. \tag{8}$$

It is sometimes possible to construct vector bundles on M from vector bundles on A.[10] Given a locally free sheaf $\tilde{E}$ on A we can obtain a locally free sheaf on M if $p^*\tilde{E}$, restricted to $q^{-1}(x)$, is free for all points $x \in M$.[11] This means that $E = q_* p^* \tilde{E}$ exists, and E_x is given by the global sections of $p^*\tilde{E}$ restricted to $q^{-1}(x)$. Note also that $\tilde{E}$ and E (as vector bundles) have the same rank. Finally, we can give E a connection ∇ induced by the vertical connection

$$\nabla_v : p^*\tilde{E} \to p^*\tilde{E} \otimes \Omega^1 F/A,$$

[10] This construction is a mild generalisation of an idea given by Ward for twistor space.

[11] We will identify a vector bundle E with its locally free sheaf of germs, $\mathcal{O}(E)$, which we denote by E.

where $\Omega^1 F/A$ are the vertical forms with respect to p. In order for this to give a connection on E, we need that $q_*\Omega^1 F/A \simeq \Omega^1 M$ is an isomorphism. If, in addition, the fibres of p are simply connected and the fibres of q are compact, we will call the double fibration good.

Theorem 3. *If the fibration, equation* (8)*, is good, there is an equivalence between vector bundles on A (whose pullbacks are trivial on the fibres of q) and vector bundles with connection on M (whose pullbacks have trivial curvature and monodromy on the fibres of p).*[12]

Remark. The requirement that the bundles on M have trivial curvature becomes vacuous when the fibres of p are one-dimensional, as they are in the case that A is the space of null rays.

Remark. The construction is not symmetric. We obtain a bundle with connection (E, ∇) on M from a bundle $\tilde{E}$ on A. There is no reason to expect the same to work in the opposite direction. The origin of the asymmetry is that the fibres of q are compact (e.g. quadrics in the projective tangent bundle), whereas the fibres of p are non-compact (e.g. null rays).

Given a good double fibration, the cohomology on A can be related to groups on M. The key fact which gives this result is that $p^*\tilde{E} \otimes \Omega^\bullet F/A$ with differential $\nabla_{F/A}$ is a resolution of the inverse image sheaf $p^{-1}\tilde{E}$. In other words, the following sequence is exact:

$$0 \to p^{-1}\tilde{E} \to p^*\tilde{E} \xrightarrow{\nabla_v} p^*\tilde{E} \otimes \Omega^1 F/A \to 0. \tag{9}$$

The associated exact sequence in cohomology gives

$$... \to H^{k-1}(F, p^{-1}\tilde{E}) \to H^{k-1}(F, p^*\tilde{E}) \to H^{k-1}(F, p^*\tilde{E} \otimes \Omega^1 F/A) \xrightarrow{\delta} H^k(F, p^{-1}\tilde{E}) \to ... \tag{10}$$

In good circumstances, the connecting homomorphism δ gives rise to explicit isomorphisms between $H^k(A, \tilde{E})$ and groups on M, in which case we call δ the Penrose transform. The relation to cohomology on M is given by the Leray spectral sequence, which 'usually' gives[13]

$$H^i(F, S) = H^0(M, R^i q_* S).$$

Likewise, the relation to cohomology on A is given by

$$H^i(F, p^{-1}\tilde{E}) = H^i(A, E),$$

which follows by the same argument, since p^{-1} is adjoint to p_*. Concrete examples of the Penrose transform are given in section 2.5.

2.4 Yang-Mills in four dimensions

We now return to four dimensions and very briefly mention the following result for Yang-Mills. Suppose that $\tilde{E}$ and (E, ∇) is a pair of bundles on A_4 and M_4 constructed by Theorem 3. The relation to the Yang-Mills equation follows by putting $\tilde{S} = \mathrm{End}(\tilde{E}) \otimes \mathcal{I}^k/\mathcal{I}^{k+1}$ in the long exact sequence, equation (10), where $\mathcal{I}$ is the ideal sheaf of A as a quadric in $Gr(1) \times Gr(3)$.

[12] The original reference is [34].

[13] The full statement is that $H^i(M, R^j q_* S)$ abuts to $H^{i+j}(F, S)$. If we have $H^i(M, R^j q_* S) = 0$ for all $i > 0$, then we need do no calculations and the result follows. The sheaves $R^i q_* S$ are modelled on the pre-sheaves $H^i(q^{-1}(U), S)$.

Theorem 4. *(Manin[14]) The bundle $\tilde{E}$ has a unique extension to the second formal neighbourhood of A. The obstruction to a third order extension is given by $\nabla \star F$, regarded as a class in*

$$H^2(A, End(\tilde{E}) \otimes \mathcal{I}^3/\mathcal{I}^4) \simeq ker(\nabla) \subset H^0(M, \Omega^3_M \otimes EndE).$$

The isomorphisms in the theorem follow essentially from equation (10). However, some key identifications—such as $R^2 q_* p^*(\mathcal{I}^1/\mathcal{I}^2) = \Omega^2_M$—rely on the presentation of F as $F(1,2,3;4)$. For this reason, the cohomological construction has not been extended to any more general cases. We note that Witten gave a concrete derivation of this result in ref. [17].

2.5 Penrose transform

We conclude our discussion of bosonic ambitwistor space in four dimensions by revisiting the exact sequence given in equation (2.5). For certain choices of $\tilde{E}$, the associated long exact sequence gives isomorphisms which are known as 'Penrose transforms'. In this section we will compute examples of these isomorphisms. We begin with some definitions. Let $\mathcal{O}(a,b)$ be the sheaf whose sections are functions on A regarded as a quadric in $\mathbb{CP}^3 \times \mathbb{CP}^3$ with homogeneity (a,b). We write $\mathcal{O}(a,b)_F = p^*\mathcal{O}(a,b)$ for the pull back bundle: its sections are functions on $(\mathbb{P}S \oplus \mathbb{P}\tilde{S})M$ with homogeneity (a,b). We may identify $\Omega^1 F/A$ with $\mathcal{O}(1,1)_F$ (a vertical p-form can be identified with a function homogeneous in P of weight p). Notice that the fibres of $F \to M$ are $\mathbb{CP}^1 \oplus \mathbb{CP}^1$. We will make crucial use of the fact that

$$H^0(\mathbb{CP}^1, \mathcal{O}) = \mathbb{C}, \qquad H^k(\mathbb{CP}^1, \mathcal{O}(n)) = 0, \text{ for all } n \geq 0,\ k \geq 1. \tag{11}$$

We also use that $H^0(\mathbb{CP}^1, \mathcal{O}(k))$ is given by the k-homogeneous functions on $\mathbb{CP}^1$ which we represent by symmetric spinor tensors: i.e. $\phi_{\alpha_1 \ldots \alpha_k}$ represents the function $\phi_{\alpha_1 \ldots \alpha_k}\lambda^{\alpha_1} \ldots \lambda^{\alpha_k}$, if λ^α are homogenous coordinates on $\mathbb{P}S$. Given all this, the exact sequence reads

$$0 \to p^{-1}\mathcal{O}(a,b) \to \mathcal{O}(a,b)_F \xrightarrow{\nabla_v} \mathcal{O}(a+1,b+1)_F \to 0.$$

We identify ∇_v with $\lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}$. For $a=0, b=0$, we find a long exact sequence

$$\begin{aligned} \ldots \to H^0(M, R^0 q_* \mathcal{O}(a,b)_F) \xrightarrow{\nabla_v} H^0(M, R^0 q_* \mathcal{O}(a+1,b+1)_F) \to H^1(A, \mathcal{O}(a,b)) \\ \to H^0(M, R^1 q_* \mathcal{O}_F(a,b)) \xrightarrow{\nabla_v} H^0(M, R^1 q_* \mathcal{O}_F(a+1,b+1)) \to \end{aligned} \tag{12}$$

We would like to know more about the sheaves $R^i q_* \mathcal{O}_F(a,b)$ appearing in this sequence. These are the sheaves modelled on the pre-sheaves $H^i(q^{-1}(U), \mathcal{O}_F(a,b))$. So, looking at the stalks, we are reduced to computing $H^i(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(a,b)_F)$ which has a Künneth decomposition as

$$H^n(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(a,b)_F) = \bigoplus_{i=0}^{n} H^i(\mathbb{CP}^1, \mathcal{O}(a)) \otimes H^{n-i}(\mathbb{CP}^1, \mathcal{O}(b)).$$

We now set about computing $H^1(A, \mathcal{O}(a,b))$.

2.5.1 Computation of $H^1(A, \mathcal{O}(a,b))$ for $a \geq 0$ and $b \geq 0$

If $a \geq 0$ and $b \geq 0$, the fourth group appearing in equation (12) vanishes (using the Künneth formula and equation (11)). So, in this case

$$H^1(A, \mathcal{O}(a,b)) = \text{coker}\nabla_v \subset H^0(M, R^0 q_* \mathcal{O}(a+1,b+1)_F).$$

[14] See Buchdahl's paper for a clear exposition. [35] The theorem is given in section 2.4 of his paper.

To be explicit, the stalks of the sheaf on the right-hand-side are

$$\left(R^0q_*\mathcal{O}(a+1,b+1)_F\right)_x = H^0(\mathbb{CP}^1,\mathcal{O}(a+1))\otimes H^0(\mathbb{CP}^1,\mathcal{O}(b+1)).$$

The sections of these groups are the degree $a+1$ and degree $b+1$ polynomials, respectively. So $H^1(A,\mathcal{O}(a,b))$ is represented by symmetric spinor fields of the form

$$\phi_{(\alpha_1\ldots\alpha_{a+1})}{}^{(\dot\beta_1\ldots\dot\beta_{b+1})}(x),$$

modulo the image of ∇_v, which is given by fields of the form

$$\nabla_{(\alpha_1}{}^{(\dot\beta_1}\phi_{\alpha_2\ldots\alpha_{a+1})}{}^{\dot\beta_2\ldots\dot\beta_{b+1})}.$$

When $a=b$, we may instead identify these as $a+1$-forms modulo exact $a+1$-forms. In any case, we conclude that

Proposition. *For $a\geq 0$ and $b\geq 0$,*

$$H^1(A,\mathcal{O}(a,b))\simeq\frac{\{\textit{spinor fields }\phi_{(\alpha_1\ldots\alpha_{a+1})}{}^{(\dot\beta_1\ldots\dot\beta_{b+1})}\}}{\{\textit{exact spinor fields }\nabla_{(\alpha_1}{}^{(\dot\beta_1}\psi_{\alpha_2\ldots\alpha_{a+1})}{}^{\dot\beta_2\ldots\dot\beta_{b+1})}\}}.$$

It is instructive to explicitly write down the isomorphism appearing in the proposition. This amounts to implementing the long exact sequence homomorphism. We use the fourier decomposition of fields on spacetime and consider a single mode,

$$\phi_k(x) = a_{(\alpha_1\ldots\alpha_{a+1})(\dot\beta_1\ldots\dot\beta_{b+1})}\lambda^{\alpha_1}\ldots\lambda^{\alpha_{a+1}}\tilde\lambda^{\dot\beta_1}\ldots\tilde\lambda^{\dot\beta_{b+1}}e^{ik\cdot X}\ \in\ H^0(M,R^0q_*\mathcal{O}(a+1,b+1)_F),$$

where a is a constant tensor with respect of X. The pre-image of this mode under ∇_v is clearly

$$\alpha_k(x)=\frac{a(\lambda,\ldots,\tilde\lambda,\ldots)}{i\lambda\not{k}\tilde\lambda}e^{ik\cdot X}.$$

The connecting homomorphism is then explicitly given by

$$\phi_k(x)\mapsto\bar\partial\alpha_k = 2\pi\bar\delta(k\cdot P)a(\lambda,\ldots,\tilde\lambda,\ldots)e^{ik\cdot X}\in H^1(A,\mathcal{O}(a,b)),$$

where we have written $\lambda\not{k}\tilde\lambda$ as $k\cdot P$ and we use that

$$\bar\delta(k\cdot P)=\frac{1}{2\pi i}\bar\partial\frac{1}{k\cdot P}.$$

2.5.2 Computation of $H^1(A,\mathcal{O}(a,b))$ for $a\geq 0$ and $b\leq -2$

When $a\leq -2$ or $b\leq -2$ the second group in equation (12) vanishes because $\mathcal{O}(k)_{\mathbb{CP}^1}$ has no global sections if $k\leq -2$. It follows that $H^1(A,\mathcal{O}(a,b))$ can be written in terms of the fourth and fifth groups in the long exact sequence:

$$H^1(A,\mathcal{O}(a,b)) = \ker\nabla_v\subset H^0(M,R^1q_*\mathcal{O}_F(a,b)).$$

Assuming that $a\geq 0$, the Künneth decomposition gives

$$\left(R^1q_*\mathcal{O}(a,b)_F\right)_x = H^0(\mathbb{CP}^1,\mathcal{O}(a))\otimes H^1(\mathbb{CP}^1,\mathcal{O}(b)).$$

Now we use Serre duality, which gives

$$H^1(\mathbb{CP}^1,\mathcal{O}(-k)) = H^0(\mathbb{CP}^1,\mathcal{O}(-k)^*\otimes K) = H^0(\mathbb{CP}^1,\mathcal{O}(k-2)).$$

Then we can identify the sections of $R^1q_*\mathcal{O}(a,b)_F$ with symmetric spinor fields having a α indices and $-b-2$ $\dot\beta$ indices. We are interested in the kernel of

$$\nabla_v : H^0(M, R^1q_*\mathcal{O}_F(a,b)) \to H^0(M, R^1q_*\mathcal{O}_F(a+1,b+1)).$$

In terms of our representation with spin fields, the group on the right has $a+1$ α indices and $-b-3$ $\dot\beta$ indices. So ∇_v acts as

$$\phi_{(\alpha_1\ldots\alpha_a)(\dot\beta_1\ldots\dot\beta_{-b-2})} \mapsto \nabla_{(\alpha_1}{}^{\dot\beta_1}\phi_{\alpha_2\ldots\alpha_{a+1})(\dot\beta_1\ldots\dot\beta_{-b-2})}.$$

We thus identify $H^1(A,\mathcal{O}(a,b))$, with $a \geq 0$ and $b \leq -2$ with spinor fields satisfying

$$\nabla_{(\alpha_1}{}^{\dot\beta_1}\phi_{\alpha_2\ldots\alpha_{a+1})(\dot\beta_1\ldots\dot\beta_{-b-2})} = 0.$$

This is not a dynamical field equation, since ϕ has $(a+1)(-b-1)$ components, while we are imposing only $a+2$ first order equations on these components. This is to be contrasted with the construction of fields satisfying the wave equation using the analogous methods for the twistorial fibration.

2.6 Super Yang-Mills

The prominent role of formal neighbourhoods in section 2.4 suggests supersymmetry.[15] In fact, everything we have said so far can be related to a result for $N=3$ super Yang-Mills. In four dimensions, the superspace $M_{4|N}$ of type (N,N) is $\Pi(S\oplus S')M$, where S and S' are the primed and unprimed spinor bundles on M_4. We can also present $M_{4|N}$ as a flag variety. Recall that M_4 is an open subset of $Gr(2;4)$. In place of $\mathbb{C}^4$, we may consider $\mathbb{C}^{4|N}$. (This is the base manifold $\mathbb{C}^4$ with functions $Sym^\bullet\mathbb{C}[x_1,...,x_4]\otimes Asym^\bullet\mathbb{C}[\theta_1,...,\theta_N]$.) The superspace $M_{4|N}$ can then be presented as an open subset of the flag variety $F(2|0,2|N;4|N)$. To see this, recall that there is a coordinate patch of $Gr(2;4)$ for which subspaces correspond to the matrices

$$\begin{bmatrix} \mathbb{1}_2 \\ x^{\alpha\dot\alpha} \end{bmatrix},$$

where the columns span the subspace. Likewise, a flag of type $2|0 \subset 2|N$ is given by the columns of a matrix

$$\begin{bmatrix} \mathbb{1}_{2\times 2} & 0_{2\times N} \\ x^{\alpha\dot\alpha} & \tilde\theta^{\dot\alpha}_i \\ \theta^\alpha_j & \mathbb{1}_{N\times N} \end{bmatrix}. \tag{13}$$

The first two columns span a $2|0$ subspace, while all columns span a $2|N$ subspace. We identify $M_{4|N}$ as the subset of $F(2|0,2|N;4|N)$ given by matrices of this form.[16] A null geodesic through x^a with tangent $\lambda^\alpha\tilde\lambda^{\dot\alpha}$ lifts to a super null geodesic of dimension $1|2N$. The super null geodesic through $x^a|\theta^\alpha_j,\tilde\theta^{\dot\alpha}_i$ with tangent $\lambda^\alpha\tilde\lambda^{\dot\alpha}$ comprises points of the form

$$x^a + tk^a \,|\, \theta^\alpha_j + \phi_j\lambda^\alpha,\, \tilde\theta^{\dot\alpha}_i + \psi_i\tilde\lambda^{\dot\alpha}.$$

All this data fits into a flag. The matrix, equation (13), can be enlarged to form

$$\begin{bmatrix} \lambda^\alpha & \mathbb{1}_{2\times 2} & 0_{2\times N} & 0_{2\times 1} \\ x^{\alpha\dot\alpha}\tilde\lambda_\alpha & x^{\alpha\dot\alpha} & \tilde\theta^{\dot\alpha}_i & \tilde\lambda^{\dot\alpha} \\ \theta^\alpha_j\lambda_\alpha & \theta^\alpha_j & \mathbb{1}_{N\times N} & \tilde\theta^{\dot\alpha}_i\tilde\lambda_{\dot\alpha} \end{bmatrix}.$$

[15] The coordinate rings of formal neighbourhoods can be considered to have 'odd coordinates'. The textbook example is the formal neighbourhood of a point $Spec(\mathbb{C}) \hookrightarrow Spec\,(\mathbb{C}[x])$ which is given by $Spec\,(\mathbb{C}[x]/x^2)$. Now x is an odd coordinate, squaring to zero, in the coordinate ring of the first neighbourhood of $Spec(\mathbb{C})$.

[16] Notice that we have been lead to introduce N pairs of odd-degree spinors as coordinates, as in the conventional description. This is implicit, for instance, in Chapter 4 of Wess and Bagger's book. [WessBagger]

The new column on the left represents a $1|0$ subspace inside the $2|0$ subspace. The new column on the right enlarges the $2|N$ subspace to a $3|N$ subspace. The flag spanned by the first and last columns is not altered if we add $\lambda^\alpha\tilde{\lambda}^{\dot\alpha}$ to $x^{\alpha\dot\alpha}$. Nor if we add λ^α to any θ^α_j or $\tilde{\lambda}^{\dot\alpha}$ to any $\tilde{\theta}^{\dot\alpha}_i$. The full flag $1|0 \subset 2|0 \subset 2|N \subset 3|N$ contains too much information, since it determines both a super null ray and a point on the ray. So, forgetting the particular point, we obtain super ambitwistor space, $\tilde{A}_{4|N}$, as the flag variety $F(1|0, 3|N; 4|N)$, which fits into the double fibration

$$F(1|0, 3|N; 4|N) \xleftarrow{p} F(1|0, 2|0, 2|N, 3|N; 4|N) \xrightarrow{q} F(2|0, 2|N; 4|N). \tag{14}$$

The fibres of p project, via q, to super null rays in $M_{4|N}$. By construction the tangent vectors to this distribution of super null rays are

$$\lambda^\alpha\tilde{\lambda}^{\dot\alpha}e_{\alpha\dot\alpha}, \qquad \lambda^\alpha e^j_\alpha, \qquad \text{and} \qquad \tilde{\lambda}^{\dot\alpha}e^i_{\dot\alpha},$$

where the frame vectors

$$e_{\alpha\dot\alpha} = \frac{\partial}{\partial x^{\alpha\dot\alpha}}, \quad e^j_\alpha = \frac{\partial}{\partial\theta^\alpha_j} - \tilde{\theta}^{\dot\alpha\, j}\frac{\partial}{\partial x^{\alpha\dot\alpha}}, \quad e^i_{\dot\alpha} = \frac{\partial}{\partial\tilde{\theta}^{\dot\alpha}_i}$$

satisfy

$$[e^j_\alpha, e^i_{\dot\alpha}] = \delta^{ij}e_{\alpha\dot\alpha}.$$

Given a vector bundle $\tilde{E}$ on $M_{4|N}$ with connection ∇, we see that $p_*q^*\tilde{E}$ exists if $\tilde{E}$ is integrable on the distribution of super null lines. Let $\nabla^i_\alpha = \nabla(e^i_\alpha)$, and so on. Then $\tilde{E}$ is integrable if,

$$[\lambda^\alpha\nabla^i_\alpha, \tilde{\lambda}^{\dot\alpha}\nabla^j_{\dot\alpha}] = \delta^{ij}\lambda^\alpha\tilde{\lambda}^{\dot\alpha}\nabla_{\alpha\dot\alpha},$$

while everything else commutes. These are the constraints for $D = 4$ super Yang-Mills. It is known that if $N = 3$, these constraints are precisely equivalent to the field equations. In other words, for $N = 3$ super Yang-Mills, the field equations are equivalent to the integrability of $\tilde{E}$ with respect to the double fibration of super null rays, equation (14). The $N = 3$ is significant for the following reason. The flag space $F(1|0, 3|N; 4|N)$ admits an embedding into $Gr(1|N; 4|N) \times Gr(3|N, 4|N)$. If $v^i|\xi^a_\alpha$ and $w_i|\chi^\alpha_a$ are homogeneous coordinates for these Grassmannians, then the Flag space is the quadric $v^iw_i + \xi\cdot\chi = 0$. But, for $N = 3$ we have $(\xi\cdot\chi)^4 = 0$. It follows that the even-degree functions on $F(1|0, 3|N; 4|N)$ are simply the functions on the third order neighbourhood of $F(1, 3; 4) \subset Gr(1; 4) \times Gr(3; 4)$. In this way, our result for super Yang-Mills implies the Yang-Mills result given in section 2.4.

