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AGNALDO ALESSANDRO DA SILVA JUNIOR

Generalized Ricci curvature on contact Calabi-Yau 7-manifolds

Curvatura generalizada de Ricci em 7-variedades Calabi-Yau de contato

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Agnaldo Alessandro da Silva Junior

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Supervisor: Henrique Nogueira de Sá Earp

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- ORCID do autor: https://orcid.org/0000-0003-2603-536X
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Resumo

Esta dissertação explora a relação entre as soluções para o G_2-sistema heterótico e as condições para Ricci generalizado plano e para a satisfazer as equações de instantons acoplados, no contexto de algebroides de Courant transitivos em geometria generalizada. Motivado por desenvolvimentos recentes em física teórica, o estudo reinterpreta as equações do G_2-sistema heterótico, examinando as interconexões entre instantons acoplados, métricas Ricci-planas generalizadas e espinores de Killing em um algebroide de Courant.

A pesquisa aborda dois problemas principais relacionados a desenvolvimentos recentes tanto em física, notavelmente por De la Ossa, Larfors e Svanes [dlOLS18a], quanto em matemática sobre variedades Calabi-Yau, como visto em trabalhos recentes por Garcia-Fernandez e González Molina [GFGM23]. A dissertação apresenta a resolução completa do primeiro problema e, especificamente para G_2-estruturas em sete dimensões, é feito um progresso significativo na abordagem do segundo problema. Além disso, o trabalho examina soluções aproximadas para o G_2-sistema heterótico, particularmente aquelas desenvolvidas em variedades Calabi-Yau de contato 7-dimensionais por Sá Earp e Lotay [LSE23]. Aqui, confirmamos a existência de 7-dimensionais-instantons acoplados aproximados e métricas aproximadamente Ricci generalizadas planas.

Por fim, a dissertação apresenta uma abordagem não-espinorial para definir instantons