3 A classical construction for $N = 1$ super Yang-Mills

In this section we derive the $D = 10$ super Yang-Mills equation from an integrability condition analogous to the one we encountered in four dimensions in section 2.6. $N = 1$ super spacetime $\tilde{M}$ is modelled on one chiral spinor bundle: $\tilde{M} = \Pi SM$. Given a bundle $\tilde{E}$ with connection ∇ and curvature F, the SYM Lagrangian, found by Brink, Schwarz, and Scherk [36] is

$$\mathcal{L} = \mathrm{tr}\left(-\frac{1}{4}F_{mn}F^{mn} + \frac{i}{2}\bar{\psi}\nabla\!\!\!/\,\psi\right),$$

which gives rise to the Yang-Mills and Dirac equations,

$$\nabla^a F_{ab} + \frac{1}{2}\Gamma_{a\alpha\beta}\psi^\alpha\psi^\beta = 0 \qquad \text{and} \qquad \Gamma_{a\alpha\beta}\nabla^a\psi^\beta = 0.$$

Now choose a frame on $\tilde{M}$ such that

$$[e_\alpha, e_\beta] = 2\Gamma^a_{\alpha\beta} e_a,$$

and let $\nabla_\alpha = \nabla(e^\mu)$, and so on. Then it can be shown that the super Yang-Mills equations are equivalent to the constraint equation[17]

$$[\nabla_\alpha, \nabla_\beta] = 2\Gamma^a_{\alpha\beta} D_a.$$

We emphasise here that brackets, [,], are to be regarded as graded commutators. In the present instance, since both of the derivatives are odd, the brackets denote an anti-commutator. A consequence of this is that, for any pure spinor λ^α,

$$[\lambda^\alpha \nabla_\alpha, \lambda^\beta \nabla_\beta] = 0. \tag{16}$$

This means that a super Yang-Mills connection is integrable on lines in $\tilde{M}$ tangent to pure spinors. The idea of our construction is to impose a condition of this kind on the Yang-Mills bundle obtained by pulling back to the ambitwistor space of $\hat{M}$. We will see that this is equivalent to the ordinary super Yang-Mills constraints in section 3.2. Following this, we give a number of homological computations which relate the $\mathcal{Q}$-cohomology of functions on $\hat{M}$ to sheaf cohomology groups on ambitwistor space.

3.1 Pure spinor ambitwistor space

We now construct an extended version of ambitwistor space. In the following section, we will demonstrate its relationship to $N = 1$ super Yang-Mills. The idea is to add to $\tilde{M}$ a bundle of pure spinors. So consider a bosonic spinor bundle $S \to \tilde{M}$. We will take a symplectic approach (similar to [37]). Consider, then, the total space of its cotangent bundle $T^*\mathbb{S}\tilde{M}$ which has a natural symplectic 2-form,

$$\omega = \mathrm{d}P_M \wedge \mathrm{d}Z^M + \mathrm{d}w_\alpha \wedge \mathrm{d}\lambda^\alpha,$$

where λ is a coordinate on the fibres of S and $Z^M = (X^m, \theta^\mu)$ are local coordinates on $\hat{M}$. We denote the pseudo-Poisson bracket associated to ω by [,]. There is a connection ∇ on S and we write, for example,

$$\nabla\lambda^\alpha = \mathrm{d}\lambda^\alpha + \mathrm{d}Z^M \Omega_{M\beta}{}^\alpha \lambda^\beta$$

See appendix B for conventions. In order to describe covariant derivatives as Poisson brackets we introduce

$$D_M = P_M - \Omega_{M\alpha}{}^\beta \lambda^\alpha w_\beta$$

so that, for example,

$$[D_M, V_\alpha \lambda^\alpha] = (\nabla_M V_\alpha)\lambda^\alpha.$$

We will constrain the spinors λ to be pure. To impose purity we introduce the constraints

$$\mathcal{K}^m = \frac{1}{2}\lambda\gamma^m\lambda,$$

which, as hamiltonians, generate

$$[\mathcal{K}^m, w_\alpha] = \gamma^m{}_{\alpha\beta}\lambda^\beta.$$

[17] Given that the connection satisfies the constraint, Witten showed in [19] that one can construct

$$\lambda^\alpha = \frac{1}{10}\Gamma^{a\alpha\beta} F_{a\beta}, \tag{15}$$

such that λ^α solves the Yang-Mills and Dirac equations. The converse direction was proved by Harnard and Schnider. [20]

So we should regard w_α as defined only up to the addition of $\not{V}\lambda$, for any V. After imposing purity we obtain a space which we call $F = T^*S_0\hat{M}$. We now describe super-null-geodesics that we lift (from $T^*\hat{M}$) to F such that the fermionic translations are tangent to the pure spinor λ. In the total space F, these have dimension $(1|1)$ and are generated by one bosonic and one fermionic hamiltonian:

$$\mathcal{H} = \frac{1}{2}P_a P_b \eta^{ab}, \quad \text{and} \quad \mathcal{Q} = \lambda^\alpha E_\alpha^M D_M.$$

For flat superspace these generate

$$[\mathcal{H}, Z^M] = \eta^{ab} P_a E_b^M$$
$$[\mathcal{Q}, Z^M] = \lambda^\alpha E_\alpha^M, \quad [\mathcal{Q}, w_\alpha] = E_\alpha^M P_M.$$

The hamiltonians are in involution since

$$[\mathcal{Q}, \mathcal{Q}] = \mathcal{K}^m P_m,$$

while all other Poisson brackets vanish. We call the reduction of F by $\mathcal{H}$ the pure-spinor ambitwistor space A_{ps}. We regard it as being a super manifold equiped with the fermionic derivation $[\mathcal{Q}, \ ]$. This derivation plays a key role in imposing the super Yang-Mills equations.

3.2 Integrability

In section 2.6, we described how the $D = 4$ super Yang-Mills constraints arise from the condition that a super Yang-Mills bundle on superspacetime $M_{4|N}$ can be pulled back to give a bundle on super ambitwistor space $A_{4|N}$. In this section, we do the same thing for ten dimensional super spacetime $\tilde{M}$ and pure spinor ambitwistor space A_{ps}. Let E be a super Yang-Mills bundle on $\hat{M}$ with connection ∇ and Lie group G. We will leave the Lie bracket implicit in our expressions.[18] Since A_{ps} can be presented as the reduction of $T^*S_0\hat{M}$ by constraints $\mathcal{Q}$ and $\mathcal{H}$, the pull back of the bundle E to $T^*S_0\hat{M}$ will push forward to A_{ps} if the curvature of ∇ vanishes on the pure spinor null rays. So we consider

$$\mathcal{Q}' = \lambda^\alpha E_\alpha^M \left(D_M + A_M\right),$$

and impose

$$[\mathcal{Q}', \mathcal{Q}'] = 0, \tag{17}$$

which is just the constraint, equation (16), that we encountered earlier. The constraints coming from $[\mathcal{Q}', \mathcal{H}]$ are too restrictive, and we consider instead the ansatze

$$\mathcal{H}' = \frac{1}{2}P^2 + P \cdot A + W^\alpha P_\alpha + U_\alpha^\beta \lambda^\alpha w_\alpha.$$

The relation

$$[\mathcal{Q}', \mathcal{H}'] = 0, \tag{18}$$

can be expanded to give

$$0 = \left(F_\alpha{}^a + T_{\alpha\beta}{}^a W^\beta\right) \lambda^\alpha P_a + \left(\nabla_\alpha W^\beta - U_\alpha{}^\beta\right) \lambda^\alpha d_\beta + \nabla_\gamma U_\alpha{}^\beta \lambda^\gamma \lambda^\alpha w_\beta.$$

[18] If so desired, it could be made explicit through the introduction of fields J^I whose Poisson brackets satisfy the Lie algebra relations—mimicking what is done in the heterotic string.

The third term vanishes if the second term vanishes, by the Bianchi identity. So, for flat superspace, the SYM equations are

$$F_{\alpha\beta}\lambda^\alpha\lambda^\beta = 0, \qquad F_{a\alpha} = \gamma_{a\alpha\beta}W^\beta, \qquad \nabla_\alpha W^\beta = U_\alpha{}^\beta.$$

We see that W^α (a superfield) contains the spinor satisfying the Dirac and Yang-Mills equations, equation (15). Given a solution to the first of these equations, the second two equations subsequently determine the super Yang-Mills superfields, W and U. So, by our opening discussion, the super Yang-Mills equations are equivalent to equations (17) and (18). We may state this as a pseudo-theorem.

Proposition. *A bundle E is super Yang-Mills if its pull back $\tilde{E}$ to A_{ps} satisfies* $[\mathcal{Q}', \mathcal{Q}'] = 0$.

The condition (18) shows that $\mathcal{Q}'$ defines a derivation on A_{ps} upon quotienting by $[\mathcal{H}', \ ]$. However, the interpretation of $[\mathcal{H}', \ ]$ remains somewhat unclear—see the discussion in section 3.4

3.3 Penrose transform

We would like to relate the cohomology on $\mathbb{A}_{ps}$ to groups defined on M. On the correspondence space F we can define sheaves $\mathcal{O}(n)_F^{[m]}$ whose sections are functions of degree n, m in the fibres of q. In other words, a section has weight n in P and weight m in λ. For fixed n, we have a short exact sequence of complexes,

$$0 \to \mathcal{O}(n)_A^{[\bullet]} \to \mathcal{O}(n)_F^{[\bullet]} \xrightarrow{\mathcal{H}} \mathcal{O}(n+1)_F^{[\bullet]} \to 0,$$

where the vertical derivatives are given by the action of $\mathcal{Q}$. The complexes of sheaves, $\mathcal{O}(n)_F^{[\bullet]}$, are exact.[19] This proves useful for the following reason. We have the following diagram, with exact columns,

$$\begin{array}{ccccccc}
 & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{O}(0)^{[0]} & \longrightarrow & \mathcal{O}(1)^{[0]} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{O}(0)^{[1]} & \longrightarrow & \mathcal{O}(1)^{[1]} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & &
\end{array}$$

[19] Let $\bar{\lambda}$ be a spinor such such $\lambda \cdot \bar{\lambda} = 1$, and suppose that $g \in \mathcal{O}(n)_F^{[m+1]}$ is $\mathcal{Q}$-closed. Using a Fourier decomposition, we restrict to the case that g is a momentum eigenstate with momentum k. In this case, $\mathcal{Q}$ acts as

$$\mathcal{Q} = \lambda^\alpha \frac{\partial}{\partial\theta^\alpha} + k \cdot \psi, \qquad \text{where} \qquad \psi^m = \lambda^\alpha\theta^\beta\gamma^m_{\alpha\beta}.$$

We claim that there exists $h \in \mathcal{O}(n)_F^{[m]}$ such that $\mathcal{Q}h = g$. Indeed, since $\mathcal{Q}g = 0$, we have

$$\lambda \cdot \partial_\theta g + k \cdot \psi g = 0.$$

Given that $\lambda \cdot \bar{\lambda} = 1$, we then solve $Qh = g$ by

$$h = \bar{\lambda} \cdot \theta\, g.$$

So, given a $\mathcal{Q}$-closed function $U^1 \in \mathcal{O}(0)_F^{[1]}$, this diagram shows that there exists some $V^0 \in \mathcal{O}(1)_F^{[0]}$ such that $\mathcal{Q}V^0 = \mathcal{H}U^1$. In practice, this relation can indeed be solved for momentum eigenstates (see below). As a $\mathcal{Q}$-cohomology class, U^1 is defined only up to the addition of $\mathcal{Q}$-exact forms. So the class $[U^1]$ defines a class $[V^0]$ in the cokernel of $\mathcal{H}$. That is, we identify

$$H^1_{\mathcal{Q}}(F, \mathcal{O}(0)_F) \simeq \mathrm{coker}(\mathcal{H}) \subset H^0(F, \mathcal{O}(1)_F).$$

Here, $H^i_{\mathcal{Q}}(F, \mathcal{O}(n)_F) = \ker \mathcal{Q}/\mathrm{im}\mathcal{Q}$, where $\ker \mathcal{Q} \subset \mathcal{O}(n)_F^{[i]}$. On the other hand, the long exact sequence

$$0 \to \mathcal{O}(n)_A \to \mathcal{O}(n)_F \xrightarrow{\mathcal{H}} \mathcal{O}(n+1)_F \to 0,$$

gives an isomorphism

$$H^1(A, \mathcal{O}(0)) \simeq \mathrm{coker}(\mathcal{H}) \subset H^0(F, \mathcal{O}(1)_F).$$

This isomorphism is derived in precisely the same fashion as described in section 2.5.1. (The key result is that $H^1(V, \mathcal{O}(1)) = 0$, where V is the quadric $P^2 = 0$ in the projective space $\mathbb{CP}^D$.) So, we find an isomorphism

$$\iota : H^1_{\mathcal{Q}}(F, \mathcal{O}(0)_F) \to H^1(A, \mathcal{O}(0)).$$

In what remains of this section, we describe this isomorphism explicitly. We begin on the right hand side with a $\mathcal{Q}$-cohomology class $[U^1]$ represented by some function in $\mathcal{O}(0)_F^{[1]}$,

$$U^1 = a_\alpha \lambda^\alpha.$$

Here, a can be regarded as a 1-form on $\hat{M}$ (or, if you prefer, a 'superfield'). (We regard it as taking values in $\mathfrak{g}$, though we suppress this in the formulas.) The requirement that U^1 is $\mathcal{Q}$-closed ($\mathcal{Q}U^1 = 0$) is the following linearised background-coupled super-Yang-Mills equation

$$\nabla_{(\alpha} a_{\beta)} \lambda^\alpha \lambda^\beta = 0. \tag{19}$$

Now we would like to find a function $V^0 \in \mathcal{O}(1)_F^{[0]}$ representing the class $[V^0] \in \mathrm{coker}(\mathcal{H})$. We will take the following ansatze for V^0 (recalling the results we found in the previous subsection)

$$V^0 = a_m P^m + w^\alpha P_\alpha + u_\alpha^\beta \lambda^\alpha w_\beta.$$

Here a, w, and u are forms—or 'superfields'—on $\hat{M}$. We impose the relation $\mathcal{Q}V^0 = \mathcal{H}U^1$, which will hold provided that the following equations are satisifed

$$\nabla_{[\alpha} a_{m]} = \gamma_{m\alpha\beta} w^\beta, \qquad \nabla_\alpha w^\beta + u_\alpha^\beta = 0, \qquad \text{and} \qquad \nabla_\alpha u_\beta^\gamma \lambda^\alpha \lambda^\beta = 0. \tag{20}$$

These are the linearised super-Yang-Mills equations. In principle, given an explicit choice of a_α, we could determine all of a_m, w^α, and u_α^β from the linearised equations—equations (19) and (20)—appearing here. For instance, suppose that a_α is a momentum eigenstate with momentum k. Then the final step of the isomorphism gives the Dolbeault representative

$$\iota(U^1) = \bar{\delta}(k \cdot P) V^0 \in H^1(A, \mathcal{O}(0)).$$

As a by-product of the construction, we notice that this class $[\iota(U^1)]$ is $\mathcal{Q}$-closed. It is not, however, a representative of a $\mathcal{Q}$ cohomology class—since the isomorphism does not permit us to add to $\iota(U^1)$ any $\mathcal{Q}$-exact $(0,1)$-form. We have thus derived the following.

Proposition. *The presentation above defines a homomorphism*

$$\iota : H^1_{\mathcal{Q}}(F, \mathcal{O}(0)_F) \to H^1(A, \mathcal{O}(0)),$$

such that the image of ι is closed under the action of $\mathcal{Q}$ on $\mathcal{O}(0)_A^{[0]}$. In terms of fields on space-time, this establishes a correspondence between $\mathcal{Q}$-closed classes in $H^1(A, \mathcal{O}(0))$ and linearised super-Yang-Mills fields.

3.4 Discussion

In section 3.2 we formulated $D = 10$ super-Yang-Mills as an integrability condition on supersymmetric ambitwistor space. We used this to find, in section 3.3, a new Penrose transform relating $H^1(A, \mathcal{O}(0))$ to linearised super-Yang-Mills fields on spacetime. An important question concerns the construction in section 3.2. There we saw that it was important to consider the quotient of the correspondence space F by the derivation $[\mathcal{H}',\ ]$, where $\mathcal{H}'$ is a deformation of the hamiltonian $\mathcal{H} = P^2/2$. We do not have a good geometric understanding of $[\mathcal{H}',\ ]$. In Ward's original construction for bosonic ambitwistor space, a modification of $\mathcal{H}$ is not needed. I believe the modified $\mathcal{H}$ appears naturally in our present context on the grounds that $\hat{M}$ is a supermanifold—but the details are unclear. To this end, it might be helpful to answer the following questions. These do not directly involve the pure spinor bundle we have been discussing, but are interesting in their own right and do not appear to be answered in the literature.[20]

i. Show that the supermanifold A obtained by quotienting $F = T_0^*\hat{M}$ by $[\mathcal{H},\ ]$ is not split.
ii. Relate the splitting obstruction groups of A to spacetime fields on $\hat{M}$ (i.e. an abstract penrose transform).
iii. Compute all the cohomology groups of A with values in sheaves of homogeneous functions on the fibres of $F \to \hat{M}$.

The benefit of these questions is that they can be precisely formulated—and one expects that work along these lines would clarify my earlier questions as well.

4 A classical construction for IIB supergravity

We now turn to type IIB supergravity. In this section we find that the constraints for IIB supergravity are equivalent to integrability for a twice extended ambitwistor space. A similar result holds for IIA, though these calculations are not sufficiently different to warrant a separate discussion. For clarity, we work exclusively with IIB. Similar to our results in section 3, the idea is to consider two auxiliary bundles of pure spinors. The IIB constraints were first given in [39] and were rederived, in a slightly different form, from the pure spinor superstring in [16]. The constraints we find match those given in [16]. To begin, lets describe the pure spinor ambitwistor space—analogous to section 3.1. Consider two spin bundles of the same chiral representation on superspace, $(\mathbb{S} \oplus \mathbb{S})\hat{M}$. On the cotangent bundle we take a symplectic form

$$\omega = \mathrm{d}P_M \mathrm{d}Z^M + \mathrm{d}w_\alpha \mathrm{d}\lambda^\alpha + \mathrm{d}w_{\hat{\alpha}} \mathrm{d}\lambda^{\hat{\alpha}}.$$

We emphasise here that Z^M denotes local coordinates of type $(10|16, 16)$. That is, $Z^M = (x^m, \theta^\alpha, \hat{\theta}^{\dot{\alpha}})$. We introduce two pure spinor translations,

$$\mathcal{Q} = \lambda^\alpha E^M_\alpha d_M \text{ and } \hat{\mathcal{Q}} = \lambda^{\hat{\alpha}} E^M_{\hat{\alpha}} d_M.$$

Here, d_M implements covariant derivatives under the Poisson bracket and is given by

$$d_M = P_M - \Omega_{M\alpha}{}^\beta \lambda^\alpha w_\beta - \Omega_{M\hat{\alpha}}{}^{\hat{\beta}} \lambda^{\hat{\alpha}} w_{\hat{\beta}}.$$

When the connection is curved we find

$$[d_M, d_N] = R_{\gamma MN}{}^\delta \lambda^\gamma w_\delta + \hat{R}^{\hat{\delta}}_{\hat{\gamma} MN} \lambda^{\hat{\gamma}} w_{\hat{\delta}}.$$

[20] Though relevant results appear in Eastwood and LeBrun. [38]

When convenient we abbreviate

$$R_{MN} \equiv R_{\gamma MN}{}^{\delta}\lambda^{\gamma}w_{\delta}, \quad \text{and} \quad \hat{R}_{MN} \equiv \hat{R}^{\hat{\delta}}_{\hat{\gamma}MN}\lambda^{\hat{\gamma}}w_{\hat{\delta}}.$$

Given this, we compute

$$[\mathcal{Q},\mathcal{Q}] = \lambda^{\alpha}\lambda^{\beta}\left(T_{\alpha\beta}{}^{M}d_M + R_{\alpha\beta} + \hat{R}_{\alpha\beta}\right).$$

Likewise,

$$[\mathcal{Q},\hat{\mathcal{Q}}] = \lambda^{\alpha}\lambda^{\hat{\beta}}\left(T_{\alpha\hat{\beta}}{}^{M}d_M + R_{\alpha\hat{\beta}} + \hat{R}_{\alpha\hat{\beta}}\right).$$

The vanishing of these two Poisson brackets gives the following integrability constraints

$$T_{\alpha\beta}{}^{M}\lambda^{\alpha}\lambda^{\beta}, \quad R_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}, \quad \hat{R}_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}, \quad T_{\alpha\hat{\beta}}{}^{M}d_M, \quad R_{\alpha\hat{\beta}}, \quad \hat{R}_{\alpha\hat{\beta}}. \tag{21}$$

The curved massless constraint is

$$\mathcal{H} = \frac{1}{2}P^2 = \frac{1}{2}P_M P_N E^M_m E^N_n \eta^{mn}.$$

We demand that this is involution with $\mathcal{Q}$ (and similarly with $\hat{\mathcal{Q}}$).

$$[\mathcal{Q},\mathcal{H}] = T_{\alpha m}{}^{n}\lambda^{\alpha}P^m P_n + T_{m\alpha}{}^{\beta}\lambda^{\alpha}P^m d_{\beta} + T_{m\alpha}{}^{\hat{\beta}}\lambda^{\alpha}P^m d_{\hat{\beta}} - R_{m\alpha}\lambda^{\alpha}P^m - \hat{R}_{m\alpha}\lambda^{\alpha}P^m.$$

This yields a number of constraints:

$$T_{\alpha(mn)}, \quad T_{m\alpha}{}^{\beta}, \quad T_{m\alpha}{}^{\hat{\beta}}, \quad R_{m\alpha}\lambda^{\alpha}, \quad \hat{R}_{m\alpha}\lambda^{\alpha}. \tag{22}$$

As it stands, we have obtained the $N=2$ supergravity constraints truncated by setting the 3-form H and all the superfields to zero. Except for the first constraint, all of these will be modified once we deform $\mathcal{Q}$ and $\mathcal{H}$ to include superfields and the B-field.

4.1 IIB as integrability

We now seek to recover the non-linear supergravity constraints from integrability equations. We will consider an ansatse for deformed constraints $\mathcal{Q}', \hat{\mathcal{Q}}', \mathcal{H}'$, by adding terms of the appropriate weights. The fields that appear in these terms will turn out to be the supergravity superfields. We take a 2-form B-field to be given, and, in fact, the relations we find may be considered to 'descend' from the B-field, as we explain at the end. Since the computations are cumbersome, we do no more than summarise the results. As our ansatse for the deformed constraints we take

$$\mathcal{Q}' = \mathcal{Q} + V^{1,0}, \quad \hat{\mathcal{Q}}' = \hat{\mathcal{Q}}' + V^{0,1}, \quad \mathcal{H}' = \mathcal{H} + V^{0,0},$$

where the new terms take the form

$$V^{0,0} = P^{\alpha\hat{\alpha}}d_{\alpha}d_{\hat{\alpha}} + C_{\alpha}{}^{\beta\hat{\gamma}}\lambda^{\alpha}w_{\beta}d_{\hat{\gamma}} + S_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}}\lambda^{\alpha}\lambda^{\hat{\alpha}}w_{\beta}w_{\hat{\beta}} \\ + P^{\alpha\hat{\beta}}B_{\hat{\beta}m}d_{\alpha}P^m + C_{\alpha}{}^{\beta\hat{\gamma}}B_{\hat{\gamma}m}\lambda^{\alpha}w_{\beta}P^m + \text{hatted},$$

and,

$$V^{0,1} = B_{m\hat{\alpha}}P^m\lambda^{\hat{\alpha}} + P^{\alpha\hat{\beta}}B_{\hat{\beta}\hat{\alpha}}d_{\alpha}\lambda^{\hat{\alpha}} + C_{\alpha}{}^{\beta\hat{\gamma}}B_{\hat{\gamma}\hat{\alpha}}\lambda^{\alpha}w_{\beta}\lambda^{\hat{\alpha}}.$$

The function $V^{1,0}$ is similar to $V^{0,1}$. We impose that the deformed constraints Poisson commute. Consider first $[\hat{\mathcal{Q}}', \mathcal{H}'] = 0$. We find that the terms involving $P^{\alpha\hat{\alpha}} d_\alpha \lambda^\beta P^m$ combine to impose

$$T_{\hat{\alpha}\hat{\beta}m} + H_{\hat{\alpha}\hat{\beta}m} = 0.$$

Similarly,

$$T_{\hat{\alpha}\beta m} - H_{\hat{\alpha}\beta m} = 0,$$

follows from looking at $P^{\alpha\hat{\alpha}} d_\alpha \lambda^\beta P^m$. The coefficient of $P^m \lambda^\alpha D_\beta$ gives

$$T_{m\alpha}{}^{\beta} + T_{\alpha\hat{\alpha}m} P^{\beta\hat{\alpha}},$$

which modifies the $T_{m\alpha}{}^{\beta}$ constraint that we obtained in the previous section. We can take a derivative to find (using the Bianchi identity)

$$R_{m\alpha}\lambda^\alpha + T_{\alpha\hat{\alpha}m} C_\beta{}^{\gamma\hat{\alpha}} \lambda^\beta w_\gamma,$$

which also follows from the coefficient of $P\lambda\lambda w$ in $[\hat{\mathcal{Q}}', \mathcal{H}'] = 0$. The superfield equations of motion (see appendiix C for an explicit list) follow from $[\hat{\mathcal{Q}}', \mathcal{H}'] = 0$ by imposing Q-closure of the four terms in $V^{0,0}$ that do not involve the B-field. Next, we turn our attention to $[\hat{\mathcal{Q}}', \hat{\mathcal{Q}}'] = 0$, which earlier yielded the integrability constraints (22). The torsion constraint is now modified. For instance,

$$T_{\alpha\beta}{}^{\hat{\gamma}} - \frac{1}{2} H_{\alpha\beta\gamma} P^{\gamma\hat{\gamma}}.$$

We also have the derivatives of these (which can be found from the Bianchi identity and the superfield equations). Finally, the relation $[\hat{\mathcal{Q}}', \hat{\mathcal{Q}}'] = 0$ implies that $\mathcal{Q}V^{0,1} = \hat{\mathcal{Q}}V^{1,0}$. Indeed, defining

$$V^{1,1} = B_{\alpha\hat{\alpha}} \lambda^\alpha \lambda^{\hat{\alpha}},$$

we see that $\mathcal{Q}V^{0,1} = \mathcal{H}V^{1,1}$, or $\hat{\mathcal{Q}}V^{1,0} = \mathcal{H}V^{1,1}$, are equivalent to the constraints

$$H_{\alpha\beta M} \lambda^\alpha \lambda^\beta, \qquad \text{and} \qquad H_{\alpha\hat{\alpha}M}.$$

The equations given here, together with the superfield equations in appendix C, are the IIB constraints given by [16]. So we arrive at the following pseudo-theorem.

Proposition. *The IIB supergravity constraints are equivalent to the nonlinear descent of the class $V^{1,1}$ associated to the B-field. (By which we mean, the relations $\mathcal{Q}V^{0,1} = \mathcal{H}V^{1,1}$ and $\hat{\mathcal{Q}}V^{1,0} = \mathcal{H}V^{1,1}$, together with the involutivity of all the $\mathcal{Q}', \hat{\mathcal{Q}}', \mathcal{H}'$.)*

This is similar to the manner in which the super-Yang-Mills equations arose from the class associated to A. The fundamental structure in both cases is the extended superspace and the integrability of null, pure-spinor lines.

4.2 Penrose transform

We would now like to relate cohomology classes on A_{ps} to fields appearing on superspacetime. In order to this, we can use many of the results we first derived in section 3.3. The main difference is that we now have two derivatives $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ which descend to the bundles defined on ambitwistor space. On the total space F, we can define bundles $\mathcal{O}(n)^{[i,j]}$, whose sections have weight i in λ and j in $\hat{\lambda}$. Given this, precisely the same reasoning as in section 3.3 suffices to show that, on the one hand,

$$H^{i+1}_{\mathcal{Q}}(F, \mathcal{O}(n)^{[\bullet,j]}_F) \simeq \operatorname{coker} \mathcal{H} \subset H^0(F, \mathcal{O}(n+1)^{[i,j]}_F),$$

and, in addition,

$$H^1(A,\mathcal{O}(n)^{[i,j]}) \simeq \operatorname{coker}\mathcal{H} \subset H^0(F,\mathcal{O}(n+1)_F^{[i,j]}).$$

(There are similar isomorphisms for the $\hat{\mathcal{Q}}$ cohomology classes.) These two isomorphisms include two very interesting cases. In this section we will discuss two groups. The first is

$$H^1_{\mathcal{Q}}(F,\mathcal{O}(1)^{[\bullet 0]}).$$

Classes, U^1, in this group correspond to linearised variations of the supergravity vierbein. We can regard U^1 as a linearised variation of $\mathcal{Q}$. The second group we consider is

$$H^2_{\mathcal{Q}}(F,\mathcal{O}(0)^{[\bullet 2]}),$$

whose classes correspond to linearised variations of the supergravity 4-form potential. In sum, we have the following proposition.

Proposition. *We have homomorphisms*

$$\iota_1 : H^1_{\mathcal{Q}}(F,\mathcal{O}(1)^{[\bullet 0]}) \to H^1(A,\mathcal{O}(1)^{[0,0]}) \qquad \textit{and} \qquad \iota_2 : H^2_{\mathcal{Q}}(F,\mathcal{O}(0)^{[\bullet 2]}) \to H^1(A,\mathcal{O}(0)^{[1,2]}),$$

such that the image of ι is $\mathcal{Q}$-closed. In terms of linearised fields, this establishes a correspondence between

i. $H^1(A,\mathcal{O}(1)^{[0,0]})$ *and linearised variations of the vierbein,*

ii. $H^1(A,\mathcal{O}(0)^{[1,2]})$ *and linearised variations of the 4-form potential.*

Let us now describe ι_1 explicitly. A variation, H, of the veirbein gives a variation of $\mathcal{Q}$,

$$U^1 = \lambda^\alpha (H_\alpha^m P_m + H_\alpha^\beta P_\beta + \Omega_\alpha + \text{hatted})$$

Imposing that U^1 is $\mathcal{Q}$-closed is equivalent to imposing the linearised integrability equations on H. These equations are given explicitly in appendix C). We would like to find $\iota_1(U^1)$. The first step is to find the function $V^0 \in H^0(F,\mathcal{O}(n+1)_F^{[i,j]})$ which is in the cokernel of $\mathcal{H}$. The function is related to a representative of U^1 by the relation

$$[\mathcal{Q},V^0] = [\mathcal{H},U^1]. \tag{23}$$

To solve this, consider an ansatze for V^0,

$$V^0 = P^a P^b H_{ab} + P^a H_a^\alpha P_\alpha + P^a \Omega_a + \mathcal{P}^{\alpha\hat{\alpha}} P_\alpha P_{\hat{\alpha}} + \hat{\mathcal{C}}^\alpha P_\alpha + \mathcal{C}^{\hat{\alpha}} P_{\hat{\alpha}} + \mathcal{S}$$

The caligraphic letters are arbitrary superfields that we will determine shortly. Notice that our choice is highly motivated by the non-linear construction given in the previous section. Given this ansatze, equation (23) expands to give the following expression,

$$\begin{aligned}
\mathcal{Q}V^0 - \mathcal{H}V^1 = {}& \lambda^\alpha P^m P^n \left(\nabla_{[\alpha} H_{m]n} + \gamma_{b\alpha\hat{\alpha}} H_a^{\hat{\alpha}}\right) \\
&+ \lambda^\alpha P^a D_\beta \left(\nabla_{[\alpha} H_{a]}^\beta - \Omega_{a\alpha}{}^\beta + \gamma_{a\alpha\hat{\beta}} \mathcal{P}^{\beta\hat{\beta}}\right) + \lambda^\alpha P^a D_{\hat{\beta}} \left(\nabla_{[\alpha} H_{a]}^{\hat{\beta}}\right) \\
&+ \lambda^\alpha P^a \lambda^\beta w_\gamma \left(\nabla_{[\alpha} \Omega_{a]\beta}{}^\gamma + \gamma_{a\alpha\hat{\gamma}} \mathcal{C}_\beta{}^{\gamma\hat{\gamma}}\right) + \lambda^\alpha P^a \lambda^{\hat{\beta}} w_{\hat{\gamma}} \left(\nabla_{[\alpha} \Omega_{a]\hat{\beta}}{}^{\hat{\gamma}}\right) \\
&+ \lambda^\alpha P_\beta P_{\hat{\beta}} \left(\nabla_\alpha \mathcal{P}^{\beta\hat{\beta}} + \mathcal{C}_\alpha{}^{\beta\hat{\beta}}\right) + \lambda^\alpha \lambda^\beta w_\gamma P_{\hat{\gamma}} \left(\nabla_\alpha \mathcal{C}_\beta{}^{\gamma\hat{\gamma}}\right) \\
&+ \lambda^\alpha \lambda^{\hat{\beta}} w_{\hat{\gamma}} P_\gamma \left(\nabla_\alpha \mathcal{C}_{\hat{\beta}}{}^{\hat{\gamma}\gamma} + \mathcal{S}_{\alpha\hat{\beta}}{}^{\gamma\hat{\gamma}}\right) + \lambda^\alpha \left(\nabla_\alpha \mathcal{S}_{\beta\hat{\beta}}{}^{\gamma\hat{\gamma}}\right) \lambda^\beta w_\gamma \lambda^{\hat{\beta}} w_{\hat{\gamma}}
\end{aligned}$$

Each term appearing in this expression corresponds to a linearised supergravity equation. The derivation of the linearised supergravity equations is carried out in appendix C. We see, therefore, that the caligraphic letters $\mathcal{P}, \mathcal{C}, \mathcal{S}$, must be taken to correspond with the linearised supergravity superfields. Given this, equation (23) is satisfied identically and we have found our desired function V^0. Assuming that U^1 is a momentum eigenstate of momentum k, the final part of the isomorphism is easily implemented and gives

$$\iota_1(U^1) = \bar{\delta}(k \cdot P)V^0.$$

This is, once again, closely related to the vertex operators of the pure spinor ambitwistor string. We now describe the second isomorphism, ι_2. We begin with a class

$$U^{2,2} \in H^2_{\mathcal{Q}}(F, \mathcal{O}(0)^{[\bullet,2]}),$$

which is naturally associated to any variation b of the 4-form potential. That is,

$$U^{2,2} = b_{\alpha\beta\hat{\alpha}\hat{\beta}}\lambda^\alpha\lambda^\beta\lambda^{\hat{\alpha}}\lambda^{\hat{\beta}}.$$

This is $\mathcal{Q}$-closed because the only non-vanishing fermionic component of the 5-form field strength $H = \mathrm{d}b$ is

$$H_{abc\alpha\hat{\alpha}} = (\gamma_{abc})_{\alpha\hat{\alpha}}. \tag{24}$$

Implementing the first step of ι_2, we notice that the function

$$V^{1,2} = b_{a\alpha\hat{\alpha}\hat{\beta}}P^a\lambda^\alpha\lambda^{\hat{\alpha}}\lambda^{\hat{\beta}},$$

satisfies the relation

$$[\mathcal{Q}, V^{1,2}] = [\mathcal{H}, V^{2,2}].$$

Once again, this holds because of the constraint, equation (24), on the field strength H. Then, if $U^{2,2}$ is a momentum eigenstate with momentum k,

$$\iota_2(U^{2,2}) = \bar{\delta}(k \cdot P)V^{1,2}.$$

There is similarly a hatted version of this map.

4.3 Discussion

A possible application of the results given in this section is to find an ambitwistor interpretation of $D = 4$, $N = 8$ supergravity. By dimensional reduction, the field equations of type IIB supergravity give $N = 8$ supergravity in four dimensions. There is no ambitwistor construction for the $N = 8$ supergravity constraints. An approach along these lines can be taken for $N = 4$ super Yang-Mills as a reduction of $D = 10, N = 1$ super Yang-Mills. The ambitwistorial construction for this reduction is described in great detail in [20]. Moreover, in a somewhat different direction, there may be other applications of the pure spinor construction to the ambitwistor string programme. For instance, in the pure spinor ambitwistor string, the vertex operators factorise neatly into two parts—realising the perturbative double copy structure of gauge and gravity theories. It may be possible to realise a classical, non-perturbative version of this supersymmetric double copy. However, besides these two applications of our results, it would be good to answer the following three questions.

i. What are the cohomology groups $H^k(A, \mathcal{O}(n)^{[i,j]})$ for all n, i, j, k?

ii. Does deformation theory guarantee the existence of the pure spinor ambitwistor space in small regions of superspacetime? Is it possible to establish that the correspondence is stable under deformations?

iii. What is the homological interpretation of the relation $\langle \lambda^3\theta^5 \rangle = 1$, which has proved so important in the program to compute tree amplitude in the pure spinor superstring?

Let us finish by discussing the second question in greater detail. First, we will recall Le Brun's theorem for the bosonic case—recall theorem 1, given at the beginning of section 2. His key argument proceeds as follows. Take the fibration

$$A \xleftarrow{q} F \xrightarrow{p} M,$$

where F is the bundle of null quadrics over M, and let $Q_x \subset A$ be the direct image of $p^{-1}(x)$ under q. Then, if N is the normal bundle to Q_x in A, a computation (proposition 1.6.1 in Le Brun's thesis [30]) establishes that

$$H^1(Q_x, \mathcal{O}(N)) = 0, \qquad \text{and} \qquad \dim H^0(Q_x, \mathcal{O}(N)) = n,$$

where n is the dimension of M. By deformation theory, this establishes that Q_x fits into a dimension n family of quadrics which we may regard as being parameterised by $x \in M$. We should like to know if the same is true in the supersymmetric setting. Moreover, in the bosonic case, Le Brun gave an additional argument which shows that a family of spacetimes M gives rise to a corresponding family of ambitwistor spaces. He also established a correspondence between conformal metrics on M and contact structures on A. In the supersymmetric case, we should first like to establish the analogy of Le Brun's first result.[21] For the construction of $\mathbb{A}_{ps}$, M is a supermanifold, but the fibres of p are still bosonic. The fibres are given by the cartesian product of two isotropic grassmannians,

$$p^{-1}(x) \simeq Gr^0(1, 2n) \times Gr^0(n+1, 2n+2),$$

where the first factor is the projective quadric of null directions and the second factor is the space of pure spinors. The normal bundle N will now have both bosonic and fermionic parts, and we conjecture that (for the $N = 1$ case)

$$\dim H^1(Q_x, \mathcal{O}(N)) = 0, \qquad \text{and} \qquad \dim H^0(Q_x, \mathcal{O}(N)) = 2n \,|\, 2^{\frac{n}{2}-1},$$

where Q_x is the image of $p^{-1}(x)$ under q. This would establish that the submanifolds Q_x embedded in $\mathbb{A}_{ps}$ form a family of the same dimension as M. Work to this effect is ongoing. The most tractable case is when M is a split super-manifold, in which case we can regard it as a bosonic manifold with an exterior algebra of functions on the spin bundle.

Part Two—spin fields

Spin fields were introduced to the RNS superstring formalism by Freed, Martinec, and Shenker [41] in order to include the Ramond sector of the superstring in a vertex operator formalism. This approach to the superstring carries over to the ambitwistor string. For instance, a gravitino insertion can be represented by a Ramond vertex operator

$$\mathcal{V} = \left(\xi^\alpha S_\alpha \, e^{-\frac{\phi}{2}}\right) \left(\epsilon \cdot \tilde{\psi}\right) e^{ik\cdot X}.$$

Here, ξ^α is a polarisation and S_α is the spin field. ϕ is a free boson, and $e^{-\phi/2}$ is fermionic ghost in bosonised form. The $\tilde{\psi}^\mu$ are free fermions such that the bilinears $\tilde{\psi}^{[\mu}\tilde{\psi}^{\nu]}$ form a current algebra for $so(D)$. All of these fields are described in more detail in section 5. We have written

[21] Somewhat analogous work to this effect was carried out for $D = 11$ supergravity in ref. [40].

$\mathcal{V}$ in a factorised form, and will focus only on the first factor. The corresponding fixed bosonic factor is

$$\epsilon \cdot \psi \, e^{-\phi}.$$

Then—postponing details until later—the standard OPEs for these fields gives a correlator

$$\left\langle \left(\xi_1^\alpha S_\alpha \, e^{-\frac{\phi}{2}}\right) \left(\xi_2^\alpha S_\alpha \, e^{-\frac{\phi}{2}}\right) \left(\epsilon \cdot \psi e^{-\phi}\right) \right\rangle = \frac{(\xi_1 \epsilon \, \xi_2)}{\sigma_{12}\sigma_{23}\sigma_{31}}.$$

After accounting for conformal invariance on the worldsheet, this correlator gives rise to the correct three-point amplitude for supergravity.[22] To compute the correlator one needs to know that, for instance,

$$\langle S_\alpha S_\beta \psi^\mu \rangle = \frac{\gamma^\mu_{\alpha\beta}}{(\sigma_{12})^{\frac{3}{4}} (\sigma_{23})^{\frac{1}{4}} (\sigma_{31})^{\frac{1}{2}}}.$$

This formula admits easy generalisations to arbitrarily many ψ^μ and to other dimensions. In section 8, we use these generalised formulae to verify the ambitwistor string prescription for 1-loop supergravity amplitudes. More ambitiously, one could hope to provide closed formulas for the correlators of any number of Ramond and Neveu-Schwarz vertex operators in the ambitwistor string. This is difficult in ten dimensions for reasons similar to those we encountered in section 2.1. Namely, the dimension of the spin representations grows very quickly compared to the fundamental representation of $so(D)$. Nevertheless, we solve the problem in four dimensions—see section 6—and in six dimensions—see section 7. The key interest of these formulas is that they are manifestly gauge invariant with respect to shifts of the polarisation data (i.e. under $\epsilon \mapsto \epsilon + k$, where k is the momentum). Conjectured applications of these results are given in section 9. We begin in section 5 by constructing the spin fields explicitly, which we hope clarifies their otherwise mysterious OPEs. In particular, we derive the relations that we later rely on in sections 6 to 8.

5 Spin Fields

In this section we construct the spin fields, which realise the spin representations of $so(D)$ in their OPEs. This is closely related to the construction of current algebras on the worldsheet for the $so(D)$ Lie algebra, and this is where we begin. Throughout this section we specialise to even dimensions and write $D = 2n$. Let ψ^μ be D free fermions with the OPEs

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{\delta^{\mu\nu}}{z}. \tag{25}$$

One finds that the bilinears $J^{\mu\nu} =: \psi^\mu \psi^\nu :$ define a current algebra for $so(2n)$ of level 1.[23] We can express this result in a manner that mimicks the Cartan basis for $so(2n)$. Let $\epsilon_1, ..., \epsilon_n$ be a basis for the dual $\mathfrak{h}^*$ of a Cartan subalgebra $\mathfrak{h} \subset so(2n)$, so that the roots have weights $\pm\epsilon_i \pm \epsilon_j$. To each of these we associate the following worldsheet fermions

$$f^{\pm\epsilon_i} = \frac{1}{\sqrt{2}} \left(\psi^{2i-1} \mp i\psi^{2i}\right). \tag{26}$$

[22] There is a 'missing half' corresponding to the $\epsilon \cdot \tilde{\psi}$ appearing $\mathcal{V}$: but we do not show this since the amplitude factorises completely.

[23] That is to say, they have the following OPEs

$$J^{\mu_1\nu_1}(z)J^{\mu_2\nu_2}(0) \sim k\frac{\delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}}{z^2} + \frac{\delta^{\mu_1\mu_2}J^{\nu_1\nu_2} - \delta^{\mu_1\nu_2}J^{\nu_1\mu_2} - (\nu_1\mu_2)}{z},$$

with $k = 1$, and the numerator for the $1/z$ term is the usual Lie bracket relation.

Notice that the only non trivial OPEs are of the form

$$f^{+\epsilon_i}(z)f^{-\epsilon_i}(0) \sim \frac{1}{z}.$$

Then, associated to every root weight, we can associate the composite field

$$E^{\pm\epsilon_i\pm\epsilon_j} =: f^{\pm\epsilon_i}f^{\pm\epsilon_j} : .$$

These have OPEs which realise the 'raising' and 'lowering' relations in the sense that, e.g.,

$$E^{+\epsilon_i+\epsilon_j}(z)E^{-\epsilon_i+\epsilon_k}(0) \sim \frac{1}{z}E^{+\epsilon_i+\epsilon_k}.$$

On the other hand, we also have, e.g.,

$$E^{+\epsilon_i+\epsilon_j}(z)E^{-\epsilon_i-\epsilon_j}(0) \sim -\frac{1}{z^2} + \frac{1}{z}\left(J^{2i-1,2i} + J^{2j-1,2j}\right).$$

So we identify the worldsheet fields $J^{2i-1,2i}$ with a basis for the Cartan subalgebra, H^i.

5.1 Bosonisation and the spin fields

The spin fields are, heuristically, the 'square root' of the fermions $f^{\pm\epsilon_i}$. The reason that they are called spin fields is that their OPEs with ψ^μ realise the spin representation of $so(2n)$. In order to take the 'square root' of the fermions $f^{\pm\epsilon_i}$, we first put them in bosonised form. In place of $f^{\pm\epsilon_i}$ we could write

$$e^{\pm\phi_i},$$

for some free boson fields ϕ_i. By construction, these obey the same OPEs, equation (26), as before. However, they do not have the correct fermionic statistics because $e^{\pm\phi_i}$ commutes with $e^{\pm\phi_j}$ whereas $f^{\pm\epsilon_i}$ anti-commutes with $f^{\pm\epsilon_j}$. To recover the correct statistics we can add a factor of

$$c_i = (-1)^{n_1+\ldots+n_{i-1}} \qquad \text{or} \qquad c_i = e^{\pm i\pi(n_1+\ldots+n_{i-1})}.$$

Here, the n_j are number operators and we can give these explicitly as

$$n_i = \frac{1}{2\pi i}\oint \partial\phi_i.$$

On account of the number operators, $c_i e^{\pm\phi_i}$ and $c_j e^{\pm\phi_j}$ anti-commute. So we identify

$$f^{\pm\epsilon_i} = c_i e^{\pm\phi_i}.$$

We can take the square root of this, but not without an ambiguity in c_i due to the branching of the exponential. The most general choice is

$$S^{A_i} = e^{A_i\phi_i}e^{1\pi\sum_{j=1}^{n}A_iM_{ij}n_j},$$

where A_i is $\pm 1/2$ and M_{ij} is a matrix of signs with all zeroes on and above the diagonal. A basis for the spin representation in D dimensions can be identified with the vectors $A = (A_1, ..., A_n)$ with each A_i being $\pm 1/2$. Then we define the spin fields to be

$$S^A = \prod_{i=1}^{n} S^{A_i}.$$

Now we ask, what is the OPE of ψ^μ with S^A? We can write

$$\psi^{2j} = \frac{i}{\sqrt{2}}\left(f^{+\epsilon_j} - f^{-\epsilon_j}\right) \qquad \text{and} \qquad \psi^{2j-1} = \frac{1}{\sqrt{2}}\left(f^{+\epsilon_j} + f^{-\epsilon_j}\right).$$

So, for instance, the OPE of ψ^{2j-1} with $e^{\pm\phi_i/2}$ gives

$$\psi^{2j-1}(z)e^{\pm\phi_i/2}(0) \sim \frac{1}{\sqrt{2}}\frac{1}{\sqrt{z}}e^{\mp\phi_i/2}.$$

In other words, the effect of ψ^{2j-1} on S^A is to flip a sign in A. Likewise, ψ^{2j} flips a sign in A and gives a factor of i in addition. Reasoning in this way, we conclude that

$$\psi^\mu(z)S^A(0) \sim \frac{1}{\sqrt{2}}\frac{1}{\sqrt{z}}\left(\Gamma^\mu\right)^A_B S^B(0),$$

for some matrices $(\Gamma^\mu)^A_B$ with complex entries. Recalling the $\psi\psi$ OPE, equation (25), we have that

$$(\Gamma^\mu\Gamma^\mu + \Gamma^\mu\Gamma^\mu)^A_B = 2\delta^{\mu\nu}\delta^A_B,$$

and so these are gamma matrices for $so(2n)$.

5.2 Properties by dimension

Just as the action of ψ^μ flips a sign of A in S^A, the bilinears $\psi^{[\mu}\psi^{\nu]}$ flip two signs. We may thus decompose the S^A into the two chiral representations: S^α and $S^{\dot\alpha}$ where $A = \alpha$ has an even number of minus signs and $A = \dot\alpha$ has an odd number. Since ψ^μ flips one sign, we have

$$\psi^\mu(z)S^\alpha = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{z}}\left(\Gamma^\mu\right)^\alpha_{\dot\beta} S^{\dot\beta}(0). \tag{27}$$

The properties of the chiral representations depend on the dimension mod 4. Consider first $D = 0$ mod 4. Then, in the OPE of $S^\alpha S^\beta$ the most singular term occurs if $\alpha = -\beta$. This is possible because, in $D = 0$ mod 4, α and β are $D/2$ vectors with an even number of minus signs and an even number of plus signs. Using the definition of S^α we find

$$S^\alpha(z)S^\beta(0) \sim z^{-\frac{D}{8}}C^{\alpha\beta} + ...$$

where

$$C^{\alpha\beta} = \delta_{\alpha+\beta}e^{-i\pi\alpha\cdot M\cdot\alpha}.$$

We identify $C^{\alpha\beta}$ with the inner product on the chiral representations, or 'charge conjugation matrix'. This relation could be used to determine the matrix M in accordance with some convention for $C^{\alpha\beta}$. Using this, and performing contractions on equation (27), we find

$$S^\alpha(z)S^{\dot\beta}(0) \sim \frac{1}{\sqrt{2}}z^{-\frac{D-4}{8}}\left(\gamma^\mu\mathcal{C}\right)^{\alpha\dot\beta}\psi_\mu(0) + ...$$

Now we consider $D = 2$ mod 4. In this dimension, α, β are $D/2$ vectors with an even number of minus signs and an odd number of plus signs. This means that $\alpha + \beta$ can never be zero, whereas $\alpha + \dot\beta$ can. So, proceeding as before, we find

$$S^\alpha(z)S^{\dot\beta}(0) \sim\sim z^{-\frac{D}{8}}C^{\alpha\dot\beta} + ...,$$

where

$$C^{\alpha\dot\beta} = \delta_{\alpha+\dot\beta}e^{-i\pi\alpha\cdot M\cdot\alpha}.$$

Moreover, combining this with (27),

$$S^{\alpha}(z)S^{\beta}(0) = \frac{1}{\sqrt{2}} z^{-\frac{D-4}{8}} \left(\gamma^{\mu}\mathcal{C}\right)^{\alpha\beta} \psi_{\mu}(0) + ...$$

These are the relations first derived in [41]. This concludes our construction of the spin fields and their OPEs.

5.3 Relations between dimensions

The spin fields for all even dimensions are related to each other in the following sense. Given that S^A are the spin fields for $so(2n)$, we may fix $A_n = +1/2$. Then $(A_1, ..., A_{n-1})$ can be regarded as an index for $so(2n-2)$ spin fields. To arrive at the correct OPEs we could set $\phi_n = 0$, which alters all the OPEs by a factor of $z^{1/4}$. However, we cannot remove the number operator n_n from the coefficient,

$$e^{1\pi \sum_{j=1}^{n} A_i M_{ij} n_j}.$$

An example of particular interest is the relation of $D = 10$ to $D = 4$ and $D = 6$. We can write $A = (a_1, a_2, \alpha_1, \alpha_2, \alpha_3)$, such that a and α are indices for $so(4)$ and $so(6)$ spinors. The $D = 10$ spin field may then be partially factorised as

$$S^A = S^a S^{\alpha} e^{i\pi \sum_{i=3}^{5} \sum_{j=1}^{2} \alpha_i M_{ij} n_j}.$$

We have used that M_{ij} is zero on and above the diagonal. So the $D = 10$ correlators do not factorise as a simple product of $D = 4$ and $D = 6$ correlators. Nevertheless, the lower dimensional results should be related to the $D = 10$ result by dimensional reduction. In particular, fixing $\alpha = (1/2, 1/2, /12)$ and setting ϕ_3, ϕ_4, ϕ_5 to zero, we obtain the $D = 4$ spin fields from the $D = 10$ spin fields. The additional factor of

$$e^{\frac{i\pi}{2} \sum_{i=3}^{5} \sum_{j=1}^{2} M_{ij} n_j}$$

may be absorbed into the definition of the $D = 4$ matrix $C^{\alpha\beta}$.

6 Four dimensions

In this section we study correlators of the spin field CFT associated to dimension four. We find formulas for the correlators of arbitrarily many spin fields and $so(4)$ currents (the $J^{\mu\nu}$ appearing earlier). There are some constraints on the number of spin fields appearing, as we discuss. Perhaps the most interesting aspect of our formulas is that they can be summed in such a way which makes them manifestly Lie theoretic. What this means is explained in section 6.3. Our formulas could be contracted with momenta and polarisations. By themselves, they are insufficient to compute amplitudes. In particular, the formulas are not rational functions of the worldsheet positions, which means that they cannot be used as integrands in the CHY formula. However, they have the attractive property that they are manifestly gauge invariant. We derive our new formulas in section 6.2, following a review in section 6.1 of a previous result.

6.1 Spin field correlators

We first specialise the OPEs derived in section 5 to four dimensions. The OPE of two unprimed spin fields is

$$S_{\alpha}S_{\beta} \sim z^{-\frac{1}{4}} \epsilon_{\alpha\beta},$$

and likewise for $S_{\dot{\alpha}}S_{\dot{\beta}}$. The OPE $S_\alpha(z)S_{\dot{\beta}}(0)$ is not singular in $D = 4$, and so the spin correlators factorise into two parts according to chirality. Moreover, Wick's theorem gives immediately that

$$\left\langle \prod_{i=1}^{2M} S_{\alpha_i}(\sigma_i) \right\rangle = \frac{(-1)^M}{2^M M!} \sum_{\rho \in \mathfrak{S}_{2M}} |\rho| \prod_{i=1}^{M} \frac{\epsilon_{\alpha_{\rho(2i-1)}\alpha_{\rho(2i)}}}{\sqrt{\sigma_{\rho(2i-1)\rho(2i)}}}. \tag{28}$$

Here $|\rho|$ is the sign of the permutation ρ. In this way, all the $D = 4$ spin correlators are easily computed. The terms in the sum are in direct correspondence with undirected chords between $2M$ points on a circle. This formula has been vastly simplified by Schlotterer, Hartl, and Stieberger. [27] They claim that equation (28) may be rewritten as

$$\left\langle \prod_{i=1}^{M} S_{\alpha_i}(\sigma_i) S_{\beta_i}(\tau_i) \right\rangle = (-1)^M W^{\frac{1}{2}} \sum_{\rho \in \mathfrak{S}_M} |\rho| \prod_{i=1}^{M} \frac{\epsilon_{\alpha_i \beta_{\rho(i)}}}{\sigma_i - \tau_{\rho(i)}}, \tag{29}$$

where

$$W = \frac{\prod_{i,j=1}^{M} (\sigma_i - \tau_j)}{\prod_{m<n} \sigma_{mn}\tau_{mn}}.$$

Notice that this coincides with the previous formula for $M = 1$. The new formula has only $M!$ terms, each of which is in direct correspondence with a permutation on M points. As SHS point out, the terms appearing here are an over-complete basis since $(1/2, 0)^{\otimes(2M)}$ contains only

$$\frac{(2M)!}{M!(M+1)!}$$

scalars, which is strictly less than $M!$ for $M > 2$. It remains to prove the formula. This can be done inductively, and amounts to showing that their formula correctly factorises near the singularities where two points collide. There are two cases: (i) two σ's or two τ's collide, and (ii) a σ collides with a τ. Let's do case (i). Without loss of generality, consider $\sigma_{12} \to 0$. Fix a permutation ρ. Let ρ' be the permutation obtained by swapping $\rho(1)$ and $\rho(2)$. Notice that $|\rho| = |\rho'|$ since ρ' is obtained from ρ by an even number of flips. The key relation is

$$\epsilon_{\alpha_1,\beta_{\rho(1)}}\epsilon_{\alpha_2,\beta_{\rho(2)}} + \epsilon_{\alpha_1,\beta_{\rho'(1)}}\epsilon_{\alpha_2,\beta_{\rho'(2)}} = \epsilon_{\alpha_1,\alpha_2}\epsilon_{\beta_{\rho(1)},\beta_{\rho(2)}}. \tag{30}$$

It is on the basis of this identity that the formula factorises. We strip away a factor of

$$\langle S_{\alpha_1}(\sigma_1) S_{\alpha_2}(\sigma_2) \rangle = -\frac{\epsilon_{\alpha_1\alpha_2}}{\sqrt{\sigma_{12}}}.$$

This done, what remains is

$$(-1)^{M-1} W_*^{\frac{1}{2}} \sum_{\rho \in \mathfrak{S}_M/\mathbb{Z}_2} |\rho| \frac{\epsilon_{\beta_{\rho(1)}\beta_{\rho(2)}}}{\tau_{\rho(1)\rho(2)}} \prod_{i=3}^{M} \frac{\epsilon_{\alpha_i\beta_{\rho(i)}}}{\sigma_i - \tau_{\rho(i)}},$$

where

$$W_* = \frac{\prod(\sigma_1 - \tau_j)^2}{\prod(\sigma_1 - \sigma_j)^2} \frac{\prod(\tau_{\rho(1)} - \sigma_j)^2}{\prod(\tau_{\rho(1)} - \tau_j)^2} \times W_{red}.$$

To complete the factorisation computation, we must take the limit of, say, $\tau_1 \to \sigma_1$. (We could choose any other τ for this.) Notice that in this limit the factor W_* is only nonvanishing if $\rho(1) = 1$. So we restrict to those ρ satisfying $\rho(1) = 1$. W_{red} then becomes the W factor for the remaining $2M - 2$ spin fields. In this way, we recover the original expression but for $2M - 2$ spin fields. Case (ii) remains. However, this case is more straightforward and the details appear on page 16 of [27]. Together, these cases establish the result by induction.

6.2 Mixed NS and R correlators

We now employ the previous formula, equation (29), to derive new formulas for the correlators of NS and R insertions. Unfixed NS insertions will be constructed from $\psi(z)\psi(w)$ with $z \to w$, fixed NS insertions from $\psi(0)$, and fixed R insertions from $S_\alpha(0)$. As we have seen, the nonvanishing spin correlators in $D = 4$ involve F_{lh} LH fields and F_{rh} RH fields where $F_{lh} = F_{rh} = 0 \bmod 2$. The spin fields can be contracted using

$$\psi^\mu(0) = -\lim_{z\to 0} \frac{1}{\sqrt{2}} \overline{\sigma}^{\mu\,\alpha\dot{\alpha}} S_\alpha(z) S_{\dot{\alpha}}(0). \tag{31}$$

Every such contraction decreases F_{lh} and F_{rh} by one. We see that no matter how many contractions we perform, we always have $F_{lh} = F_{rh} \bmod 2$. There are, in general, two cases: $F_{lh} = 0 \bmod 2$ and $F_{lh} = 1 \bmod 2$. For the first case we will consider the correlator of N integrated NS insertions with $F_{lh} = 2m$ left-handed R insertions and $F_{rh} = 2m'$ right-handed. As an example of the second case we will consider N integrated NS insertions with one fixed NS insertion.

6.2.1 $F = 0$ mod 2

Using the results from the previous section, we have the following spin correlator in the general case,

$$\left\langle \prod_{i=1}^{N+M} S_{\alpha_i}(\sigma_i) S_{\beta_i}(\tilde{\sigma}_i) \prod_{j=1}^{N+M'} S_{\dot{\alpha}_i}(\tau_i) S_{\dot{\beta}_i}(\tilde{\tau}_i) \right\rangle$$
$$= (-1)^M W^{\frac{1}{2}} \tilde{W}^{\frac{1}{2}} \left(\sum |a| \prod_{i=1}^{N+M} \frac{\epsilon_{\alpha_i \beta_{a(i)}}}{\sigma_i - \tilde{\sigma}_{a(i)}} \right) \left(\sum |b| \prod_{i=1}^{N+M'} \frac{\epsilon_{\dot{\alpha}_i \dot{\beta}_{b(i)}}}{\sigma_i - \tilde{\sigma}_{b(i)}} \right).$$

Using equation (31), we find the following mixed correlator,

$$I = \left\langle \prod_{i=1}^{N} \psi^{\mu_i}(z_i) \psi^{\nu_i}(z_i) \prod_{j=N+1}^{M} S_{\alpha_i}(\sigma_i) S_{\beta_i}(\sigma_i) \prod_{k=N+1}^{M'} S_{\dot{\alpha}_i}(\tau_i) S_{\dot{\beta}_i}(\tilde{\tau}_i) \right\rangle$$
$$= \lim \frac{(-1)^{M-N}}{2^N} W^{\frac{1}{2}} \tilde{W}^{\frac{1}{2}} \sum |a||b| \prod_{i=N+1}^{N+M} \frac{\epsilon_{\alpha_i \beta_{a(i)}}}{\sigma_i - \tilde{\sigma}_{a(i)}} \prod_{i=N+1}^{N+M'} \frac{\epsilon_{\dot{\alpha}_i \dot{\beta}_{b(i)}}}{\sigma_i - \tilde{\sigma}_{b(i)}} \prod_{i=1}^{N} \frac{\sigma^{\mu_i}{}_{\beta_{a(i)}\dot{\beta}_{b(i)}} \sigma^{\nu_i \beta_i \dot{\beta}_i}}{(\sigma_i - \tilde{\sigma}_{a(i)})(\tau_i - \tilde{\sigma}_{b(i)})}.$$

Here, we are taking the limit $\sigma_i, \tau_i \to z_i$ and $\tilde{\sigma}_i, \tilde{\tau}_i \to z_i$ for all $i = 1, ..., N$. In this limit, we find that the prefactors become

$$W \to \prod_{i=1}^{N} (z_i - z_i) W_{red} \qquad \text{and} \qquad \tilde{W} \to \prod_{i=1}^{N} (z_i - z_i) \tilde{W}_{red},$$

where W_{red} is identical to the original expression for W, but restricted to $i = N+1, ..., M$. It follows that the only permutations which give a nonvanishing contribution in this limit are those which set either $a(i) = i$ or $b(i) = i$, but not both, for each $i = 1, ..., N$.[24] For any such nonvanishing permutation, the formula becomes a product of cycles. Some possible cycles are

[24] Permutations which set $a(i) = b(i) = i$ also vanish since this amounts to performing the contraction $k_i \cdot \epsilon_i$.

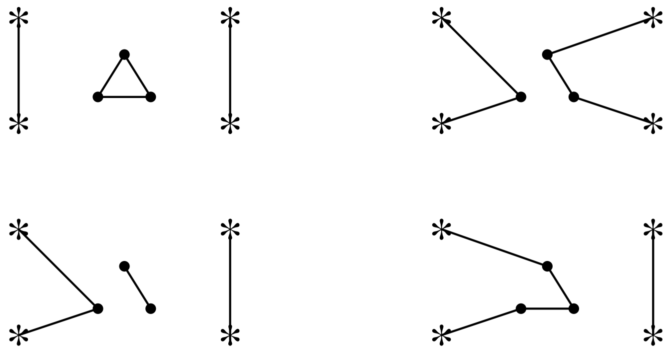

Figure 1: Some topologies for $N = 3$, $M = 2$, $M' = 2$. The asterisks are left and right-handed Ramond insertions. The bullets are NS insertions. The left-handed Ramond insertions appear to the left of the bosons, and the right-handed appear to the right.

shown in Figure 1, for the case $N = 3$, $M = 2$, $M' = 2$. Finally, by contracting with particle field strengths $F_i^{\mu_i\nu_i}$ we obtain the following cycle formula for the correlator,

$$I = (-1)^{M-N} W_{red}^{\frac{1}{2}} \tilde{W}_{red}^{\frac{1}{2}} \sum |\rho| \prod_{i=1}^{M} \frac{(\epsilon F_{[a_i]})_{\alpha_i\beta_{l(i)}}}{\sigma_{[a_i]}} \prod_{j=1}^{M'} \frac{(\bar{\epsilon} F_{[b_i]})_{\dot{\alpha}_i\dot{\beta}_{r(i)}}}{\sigma_{[b_i]}} \prod_{k=1} \frac{\mathrm{tr}(F_{(c_i)})}{\sigma_{(c_i)}},$$

where ρ has a cycle decomposition as $(a_i)(b_i)(c_i)$. I have adopted a notation so that (a_i) begins with α_i, (b_i) with $\dot{\alpha}_i$ and (c_i) is a closed cycle among the bosons. The summands can be grouped according to the maps $i \mapsto l(i)$ and $i \mapsto r(i)$ which are permutations of the left and right-handed fermions that respect the division into halves. Particularly interesting about these formulas is that the fermions of each chirality are further divided into two halves. The final result does not depend on this arbitrary division. This is a consequence of the Fierz identity, equation (30), as we showed in the proof of Hartl-Schlotterer's spin correlator formula. This gives rise to higher point consequences of the Fierz identities, which we illustrate graphically in figure 2 for the case $N = 1$, $M = 4$, $M' = 0$.

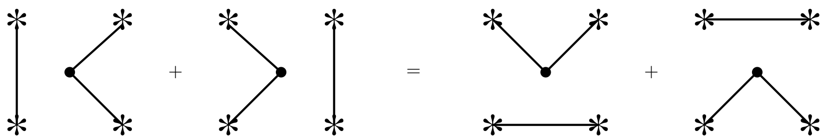

Figure 2: The manifestation of the Fierz identities for the case $N = 1$, $M = 4$, $M' = 0$. On the left, the fermions of each type are split vertically. On the right, the fermions of each type are split horizontally.

6.2.2 $F = 1$ mod 2

The effect of adding one fixed NS insertion is to provide a 'bridge' between the left-handed and right-handed fermions. To see how this works, suppose we combine the two spin fields located at σ_1 and τ_1. In the cycle representation of the correlator, we get the following new type of term

$$\frac{(...F_{a_1(2)}F_{a_1(1)}\mathcal{E}F_{b_1(1)}F_{b_1(2)}...)_{\beta_{f(1)}\dot{\beta}_{g(1)}}}{pt^{-1}(z_{f(1)}, ..., z_{g(1)})},$$

where $\mathcal{E}$ is the polarisation vector of the fixed insertion. The rest of the formula remains largely unchanged. Once again, the higher Fierz identities substantially reduce the number

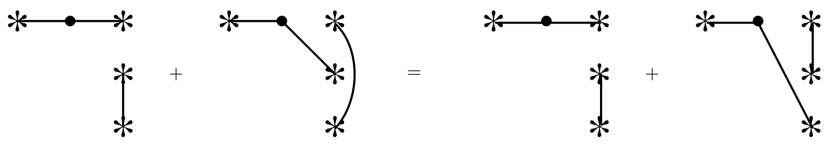

Figure 3: The manifestation of the Fierz identities for the case $N = 1$, $M = 1$, $M' = 3$.

of summands. For instance, of the M' right-handed fermions, one can choose an arbitrary $(M' - 1)/2$ subset which is never permitted to join the chain containing $\mathcal{E}$. Any two choices are equivalent, leading to many interesting relations. An example for $N = 1$, $M = 1$, $M' = 3$ is given in figure 3.

6.3 Shuffles

The formulas we have derived in this section involve the field strengths F_i. These are $so(D)$ Lie algebra elements. However, the products of field strengths that appear in the sums are not themselves $so(D)$ Lie algebra elements. In this section, we show that our formulas can be written manifestly in terms of Lie algebra elements. We begin by recalling some basic results about Lie polynomials. Generally, given some symbols $\{X_1, ..., X_n\}$, the free Lie algebra is the Lie algebra formally generated by these symbols. In concrete terms, its elements are formed by taking all possible Lie bracketings of the symbols. It is a sub-algebra of the free algebra generated by all possible words formed from the symbols. An element of the free algebra is like a polynomial in non-commuting variables. An element of the free algebra is called a Lie polynomial if it belongs to the free Lie algebra. For instance, consider a homogeneous polynomial of degree s,

$$F = \sum_{\alpha \in \mathfrak{S}_s} c(\alpha) X_{\alpha_1} .... X_{\alpha_s}.$$

The following theorem then determines a sufficient condition for F to be a Lie polynomial.

Theorem 5. *(Ree [42]) If the coefficients $c(\alpha)$ obey the shuffle identity, then F is a Lie polynomial.*

For two ordered disjoint set a, b, the set of shuffles is denoted $a \sqcup\!\sqcup\, b$. It comprises all orderings of $a \cup b$ that preserve the ordering of a and b repsectively. Then the 'shuffle identity' referred to in the theorem is the statement that

$$\sum_{\omega \in a \sqcup\!\sqcup b} c(\omega) = 0,$$

for all disjoint non-trivial partitions a, b of $\{1, ..., s\}$. Given that F is a Lie polynomial, we can apply the following theorem to write F in a way that makes it manifestly Lie. We introduce the following notation for consecutive right-sided Lie bracketings,

$$[1, 2, ..., s] = [[\, ... \,[[[1, 2], 3], 4], \, ... \,], s].$$

Then let $[F]$ denote the polynomial obtained from F be replacing every word $X_a...X_b$ with its Lie bracketing, $[X_a, ..., X_b]$.

Theorem 6. *(Dynkin-Specht-Wever) A homogeneous polynomial F of degree s is Lie iff $[F] = sF$.*[25]

[25] See, e.g., the book by Reutenauer on Free Lie Algebras for this result. [43]

Now we return to the new formulas that we have derived in this section. It is shown in appendix D that both the broken Parke-Taylor factors, $pt(\alpha)$, and the Parke-Taylor factors with one position fixed, $PT(*,\alpha)$, satisfy the shuffle identity. Combining the previous two theorems then allows us to write our correlator formulas in a way that is manifestly Lie. For instance, we have encountered the term

$$\frac{(F_{\alpha(1)} \dots F_{\alpha(k)}\epsilon)_{\alpha\beta}}{\sigma_{[a,\alpha,b]}}.$$

Here α and β are the spinor indices associated to the fermions at σ_a, σ_b. Since we sum over all permutations α, we can use Dynkin-Specht-Wever to replace this term with

$$\frac{1}{k}\frac{\left([F_{\alpha(1)}, \dots, F_{\alpha(k)}]\epsilon\right)_{\alpha\dot{\beta}}}{\sigma_{[a,\alpha,b]}}. \tag{32}$$

inside the sum. Likewise, consider the term

$$\frac{\mathrm{tr}\left(F_{\alpha(1)} \dots F_{\alpha(k)}\right)}{\sigma_{(\alpha)}}.$$

By cyclic invariance we can always move, say, F_1 to the first position. Then, in the permutation sum, we may replace this with

$$\frac{k}{k-1}\frac{\mathrm{tr}\left(F_1[F_{\alpha(2)}, \dots, F_{\alpha(k)}]\right)}{\sigma_{(\alpha)}}. \tag{33}$$

In the first case, equation (32) is a natural pairing $\langle u, Lv\rangle$ where $L \in so(D)$ is the Lie algebra element formed by the field strengths. v is a spinor polarisations in the chiral spinor representation and u is a spinor polarisation in the dual representation. Likewise, equation (33) is also such a pairing except that now $L \in so(D)$ is taken to act of the fundamental representation of $so(D)$ and its dual. Here u and v are given by the polarisation and momentum vectors of F_1.

7 Six dimensions

Having investigated four dimensions in the previous sections, there are several reasons to investigate six. One reason is that we are interested in the spin field CFT for ten dimensions, and $SO(10)$ spinors can be realised as tensor products and direct sums of $SO(4)$ and $SO(6)$ spinors. However, six dimensions is interesting for its own sake. For instance, the bi-adjoint scalar theory is conformally invariant in six dimensions, and its amplitudes can be computed using the ambitwistor string. [44] In this section, we derive formulas for six dimensions which are as exhaustive as those we found in four. We present the new formulas in section 7.2, following a review of the earlier results on spin field correlators in section 7.1

7.1 Spin field correlators

For six dimensions, the spin fields have the following OPEs,

$$S_\alpha(z)S^\beta(0) \sim z^{-\frac{3}{4}}\delta_\alpha^\beta,$$

$$S_\alpha(z)S_\beta(0) \sim \frac{1}{\sqrt{2}}z^{-\frac{1}{4}}\left(\gamma^\mu \mathcal{C}\right)_{\alpha\beta}\,\psi_\mu(0).$$

(This follows directly from section 5.2.) Hartl and Schlotterer [27] present a formula for all non-vanishing spin correlators in $D = 6$. In this section, I comment on its proof. We will use it to obtain new results in the next section. Their formula is

$$\left\langle \prod_{i=1}^{N} S_{\alpha_i}(\sigma_i) S^{\beta_i}(\tilde{\sigma}_i) \right\rangle = W^{1/4} \sum_{\rho} |\rho| \prod_{i=1}^{N} \frac{\delta_{\alpha_i}{}^{\beta_{\rho(i)}}}{\sigma_i - \tilde{\sigma}_{\rho(i)}},$$

where, as before,

$$W = \frac{\prod_{i,j=1}^{N} (\sigma_i - \tilde{\sigma}_j)}{\prod_{i<j}^{N} \sigma_{ij} \tilde{\sigma}_{ij}}.$$

The formula clearly holds for $N = 1$, since the relevant OPE in $D = 6$ is

$$S_\alpha(z) S^\beta(0) \sim z^{-\frac{3}{4}} \delta_\alpha{}^\beta.$$

To prove the formula, we need only consider the singularities for $\sigma_i - \tilde{\sigma}_j \to 0$. This is much simpler than the $D = 4$ case. It suffices to consider $i = N$, $j = N$ (due to permutation symmetry). Then the key observation is that, in this singular limit,

$$W \to (\sigma_n - \tilde{\sigma}_n) \frac{\prod_{i,j=1}^{N-1} (\sigma_i - \tilde{\sigma}_j)}{\prod_{i<j}^{N-1} \sigma_{ij} \tilde{\sigma}_{ij}} \frac{\prod_{i=1}^{N-1} (\sigma_i - \tilde{\sigma}_N)(\sigma_N - \tilde{\sigma}_i)}{\prod_{i=1}^{N-1} (\sigma_i - \sigma_N)(\tilde{\sigma}_N - \tilde{\sigma}_i)}.$$

It suffices to notice that

$$\frac{\prod_{i=1}^{N-1} (\sigma_i - \tilde{\sigma}_N)(\sigma_N - \tilde{\sigma}_i)}{\prod_{i=1}^{N-1} (\sigma_i - \sigma_N)(\tilde{\sigma}_N - \tilde{\sigma}_i)} = 1 + \mathcal{O}(\sigma_n - \tilde{\sigma}_n),$$

and this completes the proof.

7.2 Mixed NS and R correlators

We perform contractions on Hartl-Schlotterer's formula to obtain closed formulas for mixed correlators. Compared to our counting in section 6, $D = 6$ is simpler. There is no longer a chiral division of the fermions. We can have any even number of R insertions in the same representation. Finally, we recall that the $D = 6$ OPEs give, for instance,

$$\psi^\mu(0) = + \lim_{z \to 0} 2^{-\frac{3}{2}} z^{\frac{1}{4}} \left(\mathcal{C}^{-1} \bar{\gamma}^\mu \right)^{\alpha\beta} S_\alpha(z) S_\beta(0). \tag{34}$$

We use this to perform the contractions. We will consider the case of N bosons and $2M$ fermions in the same representation. As we will see, it is necessary for M of the bosons to be fixed NS insertions, while the remaining $N - M$ are unfixed. To arrive at our formula for this mixed correlator, we will begin with the Hartl-Schlotterer formula for $2N$ spin fields of each type. Contracting the first $2(N - M)$ of these and taking appropriate limits gives the following formula which I obtained in CFC section 6,

$$\begin{aligned} I &= \left\langle \prod_{i=1}^{N-M} F^i_{\mu_i \nu_i} \psi^{\mu_i}(z_i) \psi^{\nu_i}(z_i) \prod_{j=1}^{2M} S_{\alpha_j}(\sigma_j) S^{\dot{\beta}_j}(\tilde{\sigma}_j) \right\rangle \\ &= 2^{N-M} \left(\frac{\prod_{i,j} (\sigma_i - \tilde{\sigma}_j)}{\prod_{i<j} \sigma_{ij} \tilde{\sigma}_{ij}} \right)^{\frac{1}{4}} \sum_{\rho \in \mathfrak{S}_{N+M}} |\rho| \prod_{i=1}^{2M} \frac{\mathcal{C}_{\alpha_i}{}^{\dot{\beta}_{\rho(i)}}}{\sigma_i - \tilde{\sigma}_{\rho(i)}} \prod_{i=1}^{N-M} \frac{\left(\mathcal{C}^{-1} F_i \mathcal{C} \right)_{\dot{\beta}_i}{}^{\dot{\beta}_{\rho(2M+i)}}}{z_i - \tilde{\sigma}_{\rho(2M+i)}}, \end{aligned}$$

where it is understood that $\tilde{\sigma}_{2M+i} = z_i$ for $1 \leq i \leq N-M$. We proceed further by taking the limit in which $\sigma_{2i-1}, \sigma_{2i} \to y_i$, and using (34). Contracting with some polarisation data we are, on the LHS, computing

$$I = \left\langle \prod_{i=1}^{N-M} F^i_{\mu_i \nu_i} \psi^{\mu_i}(z_i) \psi^{\nu_i}(z_i) \prod_{j=1}^{2M} S^{\dot{\beta}_j}(\tilde{\sigma}_j) \prod_{k=1}^{M} \mathcal{E}_k \cdot \psi(y_k) \right\rangle .$$

We find, on the RHS,[26]

$$\sum_{\rho \in \mathfrak{S}_{N+M}} |\rho| \prod_{i=1}^{N-M} \frac{\left(\mathcal{C}^{-1} F_i\, \mathcal{C}\right)_{\dot{\beta}_i}{}^{\dot{\beta}_{\rho(2M+i)}}}{z_i - \tilde{\sigma}_{\rho(2M+i)}} \prod_{i=1}^{M} \frac{(\mathcal{E}_i\, \mathcal{C})^{\dot{\beta}_{\rho(2i-1)} \dot{\beta}_{\rho(2i)}}}{(y_i - \tilde{\sigma}_{\rho(2i-1)})(y_i - \tilde{\sigma}_{\rho(2i)})},$$

where

$$W = \frac{\prod_{i=1}^{M} \prod_{j=1}^{2M} (y_i - \tilde{\sigma}_j)^{1/2}}{\prod_{i \leq j} \tilde{\sigma}_{ij}^{1/4} \prod_{i<j}^{M} y_{ij}} .$$

We have dropped the distracting powers of 2. Fix a permutation ρ. The corresponding summand is a product of two types of cycles: closed cycles containing only bosons and open cycles connecting two fermions. The index structure imposes that

i. every closed cycle contains no fixed bosons,

ii. every open cycle contains an odd number of fixed bosons.

A counting argument then establishes the 'one fixed boson lemma': every fermion chain contains one and only one $\mathcal{E}_i$. Moreover, we claim that a fermion chain only contributes when its $\mathcal{E}_i$ is at the beginning or end of the chain.[27] Given these results, the contributing topologies at low points are shown in figure 4. This leads us also to a cycle representation of the formula,

$$I = (-1)^M W \sum_{\rho} |\rho| \prod_{i} \frac{\xi_{l(i)} \mathcal{E} F \ldots \mathcal{C} \xi_{r(i)}}{\sigma_{[a_i]}} \prod_{j} \frac{\mathrm{tr} F_{(b_i)}}{\sigma_{(b_i)}} .$$

[26] From this calculation we see that we could not have introduced further integrated vertex operators without encountering singularities that cannot be removed—except perhaps by bubbling off some of the $S^{\dot{\alpha}}$ spin fields.

[27] Our claim is that if $\mathcal{E}$ appears in the middle of a fermion chain in a particular summand this will not contribute to the final result of the permutation sum. We use,

$$\frac{1}{[1234]} + \frac{1}{[1324]} = \frac{\sigma_{14}}{[124][134]} .$$

We also need to recall that $(\gamma \mathcal{C})$ and $(\gamma^2 \mathcal{C})$ are anti-symmetric in $D = 6$, while $(\gamma^3 \mathcal{C})$ is symmetric. In the middle of the fermion chain we have, for some fixed permutation, the term

$$(\ldots) \frac{F_1 F_2 E_3 F_4}{[1234][32]} (\ldots).$$

We can consider also the term obtained from this by swapping 2 and 3. This gives a relative sign and we find, inside the permutation sum,

$$(\ldots) \left[\frac{F_1 F_2 E_3 F_4}{[1234][32]} - \frac{F_1 E_3 F_2 F_4}{[1324][23]} \right] (\ldots).$$

The spinorial indices can be rearranged such that $F_1 E_3 F_2 F_4 = F_1 F_2 E_3 F_4$ and $F_1 F_2 E_3 F_4 = F_4 F_2 E_3 F_1$. The bracketed expression is then

$$\frac{1}{[32]} \frac{\sigma_{14}}{[124][134]} F_1 F_2 E_3 F_4 .$$

Exchanging 1 and 4 does not change the order of the permutation since it takes 6 flips to swap 1 and 4. Then we can add the corresponding summands. That is, we consider

$$X_{(\ldots 1234 \ldots)} - X_{(\ldots 1324 \ldots)} + X_{(\ldots 4231 \ldots)} - X_{(\ldots 4321 \ldots)},$$

where X_ρ is the summand associated to ρ. The antisymmetry of $\sigma_{14}/[124][134]$ then shows that this vanishes.

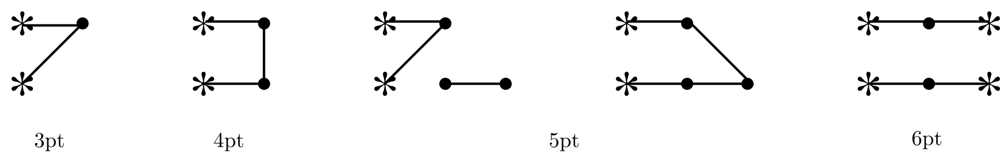

Figure 4: Contributing topologies at low points

Here we have introduced polarisation spinors ξ_i for the fermions. Just as in four dimensions, we can use the properties of the Parke-Taylor factors to infer that the permutation sums give Lie algebra elements inside the traces. Since the results in section 6.3 can easily be adapted to the present case we do not describe this in any further detail.

8 Ten dimensions

In ten dimensions, it is expected that the ambitwistor string gives the correct amplitudes at loop level. In this section, we find formulas for correlators involving two spin fields. These formulas are immediately useful: they can be used to find tree level amplitudes with one fermion line. Moreover, they allow us to directly verify that 1-loop prescription for the ambitwistor string amplitudes. In section 8.1 we review known results for the two spin field correlator. We relate this to an ambitwistor string correlator in section 8.2, and rewrite the integrand in section 8.3 to be manifestly gauge invariant. Finally, in section 8.4, we study the connection to the 1 loop amplitude formulas.

8.1 Two spin fields

We begin by giving the specialisation of our formulas in section 5.2 to ten dimensions. The OPEs are now

$$S_\alpha(z)S_{\dot\beta}(0) \sim z^{-\frac{5}{4}}\mathcal{C}_{\alpha\dot\beta},$$
$$S_\alpha(z)S_\beta(0) \sim \frac{1}{\sqrt{2}}z^{-\frac{3}{4}}\,(\gamma^\mu\mathcal{C})_{\alpha\beta}\;\psi_\mu(0).$$

Härtl and Schlotterer [27] give a formula for the correlator of two spin fields with arbitrarily many fermions ψ^μ in ten dimensions. They gave their result for arbitrary genus, but, for simplicity, we will state and prove it genus zero. Their formula is

$$\left\langle S_\alpha(\sigma_a)S_\beta(\sigma_b)\prod_{i=1}^{2n-1}\psi^{\mu_i}(\sigma_i)\right\rangle = \frac{1}{\sqrt{2}}\frac{1}{(\sigma_{ab})^{\frac{D-4}{8}}}\prod_{i=1}^{2n-1}\frac{1}{\sqrt{\sigma_{ia}\sigma_{ib}}}\times$$
$$\left(\sum_{s=0}^{n-1}\frac{\sigma_{ab}^s}{2^s}\sum_{\rho\in\mathfrak{S}^*_{2n-1}}|\rho|(\Gamma^{\mu_{\rho_1}}...\Gamma^{\mu_{\rho_{2s+1}}}C)_{\alpha\beta}\prod_{j=1}^{n-s-1}\frac{\eta^{\mu_*,\mu_{*+1}}}{\sigma_{*,*+1}}\sigma_{*,a}\sigma_{*+1,b}\right). \quad (35)$$

The appearance of an odd number of ψ's and an odd number of Γ matrices will become clear shortly—it is a consequence of the two spin fields having the same chirality. We will treat the opposite chirality case shortly. In the rightmost product, we have introduced an abbreviation: '$*$' is short for $\rho(2(s+j))$ and '$*+1$' for $\rho(2(s+j)+1)$. Finally, the permutation group $\mathfrak{S}^*_{2n-1}$ that appears in the formula is the quotient of $\mathfrak{S}_{2n-1}$ by the subgroup generated by

i. permutations of the indices of the Γ matrices,

ii. permutations of the ordering of the η pairings,

iii. flips of the indices on the factors of η.

The formula may seem unweildy, but it is easily verified. It is certainly correct for $n=1$ where

$$\langle S_\alpha(\sigma_a)S_\beta(\sigma_b)\psi^\mu(\sigma_1)\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sigma_{ab}^{3/4}\sigma_{1a}^{1/2}\sigma_{1b}^{1/2}}\,(\gamma^\mu\mathcal{C})_{\alpha\beta}\,.$$

The proof then proceeds by induction. Assuming the formula is correct for $n-1$, the pole at $\sigma_{ij}=0$ in equation (35) is given by

$$\left\langle S_\alpha(\sigma_a)S_\beta(\sigma_b)\prod_{i=1}^{2n-1}\psi^{\mu_i}(\sigma_i)\right\rangle \simeq \frac{\eta^{\mu_i\mu_j}}{\sigma_{ij}}\left\langle S_\alpha(\sigma_a)S_\beta(\sigma_b)\prod_{k\neq i,j}^{2n-1}\psi^{\mu_k}(\sigma_k)\right\rangle + \ldots$$

Moreover, the pole at $\sigma_{ab}=0$ is given by

$$\left\langle S_\alpha(\sigma_a)S_\beta(\sigma_b)\prod_{i=1}^{2n-1}\psi^{\mu_i}(\sigma_i)\right\rangle \simeq \frac{1}{\sqrt{2}}(\sigma_{ab})^{-\frac{D-4}{8}}\,(\gamma_\mu\mathcal{C})_{\alpha\beta}\left\langle \psi^\mu(\sigma_a)\prod_{i=1}^{2n-1}\psi^{\mu_i}(\sigma_i)\right\rangle.$$

To see this, observe that the all ψ correlator follows by Wick's theorem. The formula is

$$\left\langle\prod_{i=1}^{2n}\psi^{\mu_i}(\sigma_i)\right\rangle = \sum_{\rho\in\mathfrak{S}^*_{2n}}\operatorname{sgn}(\rho)\prod_{i=1}^{n}\frac{\eta^{\mu_*,\mu_{*+1}}}{\sigma_{*,*+1}}.$$

To complete the proof, one must also consider the poles at $\sigma_{ai}=0$ and $\sigma_{bi}=0$. However, recall the OPE, equation (**??**), of S_α and ψ^μ. To verify equation (35), we must also give a formula for the opposite chirality case. Indeed, taking the $\sigma_{bi}=0$ pole of (35), we arrive at a conjecture,

$$\left\langle S_\alpha(\sigma_a)S^\beta(\sigma_b)\prod_{i=1}^{2n-2}\psi^{\mu_i}(\sigma_i)\right\rangle = \frac{1}{(\sigma_{ab})^{\frac{D-4}{8}}}\prod_{i=1}^{2n-2}\frac{1}{\sqrt{\sigma_{ia}\sigma_{ib}}}\times$$
$$\left(\sum_{s=0}^{n-1}\frac{\sigma_{ab}^s}{2^s}\sum_{\rho\in\mathfrak{S}^*_{2n-1}}|\rho|(\Gamma^{\mu_{\rho_1}}\ldots\Gamma^{\mu_{\rho_{2s}}}C)_\alpha{}^{\dot\beta}\prod_{j=1}^{n-s-1}\frac{\eta^{\mu_*,\mu_{*+1}}}{\sigma_{*,*+1}}\sigma_{*,a}\sigma_{*+1,b}\right). \quad (36)$$

The proof by induction then establishes (35) and (36) simultaneously. Assuming, for instance, that (35) holds for $n-1$, we find that (36) has the correct pole

$$\left\langle S_\alpha(\sigma_a)S^\beta(\sigma_b)\prod_{i=1}^{2n-2}\psi^{\mu_i}(\sigma_i)\right\rangle = \frac{1}{\sqrt{2}}(\sigma_{jb})^{-\frac{1}{2}}(\gamma^{\mu_j})^{\beta\gamma}\left\langle S_\alpha(\sigma_a)S_\gamma(\sigma_b)\prod_{i\neq j}^{2n-2}\psi^{\mu_i}(\sigma_i)\right\rangle.$$

The other poles are similar and this concludes the proof of both formulas.

8.2 CHY integrand

To arrive at the CHY integrand computed by the ambitwistor string, we first contract the μ_i indices in equation (35) with polarisations and momenta. To be precise, we consider the product

$$\left(\prod_{i=1}^{n-1}\epsilon_i\cdot\psi(\sigma_{2i-1})k_i\cdot\psi(\sigma_{2i})\right)\epsilon_{2n-1}\cdot\psi(\sigma_{2n-1}).$$

We also contract $S_\alpha S_\beta$ with spinor polarisations $\xi_a^\alpha \xi_b^\beta$. Next we consider the limit $\sigma_{2i-1,2i} = 0$ (for $1 \le i \le n-1$). We do not receive a contribution from singular terms in this limit since $\epsilon_i \cdot k_i = 0$. We must also add the ghost contribution,

$$\left\langle e^{-\phi_a/2} e^{-\phi_b/2} e^{-\phi_n} \right\rangle = \frac{1}{(\sigma_{ab})^{\frac{1}{4}} (\sigma_{na})^{\frac{1}{2}} (\sigma_{bn})^{\frac{1}{2}}}.$$

If we generically denote the polarisations and momenta by a collection of vectors v_i, then the correlator gives the following CHY integrand

$$I = \frac{1}{\sqrt{2}} \frac{1}{\sigma_{ab}} \frac{1}{\sigma_{na}\sigma_{nb}} \prod_{i=1}^{n-1} \frac{1}{\sigma_{2i,a}\sigma_{2i,b}} \times$$
$$\lim_{\sigma_{2i-1,2i}\to 0} \left(\sum_{s=0}^{n-1} \frac{\sigma_{ab}^s}{2^s} \sum_{\rho \in \mathfrak{S}^*_{2n-1}} |\rho| (\xi_a \not{v}_{\rho(1)} \cdots \not{v}_{\rho(2s+1)} C \xi_b) \prod_{j=1}^{n-s-1} \frac{v_* \cdot v_{*+1}}{\sigma_{*,*+1}} \sigma_{*,a}\sigma_{*+1,b} \right). \quad (37)$$

As we expect for a CHY integrand, I is a rational function of the insertion points. One disadvantage of this presentation of the correlator is that it is not manifestly gauge invariant (under $\epsilon_i \mapsto \epsilon_i + k_i$). Nevertheless, it will be useful to us in section 8.4. In particular, notice that it has a pole at $\sigma_{ab} = 0$.

8.3 A gauge invariant formula

As we discussed in section 6, it may be possible to regard the CHY formula as the unique gauge invariant possibility. It is, therefore, enlightening to consider manifestly gauge invariant presentations of the CHY formula and the closely related integrands that we have been deriving. In this section, we briefly present a manifestly gauge invariant version of the formula, equation (37), presented in the previous section. We do not give its derivation, since it is not important to our main aim in this section—which is to relate the two spin field correlator to loop amplitudes. The manifestly gauge invariant formula is

$$I = \sum_{\alpha \subset \{1\ldots n\}} \frac{(-1)^m}{\sqrt{2}} \frac{(\xi_a F_{\alpha(1)} \cdots F_{\alpha(m)} \gamma^\mu \mathcal{C}\, \xi_b)}{\sigma_{(a,\alpha(1),\ldots,\alpha(m),b)}} \frac{\partial}{\partial \epsilon^\mu} \mathrm{Pf}(M^{[bn]}).$$

If so desired, the product of field strengths can be replaced with the total Lie bracket of the field strengths (see section 6.3). Though I derived this formula from the same OPE relations, I do not have a direct algebraic proof demonstrating that it is equal to equation 37. What is certainly true is that both formulas for I have the same pole at $\sigma_{ab} = 0$. Indeed, the pole is given by

$$I = \frac{1}{\sqrt{2}} \frac{1}{\sigma_{ab}} (\xi_a \gamma^\mu \mathcal{C} \xi_b) \frac{\partial}{\partial \epsilon^\mu} \mathrm{Pf}(M^{[bn]}).$$

We mention this here because it will be useful in the following section.

8.4 The forward limit and the degenerating torus

In this section we study the forward limit of the two spin field correlators. In the forward limit we take $k_a + k_b \to 0$. Our motivation for this is that the ambitwistor string integrands for loop amplitudes can be obtained from the forward limits of tree-level correlators. By taking the forward limit of the two Ramond insertions, we will thereby obtain the fermion contribution to a 1-loop amplitude with all external particles being bosons. On its own, the forward limit of the integrand is singular. But, as we will see, this singularity cancels precisely the singularity that arises from bosons running in the loop.

8.4.1 The singular part

He and Yuan studied the solutions to the scattering equations in the forward limit. If we set $k_a + k_b = \tau q$, with $\tau \to 0$, then they found that there are $(n-2)!$ solutions for which $\sigma_{ab} \sim \tau$ and $(n-2)!$ solutions for which $\sigma_{ab} \sim \tau^2$. We call these the singular solutions. The correlator for two spin fields does have a pole in σ_{ab}. This was a key part of the proof we gave in section 8.1. As we mentioned in section 8.3, the pole of the integrand, including ghosts, is

$$I = \frac{1}{\sigma_{ab}} \mathrm{Pf}' \left(M^{[bn]}(b, 1, ..., n) \right),$$

where $M^{[bn]}$ is a CHY matrix with the momenta for b and n removed. The polarisation of b is $\epsilon^\mu = (\xi_a \gamma^\mu \mathcal{C} \xi_b)/\sqrt{2}$. The Pfaffian is linear in the polarisation, and so we can write

$$I = \frac{1}{\sigma_{ab}} \frac{1}{\sqrt{2}} (\xi_a \gamma^\mu \mathcal{C} \xi_b) \frac{\partial}{\partial \epsilon^\mu} \mathrm{Pf}' \left(M^{[bn]}(b, 1, ..., n) \right).$$

To obtain the contribution to the loop amplitude we set $\xi_a = \xi_b$ and sum over a basis of polarisation states. Let us now compare with the contribution from bosons in the loop. To begin, consider $n+2$ bosons. The corresponding integrand is the CHY pfaffian

$$J = \mathrm{Pf}' \left(M^{[an]}(a, b, 1, ..., n) \right).$$

Using the expansion of the Pfaffian we see that it has a pole in σ_{ab} given by

$$J = \frac{1}{\sigma_{ab}} \epsilon_a \cdot \epsilon_b \, \mathrm{Pf}' \left(M^{[b+n,n]}(b, 1, ..., n) \right) + ...$$

Removing $b+n$ from M leaves only the momentum, call it l^μ, associated to b. So we could just as well write this as

$$J = \frac{1}{\sigma_{ab}} \epsilon_a \cdot \epsilon_b \, l^\mu \frac{\partial}{\partial \epsilon^\mu} \mathrm{Pf}'(M^{[bn]}(b, 1, ..., n)).$$

To obtain the loop contribution we set $\epsilon_a = \epsilon_b$ and sum over a basis of polarisation states. We see that the boson contribution cancels the fermion contribution if

$$\frac{1}{\sqrt{2}} \sum_\xi (\xi \gamma^\mu \mathcal{C} \xi) = l^\mu \sum_\epsilon \epsilon \cdot \epsilon. \tag{38}$$

We will assume a normalisation so that this relation holds. Finally, we derive a brief corollary of this. In $D = 10$, the only p-forms that have symmetric index structure ${}^{\alpha\beta}$ are the 1-form and the 5-form:

$$\mathcal{C}^{-1} \bar{\gamma}^{\mu\,\alpha\beta} \qquad \text{and} \qquad \mathcal{C}^{-1} \bar{\gamma}^{5\,\alpha\beta}.$$

(The 3-form has the same index structure, but is antisymmetric.) On these grounds, one must have

$$\sum_h \xi^\alpha \xi^\beta = A_\mu \, \mathcal{C}^{-1} \bar{\gamma}^{\mu\,\alpha\beta} + B_{abcde} \mathcal{C}^{-1} \bar{\gamma}^{abcde\,\alpha\beta},$$

for some p-forms A and B. However, comparing with equation (38), we see that

$$A^\mu = \frac{\sqrt{2}}{2^{D/2-1}} l^\mu \sum_\epsilon \epsilon \cdot \epsilon.$$

If we normalise the boson polarisation sum so that $\sum \epsilon \cdot \epsilon = 2^{-(D-3)/2}$ we find that

$$\sum_h \xi^\alpha \xi^\beta = l_\mu \, \mathcal{C}^{-1} \bar{\gamma}^{\mu\,\alpha\beta} + B_{abcde} \mathcal{C}^{-1} \bar{\gamma}^{abcde\,\alpha\beta}. \tag{39}$$

We will use this in the following section.

8.4.2 Relation to degenerating torus

We expect that the forward limit of our integrand I is related to the integrand one would obtain from the ambitwistor string at genus 1. (Following, in particular, reference [24].) Taking our expression equation (37) for the integrand, we find that

$$\sum_{\xi} I = \lim_{\sigma_{2i-1,2i}\to 0} \frac{1}{\sqrt{2}}\frac{1}{\sigma_{ab}}\frac{1}{\sigma_{na}\sigma_{nb}} \prod_{i=1}^{2n-2} \frac{1}{\sqrt{\sigma_{ia}\sigma_{ib}}} \times \left(\sum_{s=0}^{n-1} \frac{\sigma_{ab}^{s}}{2^{s}} \sum_{\rho\in\mathfrak{S}^{*}_{2n-1}} |\rho| \mathrm{tr}(\not{v}_{\rho(1)}...\not{v}_{\rho(2s+1)}\not{l}) \prod_{j=1}^{n-s-1} \frac{v_{*}\cdot v_{*+1}}{\sigma_{*,*+1}} \sigma_{*,a}\sigma_{*+1,b} \right) + ... \quad (40)$$

We have used equation (39) and moved the pre-factor back inside the limit. There is a second term involving the 5-form B_{abcde} which appears in equation (39). We focus on the first term for now. Following Roehrig and Skinner (who did a similar case), the summands in this formula can be identified with Pfaffians. We write

$$\not{v}_{\rho(1)}\not{v}_{\rho(2)}...\not{v}_{\rho(2s+1)}\not{l} = \frac{1}{2}(1+\Gamma_{11})\not{v}_{\rho(1)}\not{v}_{\rho(2)}...\not{v}_{\rho(2s+1)}\not{l},$$

where the $\not{v}$ appearing on the right hand side are formed using the full gamma matrices. The factor of $(1+\Gamma_{11})/2$ is the projection onto the chiral part. The trace can be written in terms of pfaffians as

$$\mathrm{tr}(\not{v}_{1}....\not{v}_{2s+2}) = 2^{5}\mathrm{Pf}(V) \qquad \mathrm{tr}(\not{v}_{1}....\not{v}_{2s+2}\Gamma_{11}) = \frac{2^{5}}{9!!}\int \mathrm{d}^{10}\Psi \mathrm{Pf}(A),$$

where

$$V_{ij} = v_i\cdot v_j \mathrm{sgn}(i-j) \qquad A_{ij} = v_i\cdot v_j\mathrm{sgn}(i-j) + v_i\cdot\Psi v_j\cdot\Psi.$$

The integral over Ψ is a Grassmann integral, with Ψ odd. Rearranging equation (40) we have

$$\sum_{\xi} I = \lim_{\sigma_{2i-1,2i}\to 0} \frac{1}{\sqrt{2}} 2^{-(n-1)} \frac{1}{\sigma_{ab}^{2}}\frac{\sigma_{ab}}{\sigma_{na}\sigma_{nb}} \prod_{i=1}^{2n-2} \sqrt{\frac{\sigma_{ab}}{\sigma_{ia}\sigma_{ib}}} \times \left(\sum_{s=0}^{n-1} \sum_{\rho\in\mathfrak{S}^{*}_{2n-1}} |\rho| \mathrm{tr}(\not{v}_{\rho(1)}...\not{v}_{\rho(2s+1)}\not{l}) \prod_{j=1}^{n-s-1} 2v_{*}\cdot v_{*+1}\frac{\sigma_{*,a}\sigma_{*+1,b}}{\sigma_{*,*+1}\sigma_{ab}} \right) + ... \quad (41)$$

The sum given here in parantheses may be recast as a sum over pfaffians. To do this, we will write $l = v_{2n}$. Then

$$(...) = \sum_{\alpha\subset\{1,...,2n\}} \mathrm{sgn}(\alpha,\alpha^{c}) 2^{5}\mathrm{Pf}(V_{ij})_{ij\in\alpha}\mathrm{Pf}(M_{ij})_{ij\in\alpha^{c}} + ...,$$

where the elipsis is the analogous term involving the matrix A. The pfaffians combine according to

$$\sum_{\alpha\subset\{1,...,2n\}} \mathrm{sgn}(\alpha,\alpha^{c})\mathrm{Pf}(V_{ij})_{ij\in\alpha}\mathrm{Pf}(M_{ij})_{ij\in\alpha^{c}} = \mathrm{Pf}(V+M),$$

which follows from the definition of the Pfaffian, expanding the right hand side. The entire expression may thus be written as

$$\sum_{\xi} I = \lim_{\sigma_{2i-1,2i}\to 0} \frac{2^{3}}{\sqrt{2}}\frac{1}{\sigma_{ab}^{2}} \sqrt{\frac{\sigma_{ab}}{\sigma_{na}\sigma_{nb}}} \left(\mathrm{Pf}(X) + \frac{1}{9!!}\int \mathrm{d}^{10}\Psi\mathrm{Pf}(Y) \right) + \text{five-form term},$$

where

$$X_{ij} = \sqrt{\frac{\sigma_{ab}}{\sigma_{ia}\sigma_{ib}}}\sqrt{\frac{\sigma_{ab}}{\sigma_{ja}\sigma_{jb}}}\left(\frac{1}{2}\mathrm{sgn}(i-j) + \frac{\sigma_{ia}\sigma_{jb}}{\sigma_{ij}\sigma_{ab}}\right) v_i \cdot v_j.$$

and Y is the analogous matrix formed from A. (Notice that we have absorbed a factor of $2^{-(n+1)}$ into the matrix entires.) We can rearrange X_{ij} so that

$$X_{ij} = v_i \cdot v_j S(i,j),$$

where

$$S(i,j) = \frac{1}{\sigma_{ij}}\frac{1}{2}\left(\sqrt{\frac{\sigma_{ib}\sigma_{ja}}{\sigma_{ia}\sigma_{jb}}} + \sqrt{\frac{\sigma_{jb}\sigma_{ia}}{\sigma_{ja}\sigma_{ib}}}\right).$$

In this form, $S(i,j)$ is the Szegö kernel for the torus in the degenerating limit (for two of the four possible spin structures). See for example equation (2.33) of [24], where the spin structures that give rise to this limit are labelled $\alpha = 1, 2$—where these are associated to the Ramond contribution at 1-loop. In this way, $\sum_\xi I$ (less the five-form term) can be understood to arise from the Ramond sector of the ambitwistor string at one loop, in the degenerating limit. (See especially section 3.2 of [24] for a discussion of the Ramond contribution to a 1-loop amplitude with all external states bosonic.)

9 Prospects

The most immediate prospect for the work described in Part 2 is a proof of the 1-loop formula conjectured first in [5] and partially proved in [24]. (It is not clear what role the five-form term will play in such a proof—and this will require some clarification.) Recently, formulas for the correlators for two spin fields on a genus one worldsheet were studied by Schlotterer and Lee in [45]. Their formulas might be useful for us. Taking the degenerating limit of the worldsheet, and the forward limit of the two Ramond insertions might allow us to use their formulas to prove the 2-loop ambitwistor string conjecture, studied in [25].

The other prospect for the work in Part 2 concerns the correlators involving both Ramond-type and Neveu-Schwarz insertions. In sections 6 and 7 we presented new formulas for these correlators—though only for the spin fields associated to four and six dimensions. At present, we have been unable to relate these interesting formulas to amplitudes. However, if the correlators discussed in section 6 can be related to a dimensional reduction of the $D = 10$ ambitwistor string, we may arrive at formulas for $D = 4$ super Yang-Mills and supergravity tree amplitudes. These are already known, so this would be an excellent check that the Ramond sector of the ambitwistor string is giving the correct amplitudes at tree level. The $D = 4$ and $D = 6$ formulas are not, by themselves, sufficient to produce $D = 10$ formulas for the reason described in section 5.3. Namely, the $D = 10$ spin fields do not fully factorise into two parts. Nevertheless, the $D = 4$ and $D = 6$ formulas constrain what the $D = 10$ correlators can be, since the later should dimensionally reduce to the former.

A Spinors and purity

In this appendix we review results about the spin representations and pure spinors. We mention the general theory only in passing, and focus instead on mentioning specific cases which are used in the main text.

A.1 Some properties by dimension

For reasons discussed in section 5.2, the properties of the chiral spin representations vary with dimension modulo 4. When $D = 0$ mod 4 there are forms on S^+ and S^-, which we write as $C_{\alpha\beta}$ and $C_{\dot\alpha\dot\beta}$. When $D = 2$ mod 4, S^+ and S^- are dual to each other, and we write $C_\alpha^{\dot\beta}$ and $C_{\dot\alpha}^{\beta}$ for the pairing. In $D = 0$ mod 8, $C^{\alpha\beta}$ and $C^{\dot\alpha\dot\beta}$ are symmetric, while when $D = 4$ mod 8, they are anti-symmetric.[28] Moreover, the p-forms $\gamma_{(p)}^{\alpha\beta}$ are symmetric when $D-2p = 0$ mod 8 and antisymmetric when $D-2p = 4$ mod 8. If $D-2p = 6$ mod 8, $\gamma_{(p)}^{\alpha\dot\beta} = -\gamma_{(p)}^{\dot\beta\alpha}$ is odd under exchange of the indices, while if $D-2p = 2$ mod 8, it is even.[29] Combining these observations we comment on two particular cases. First, in $D = 6$,

$$(\gamma^a C)_{\alpha\beta} \quad \text{and} \quad (\gamma^{abc} C)_{\alpha\beta}$$

are the only matrices with this index structure and they are both symmetric. Second, in $D = 10$,

$$(\gamma^a C)_{\alpha\beta} \quad \text{and} \quad (\gamma^{abcde} C)_{\alpha\beta}$$

are symmetric while

$$(\gamma^{abc} C)_{\alpha\beta}$$

is skew. Generally, the chiral spin representations for $2n$ dimensions can be constructed as direct sums of the chiral representations for $2n-2$. Finally, one particular case is of interest. Namely, the spin(10) group can be broken to a spin(4) $\times$ spin(6) subgroup, and the chiral 10 dimensional spinors decomposed as tensor products of 4 and 6 dimensional spinors. If we write the 10 dimensional chiral spinor λ^A as $(\lambda_i^\alpha, \lambda^{j\dot\beta})$ under this decomposition, then the corresponding 10 dimensional gamma matrices may be written as[30]

$$\gamma^{\alpha\dot\beta}_{(i\beta)(j\dot\gamma)} = \delta_{ij}\delta^\alpha_\beta\delta^{\dot\beta}_{\dot\gamma}, \quad \gamma^{kl}_{(i\beta)(j\gamma)} = \frac{1}{2}\epsilon_{\beta\gamma}\left(\delta^k_i\delta^l_j - (kl)\right), \quad \text{and} \quad \gamma^{kl}_{(i\dot\beta)(j\dot\gamma)} = \frac{1}{2}\epsilon^{ijkl}\epsilon_{\dot\beta\dot\gamma},$$

where the pair $\alpha\dot\beta$ refers to spinor coordinates on $\mathbb{C}^4$ and kl to coordinates on $\mathbb{C}^6$.

A.2 Pure spinors

The space of pure spinors in $2n$ dimensions is the space of null n-planes in $\mathbb{C}^{2n}$ (with some quadratic form) or, equivalently, the space of complex structures on $\mathbb{C}^{2n}$. From the first definition, we see that the space of pure spinors has dimension $n(n-1)/2$. (We may decompose $\mathbb{C}^{2n} = W \oplus W^*$ so that the quadratic form is written in the form $v^i w_i$ for coordinates (v, w). Then a maximal null plane is given by a choice of skew $n \times n$ matrix X_{ij} such that $w_i = X_{ij} v^j$ is the equation of the plane in V.[31]) The second definition is equivalent to the first. The space of complex structures is $SO(2n)/U(n)$, which we identify with a choice of coordinates $v = Az + B\bar z$ (in terms of some coordinates $z, \bar z$ already given), defined up to $A \sim UA, B \sim UB$ where $UU^* = 1$.[32] The rows of the matrix $[AB]$ define an n-plane in $\mathbb{C}^{2n}$. It is null with respect to the metric $\mathrm{d}z^i \mathrm{d}\bar z^i$. Associated to a null n-plane is an n-form ω. Given this, we may consider the image of $\gamma^{abcde}\omega_{abcde}$, which is one dimensional since the plane is null. Conversely, given a spinor λ^α, we can consider the decomposition of the bilinear $\lambda^\alpha\lambda^\beta$ into p-forms. If the bilinear only has a 5-form component, then λ defines a totally null 5-plane. This turns out

[28] See [46] section 6.5.

[29] All this is described in Appendix A of [47].

[30] See equation 5.10 of [20].

[31] Appendix A of [47].

[32] An explicit discussion is given in the appendix to [48].

to be a correspondence and it was first described by Cartan. Such spinors are called pure. For example, in eight dimensions λ^α is pure if $\lambda^\alpha\lambda^\beta C_{\alpha\beta}=0$. In ten dimensions, λ^α is pure if $\lambda^\alpha\gamma^a_{\alpha\beta}\lambda^\beta=0$. In twelve dimensions, $\lambda\gamma^{ab}\lambda=0$ is sufficient. [49] In general one demands that contractions of the bilinear with the projections

$$P^{\alpha\beta}_{\gamma\delta}=\gamma_{abcd\ldots\gamma}{}^{\beta}\gamma_{\delta}{}^{abcd\ldots\gamma}$$

all vanish, except in degree n. We now give some examples of how the purity condition can be related between different dimensions. For instance, a pair of $SO(6)$ spinors $(w_\alpha,\lambda^\alpha)$ is pure as an $SO(8)$ spinor if $w_\alpha\lambda^\alpha=0$. Likewise, a pair of $SO(8)$ spinors $(w_\alpha,\lambda^{\dot\alpha})$ is pure as an $SO(10)$ spinor if $w_\alpha\lambda_{\dot\alpha}\gamma_a^{\alpha\dot\alpha}=0$. Finally, using the decomposition of $SO(10)$ spinors described in the previous section, a 10 dimensional chiral spinor λ^A written as $(\lambda^\alpha_i,\lambda^{j\dot\beta})$ is pure if

$$\lambda^{i\alpha}\lambda^{\dot\alpha}_i=0,\qquad \lambda^{i\alpha}\lambda^j_\alpha=0,\qquad \lambda^{i\dot\alpha}\lambda^j_{\dot\alpha}=0.$$

B Physics conventions for superspace

Our conventions for super-geometry are those used in the physics literature. We give examples of how these conventions relate to the calculation of Poisson brackets. Local coordinates on a supermanifold are a collection of even and odd functions, x^m,θ^μ, that we will collectively denote Z^M: the middle of the alphabet—M,m,μ,...—will be used to denote local coordinate indices. We denote the degree in the $\mathbb{Z}_2$ grading by $|\bullet|$. The super Grassmann algebra is generated by $\mathrm{d}Z^M$ where the wedge product is graded according to

$$\mathrm{d}Z^M\wedge\mathrm{d}Z^N=-(-1)^{|M||N|}\mathrm{d}Z^N\wedge\mathrm{d}Z^M.$$

For the differential, d, the convention used by [50] is that

$$\mathrm{d}(\mathrm{d}Z^M\ldots\phi_{M\ldots})=\mathrm{d}Z^M\ldots\mathrm{d}Z^P\partial_P\phi_{M\ldots}.$$

A derivation D (of pure degree) on the Grassmann algebra respects the grading;

$$D(ab)=Dab+(-1)^{|D||a|}aDb.$$

For instance, we have a basis of even, $\partial_m=\partial/\partial x^m$, and odd, $\partial_\theta=\partial/\partial\theta^\alpha$, vector fields which are derivations. We also have the odd derivation $\partial/\partial\mathrm{d}x^m$ and the even derivation $\partial/\partial\mathrm{d}\theta^\alpha$. A vector field V can be written $V=V^M\partial_M$. Then $V\lrcorner$ is a derivation on the Grassmann algebra given by [51]

$$V\lrcorner=V^M\frac{\partial}{\partial\mathrm{d}Z^M}.$$

For example, the Eulerian vector field becomes

$$V\lrcorner=\lambda^\alpha\frac{\partial}{\partial\mathrm{d}\lambda^\alpha}-w_\alpha\frac{\partial}{\partial\mathrm{d}w_\alpha},$$

such that, with the appropriate sign rules,

$$V\lrcorner\,\omega=-\mathrm{d}(\lambda^\alpha w_\alpha).$$

The coordinate basis is inconvenient, even in flat space, so we define basis 1-forms $E^A=\mathrm{d}Z^ME^A_M$, where E takes values in the Lie algebra of the super Lorentz group: the beginning of the alphabet—$A,a,\alpha,\ldots$—will be used to denote these Lie algebra indices. Notice that $|E^A_M|=|A|+|M|$ so that only E^α_m and E^a_μ are odd. The inverse vierbein, E^M_A, allows us to define $d_A = E_A^M \partial_M$ so that $\mathrm{d} = e^A d_A$. We introduce connection 1-forms $\Omega = \mathrm{d}Z^M \Omega_M$ taking values in endomorphisms of the Lie algebra. For example, $\nabla V^B = \mathrm{d}V^B + \Omega_A{}^B V^A$ is the covariant derivative on Lorentz vectors. In flat space, $\Omega = 0$, but the vierbein is not trivial since we take

$$\mathrm{d}e^a = \frac{1}{2} e^\alpha \wedge e^\beta \gamma^a{}_{\alpha\beta}, \qquad \mathrm{d}e^\alpha = 0. \tag{42}$$

A more opaque way to write this is

$$e^a = \delta^a_m \mathrm{d}x^m + \mathrm{d}\theta^\alpha \theta^\beta \gamma^a{}_{\alpha\beta}/2 \qquad e^\alpha = \delta^\alpha_\mu \mathrm{d}\theta^\mu.$$

For a super 1-form A with values in $\mathfrak{g}$, the field strength is $F = \mathrm{d}A + [A \wedge A]$, with conventions as above. The torsion and curvature of ∇ are

$$T = \mathrm{d}E + E \wedge \Omega, \quad R = \mathrm{d}\Omega + \Omega \wedge \Omega,$$

where we understand the Lie algebra endomorphisms to act from the right. We can define components of T as $T^A = e^B \wedge e^C T_{BC}{}^A/2$. For example, (42) shows that in flat space the only non-zero torsion components are

$$T_{\alpha\beta}{}^a = \gamma^a{}_{\alpha\beta}.$$

In general, the torsion components arise in Poisson brackets involving the fields D_M which implement covariant derivatives. We have

$$\lambda^\beta [D_\beta, E_\alpha^M \lambda^\alpha] D_M - D_M [E_\beta^M \lambda^\beta, D_\alpha] \lambda^\alpha = T_{\alpha\beta}{}^M \lambda^\alpha \lambda^\beta D_M.$$

This is how the torsion components arise in the Poisson bracket $[Q, Q]$.

C Linearised equations for IIB

We consider perturbations of the flat space vierbein $\overset{0}{E}$,

$$E_M^A = \overset{0}{E}_M{}^A + \overset{0}{E}_M{}^B H_B^A.$$

The torsion, $T = \mathrm{d}E + E \wedge \Omega$, recieves a first order contribution

$$\overset{1}{T}_{AB}{}^C = \overset{0}{\nabla}_{[A} H^C_{B]} + \overset{1}{\Omega}_{[AB]}{}^C + \overset{0}{T}_{AB}{}^D H^C_D + H^D_{[A} \overset{0}{T}_{D|B]}{}^C.$$

See also equation 3.13 of [39]. This is almost all we need. For example, the non-linear supergravity constraint $T_{\alpha(ab)} = 0$ becomes, in this linearised expansion,

$$T_{\alpha(ab)} \longrightarrow 0 = \nabla_{[\alpha} H_{a]b} + \gamma_{b\alpha\hat{\beta}} H^{\hat{\beta}}_a + (a \leftrightarrow b), \tag{43}$$

where we use that the only flat space torsion is $\overset{0}{T}_{\alpha\hat{\beta}}{}^a = \gamma^a{}_{\alpha\hat{\beta}}$. We now linearise the other torsion constraints.

$$T_{\alpha a}{}^\beta + T_{\alpha\hat{\gamma}a} P^{\beta\hat{\gamma}} \longrightarrow 0 = \nabla_{[\alpha} H^\beta_{a]} - \Omega_{a\alpha}{}^\beta + \gamma_{a\alpha\hat{\gamma}} P^{\beta\hat{\gamma}}, \tag{44}$$

$$T_{\alpha a}{}^{\hat{\beta}} - T_{\alpha\gamma a} P^{\gamma\hat{\beta}} \longrightarrow 0 = \nabla_{[\alpha} H^{\hat{\beta}}_{a]} \tag{45}$$

$$T_{\alpha\hat{\gamma}}{}^\beta \longrightarrow 0 = \nabla_{[\alpha} H^\beta_{\hat{\gamma}]} + \gamma^a{}_{\alpha\hat{\gamma}} H^\beta_a + \Omega_{\hat{\gamma}\alpha}{}^\beta. \tag{46}$$

$$T_{\alpha\beta}{}^{\hat{\gamma}} - \frac{1}{2} H_{\alpha\beta\gamma} P^{\gamma\hat{\gamma}} \longrightarrow 0 = \nabla_{[\alpha} H^{\hat{\gamma}}_{\beta]} \tag{47}$$

The 3-form H is zero for the flat background and so $H_{\alpha\beta\gamma}P^{\gamma\hat{\gamma}}$ does not contribute at first order. Equations (43)-(47) are all the torsion constraints that we need.[33] We now turn to the equations involving derivatives of the supergravity superfields. We linearise each of these as follows,

$$\nabla_\alpha P^{\gamma\hat{\beta}} - T_{\alpha\rho}{}^{\gamma}P^{\rho\hat{\beta}} + C_\alpha{}^{\gamma\hat{\beta}} \longrightarrow 0 = \nabla_\alpha P^{\gamma\hat{\beta}} + C_\alpha{}^{\gamma\hat{\beta}} \tag{48}$$

$$\lambda^\alpha\lambda^\beta\left(\nabla_\alpha C_\beta{}^{\delta\hat{\gamma}} - R_{\alpha\kappa\beta}{}^{\delta}P^{\kappa\hat{\gamma}}\right) \longrightarrow 0 = \lambda^\alpha\lambda^\beta\nabla_\alpha C_\beta{}^{\delta\hat{\gamma}}, \tag{49}$$

$$\nabla_\alpha C_{\hat{\gamma}}{}^{\hat{\delta}\rho} - T_{\alpha\beta}{}^{\rho}C_{\hat{\gamma}}{}^{\hat{\delta}\beta} + R_{\alpha\hat{\beta}\hat{\gamma}}{}^{\hat{\delta}}P^{\rho\hat{\beta}} + S_{\alpha\hat{\gamma}}{}^{\rho\hat{\delta}} \longrightarrow 0 = \nabla_\alpha C_{\hat{\gamma}}{}^{\hat{\delta}\rho} + S_{\alpha\hat{\gamma}}{}^{\rho\hat{\delta}} \tag{50}$$

$$\lambda^\alpha\lambda^\beta\left(\nabla_\alpha S_{\beta\hat{\gamma}}{}^{\rho\hat{\delta}} - R_{\alpha\hat{\kappa}\hat{\gamma}}{}^{\hat{\delta}}C_\beta{}^{\rho\hat{\kappa}} - R_{\alpha\kappa\gamma}{}^{\rho}C_{\hat{\gamma}}{}^{\hat{\delta}\kappa}\right) \longrightarrow 0 = \lambda^\alpha\lambda^\beta\nabla_\alpha S_{\beta\hat{\gamma}}{}^{\rho\hat{\delta}}. \tag{51}$$

The superfields $P, C, \hat{C}, U$ on the right hand side are to be understood as the first order variation around a background where all superfields vanish. Finally, each of the torsion constraints implies a constraint on the curvature using the Bianchi identity $\mathrm{d}T = R$. We will only need the curvature versions of (43) and (44), which are

$$R_{c\alpha\beta}{}^{\gamma} + T_{\alpha\hat{\delta}c}C_\beta{}^{\gamma\hat{\delta}} \longrightarrow 0 = \nabla_{[c}\Omega_{\alpha]\beta}{}^{\gamma} + \gamma_{c\alpha\hat{\alpha}}C_\beta{}^{\gamma\hat{\alpha}} \tag{52}$$

$$R_{c\alpha\hat{\beta}}{}^{\hat{\gamma}} + T_{\alpha\delta c}C_{\hat{\beta}}{}^{\hat{\gamma}\hat{\delta}} \longrightarrow 0 = \nabla_{[c}\Omega_{\alpha]\hat{\beta}}{}^{\hat{\gamma}}. \tag{53}$$

To expand R we use the definition $R = \mathrm{d}\Omega + \Omega\wedge\Omega$. Since $\Omega = 0$ for the flat background, $R = \mathrm{d}\Omega$ to first order. We have also used (48).

D Shuffles and Lie polynomials

In this section we prove two results which are first used in section 6.3 in the main text. The first proof is merely a translation from that which appears following pg. 40 in reference [52]. The second proof adapts the idea to a new case that we need in the main text.

Theorem 7. *The broken Parke-Taylor factors satisfy the shuffle identity,*

$$\sum_{\alpha\sqcup\!\sqcup\beta} pt(\omega) = 0,$$

where it is understood that the sum is over all $\omega \in \alpha \sqcup\!\sqcup \beta$.

Proof. The result holds for $|\omega| = 2$ and $|\omega| = 3$. We proceed by induction. The key observation is that $\alpha \sqcup\!\sqcup \beta$ can be decomposed into two sets. This is because every element of $\alpha \sqcup\!\sqcup \beta$ must end with either the last letter in α or the last letter in β. Let $|\alpha| = a$ and $|\beta| = b$ be the orders of α and β. Then

$$\alpha \sqcup\!\sqcup \beta = (\alpha \sqcup\!\sqcup \beta_{-1}, \beta_b) \cup (\alpha_{-1} \sqcup\!\sqcup \beta, \alpha_a).$$

We can iterate this decomposition to obtain an expansion

$$\sum_{\alpha\sqcup\!\sqcup\beta} pt(\omega) = \sum_{(\alpha\sqcup\!\sqcup\beta_{-2},\beta_{b-1},\beta_b)} pt(\omega) + \sum_{(\alpha_{-1}\sqcup\!\sqcup\beta_{-1},\alpha_a,\beta_b)} pt(\omega) + \sum_{(\alpha_{-1}\sqcup\!\sqcup\beta_{-1},\beta_b,\alpha_a)} pt(\omega) + \sum_{(\alpha_{-2}\sqcup\!\sqcup\beta,\alpha_{a-1},\alpha_a)} pt(\omega).$$

[33] For instance, we are missing $T_{\alpha\beta a} + H_{\alpha\beta a}$, but this is not needed to impose our linearised relations. We will see this constraint appearing in the non-linear version of this descent.

Using the inductive hypothesis we can combine these into two sums. This is because, for example,

$$(\alpha \sqcup\!\sqcup\, \beta_{-2}, \beta_{b-1}) \cup (\alpha_{-1} \sqcup\!\sqcup\, \beta_{-2}, \alpha_a) = (\alpha \sqcup\!\sqcup\, \beta_{-1}).$$

We also use

$$pt(1, ..., m) = pt(1, ..., m-1) pt(m-1, m).$$

Then

$$\sum_{\alpha \sqcup\!\sqcup \beta} pt(\omega) = \left(\sum_{(\alpha_{-1} \sqcup\!\sqcup \beta_{-1}, \alpha_a)} pt(\gamma) \right) (pt(\alpha_a, \beta_b) - pt(\beta_{b-1}, \beta_b)) \\ + \left(\sum_{(\alpha_{-1} \sqcup\!\sqcup \beta_{-1}, \beta_b)} pt(\gamma) \right) (pt(\beta_b, \alpha_a) - pt(\alpha_{a-1}, \alpha_a)).$$

Repeating this expansion, and then combining terms using the inductive hypothesis we obtain

$$\sum_{\alpha \sqcup\!\sqcup \beta} pt(\omega) = \left(\sum_{(\alpha_{-2} \sqcup\!\sqcup \beta_{-1}, \alpha_{a-1})} pt(\gamma) \right) C,$$

where

$$C = pt(\alpha_a, \alpha_{a-1}, \beta_b) + pt(\alpha_{a-1}, \alpha_a, \beta_b) + pt(\alpha_{a-1}, \beta_b, \alpha_a) \\ - pt(\alpha_a, \beta_{b-1}, \beta_b) - pt(\beta_{b-1}, \alpha_a, \beta_b) - pt(\beta_{b-1}, \beta_b, \alpha_a).$$

The shuffle identity for $|\omega| = 3$ then shows that $C = 0$. □

Theorem 8. *The Parke-Taylor factors $PT(*, 1, 2, ..., n)$ with one point fixed obey the shuffle identity.*

Proof. The key relation is that

$$PT(*, 1, ..., n) = PT(*, 1, ..., n-1) \Omega_{n-1,n},$$

where

$$\Omega_{n-1,n} = \frac{\sigma_{*,n-1}}{\sigma_{*,n} \sigma_{n-1,n}}.$$

Just as in the previous proof, one expands the shuffle sum to find

$$\sum_{\alpha \sqcup\!\sqcup \beta} PT(*, \omega) = \left(\sum_{(\alpha_{-2} \sqcup\!\sqcup \beta_{-1}, \alpha_{a-1})} PT(*, \gamma) \right) C,$$

where C has the form

$$C = (\Omega_{ca}\Omega_{ab} + \Omega_{cb}\Omega_{ba} - \Omega_{cb}\Omega_{ca}) - (\Omega_{da}\Omega_{ab} - \Omega_{da}\Omega_{db} + \Omega_{db}\Omega_{ba}).$$

An explicit expansion shows that both of the bracketed terms appearing here vanish independently. □

References

[1] Lionel Mason and David Skinner. Ambitwistor strings and the scattering equations. *Journal of High Energy Physics*, 2014(7), July 2014. arXiv: 1311.2564.

[2] Freddy Cachazo, Song He, and Ellis Ye Yuan. Scattering Equations and KLT Orthogonality. *Physical Review D*, 90(6), September 2014. arXiv: 1306.6575.

[3] Freddy Cachazo, Song He, and Ellis Ye Yuan. Scattering of Massless Particles: Scalars, Gluons and Gravitons. *Journal of High Energy Physics*, 2014(7), July 2014. arXiv: 1309.0885.

[4] Freddy Cachazo, Song He, and Ellis Ye Yuan. Scattering of Massless Particles: Scalars, Gluons and Gravitons. *Journal of High Energy Physics*, 2014(7), July 2014. arXiv: 1309.0885.

[5] Tim Adamo, Eduardo Casali, and David Skinner. Ambitwistor strings and the scattering equations at one loop. *Journal of High Energy Physics*, 2014(4), April 2014. arXiv: 1312.3828.

[6] Tim Adamo, Eduardo Casali, Lionel Mason, and Stefan Nekovar. Amplitudes on plane waves from ambitwistor strings. *arXiv:1708.09249 [hep-th]*, August 2017. arXiv: 1708.09249.

[7] Tim Adamo, Eduardo Casali, Lionel Mason, and Stefan Nekovar. Scattering on plane waves and the double copy. *arXiv:1706.08925 [gr-qc, physics:hep-th]*, June 2017. arXiv: 1706.08925.

[8] Nathan Berkovits. Infinite Tension Limit of the Pure Spinor Superstring. *Journal of High Energy Physics*, 2014(3), March 2014. arXiv: 1311.4156.

[9] Nathan Berkovits. Super-Poincare Covariant Quantization of the Superstring. *Journal of High Energy Physics*, 2000(04):018–018, April 2000. arXiv: hep-th/0001035.

[10] Nathan Berkovits. Covariant Map Between Ramond-Neveu-Schwarz and Pure Spinor Formalisms for the Superstring. *Journal of High Energy Physics*, 2014(4), April 2014. arXiv: 1312.0845.

[11] Nathan Berkovits. Twistor Origin of the Superstring. *arXiv:1409.2510 [hep-th]*, September 2014. arXiv: 1409.2510.

[12] Humberto Gomez and Ellis Ye Yuan. N-Point Tree-Level Scattering Amplitude in the New Berkovits' String. *Journal of High Energy Physics*, 2014(4), April 2014. arXiv: 1312.5485.

[13] Tim Adamo, Eduardo Casali, and David Skinner. A Worldsheet Theory for Supergravity. *Journal of High Energy Physics*, 2015(2), February 2015. arXiv: 1409.5656.

[14] Osvaldo Chandia and Brenno Carlini Vallilo. Ambitwistor pure spinor string in a type II supergravity background. *arXiv:1505.05122 [hep-th]*, May 2015. arXiv: 1505.05122.

[15] Thales Azevedo and Renann Lipinski Jusinskas. Background constraints in the infinite tension limit of the heterotic string. *Journal of High Energy Physics*, 2016(8):133, August 2016.

[16] Nathan Berkovits and Paul Howe. Ten-Dimensional Supergravity Constraints from the Pure Spinor Formalism for the Superstring. *Nuclear Physics B*, 635(1-2):75–105, July 2002. arXiv: hep-th/0112160.

[17] Edward Witten. An interpretation of classical Yang-Mills theory. *Physics Letters B*, 77(4):394–398, August 1978.

[18] James Isenberg, Philip B. Yasskin, and Paul S. Green. Non-self-dual gauge fields. *Physics Letters B*, 78(4):462–464, October 1978.

[19] Edward Witten. Twistor-like transform in ten dimensions. *Nuclear Physics B*, 266(2):245–264, March 1986.

[20] J. Harnad and S. Shnider. Constraints and field equations for ten-dimensional super Yang-Mills theory. *Communications in Mathematical Physics*, 106(2):183–199, 1986.

[21] Z. Bern, J. J. M. Carrasco, and H. Johansson. New Relations for Gauge-Theory Amplitudes. *Physical Review D*, 78(8), October 2008. arXiv: 0805.3993.

[22] Carlos R. Mafra, Oliver Schlotterer, and Stephan Stieberger. Explicit BCJ Numerators from Pure Spinors. *Journal of High Energy Physics*, 2011(7), July 2011. arXiv: 1104.5224.

[23] Ricardo Monteiro, Donal O'Connell, and Chris D. White. Black holes and the double copy. *Journal of High Energy Physics*, 2014(12), December 2014. arXiv: 1410.0239.

[24] Yvonne Geyer, Lionel Mason, Ricardo Monteiro, and Piotr Tourkine. One-loop amplitudes on the Riemann sphere. *Journal of High Energy Physics*, 2016(3), March 2016. arXiv: 1511.06315.

[25] Yvonne Geyer, Lionel Mason, Ricardo Monteiro, and Piotr Tourkine. Two-Loop Scattering Amplitudes from the Riemann Sphere. *Physical Review D*, 94(12), December 2016. arXiv: 1607.08887.

[26] Kai A. Roehrig and David Skinner. A Gluing Operator for the Ambitwistor String. *arXiv:1709.03262 [hep-th]*, September 2017. arXiv: 1709.03262.

[27] D. Haertl, O. Schlotterer, and S. Stieberger. Higher Point Spin Field Correlators in D=4 Superstring Theory. *Nuclear Physics B*, 834(1-2):163–221, July 2010. arXiv: 0911.5168.

[28] Oliver Schlotterer. Higher Loop Spin Field Correlators in D=4 Superstring Theory. *Journal of High Energy Physics*, 2010(9), September 2010. arXiv: 1001.3158.

[29] D. Haertl and O. Schlotterer. Higher Loop Spin Field Correlators in Various Dimensions. *Nuclear Physics B*, 849(2):364–409, August 2011. arXiv: 1011.1249.

[30] Claude LeBrun. *Spaces of complex geodesics and related structures.* Thesis, University of Oxford, 1980.

[31] Claude LeBrun. Spaces of complex null geodesics in complex-Riemannian geometry. *Transactions of the American Mathematical Society*, 278(1):209–231, 1983.

[32] Claude Le Brun. The exceptional case of three dimensions. *Twistor Newsletter*, 9, November 1979.

[33] John Harnad and Steven Shnider. *Isotropic geometry and twistors in higher dimensions. I. The generalized Klein correspondence and spinor flags in even dimensions*, volume 33. September 1992. DOI: 10.1063/1.529538.

[34] Yu I. Manin. Gauge fields and holomorphic geometry. *Journal of Soviet Mathematics*, 21(4):465–507, March 1983.

[35] N. P. Buchdahl. Analysis on analytic spaces and non-self-dual Yang-Mills fields. *Transactions of the American Mathematical Society*, 288(2):431–469, 1985.

[36] Lars Brink, John H. Schwarz, and J. Scherk. Supersymmetric Yang-Mills theories. *Nuclear Physics B*, 121(1):77–92, March 1977.

[37] John Harnad, Joel A. Shapiro, Steven Shnider, and Cyrus C. Taylor. Symplectic Reduction of the Minimally Coupled Massless Superparticle in D=10. In *Differential Geometric Methods in Theoretical Physics*, NATO ASI Series, pages 693–701. Springer, Boston, MA, 1990. DOI: 10.1007/978-1-4684-9148-7_68.

[38] Michael Eastwood and Claude LeBrun. Thickening and Supersymmetric Extensions of Complex Manifolds. *American Journal of Mathematics*, 108(5):1177–1192, 1986.

[39] P. S. Howe and P. C. West. The complete N =2, d = 10 supergravity. *Nuclear Physics B*, 238(1):181–220, May 1984.

[40] M. V. Movshev. The odd twistor transform in eleven-dimensional supergravity. *arXiv:1206.0057 [hep-th, physics:math-ph]*, May 2012. arXiv: 1206.0057.

[41] Daniel Friedan, Stephen Shenker, and Emil Martinec. Covariant quantization of superstrings. *Physics Letters B*, 160(1):55–61, October 1985.

[42] Rimhak Ree. Lie Elements and an Algebra Associated With Shuffles. *Annals of Mathematics*, 68(2):210–220, 1958.

[43] Christophe Reutenauer. *Free Lie Algebras.* London Mathematical Society Monographs. Oxford University Press, Oxford, New York, May 1993.

[44] Tim Adamo, Ricardo Monteiro, and Miguel F. Paulos. Space-time CFTs from the Riemann sphere. *Journal of High Energy Physics*, 2017(8), August 2017. arXiv: 1703.04589.

[45] Seungjin Lee and Oliver Schlotterer. Fermionic one-loop amplitudes of the RNS superstring. *arXiv:1710.07353 [hep-th]*, October 2017. arXiv: 1710.07353.

[46] V. S. Varadarajan. *Supersymmetry for Mathematicians: An Introduction*. American Mathematical Society, Providence, R.I, July 2004.

[47] Penrose/Rindler. *002: Spinors and Space Time Volume 2: Spinor and Twistor Methods in Space-time Geometry Vol 2*. Cambridge University Press, Cambridge, revised ed. edition edition, January 2008.

[48] Nathan Berkovits and Sergey A. Cherkis. Higher-Dimensional Twistor Transforms using Pure Spinors. *Journal of High Energy Physics*, 2004(12):049–049, December 2004. arXiv: hep-th/0409243.

[49] Nathan Berkovits and Nikita Nekrasov. The Character of Pure Spinors. *Letters in Mathematical Physics*, 74(1):75–109, October 2005. arXiv: hep-th/0503075.

[50] Julius Wess and Jonathan Bagger. *Supersymmetry and Supergravity*. Princeton University Press, Princeton, N.J, revised edition edition, March 1992.

[51] Edward Witten. Notes On Supermanifolds and Integration. *arXiv:1209.2199 [hep-th]*, September 2012. arXiv: 1209.2199.

[52] Jacky Cresson. Calcul Moulien. *arXiv:math/0509548*, September 2005. arXiv: math/0509548.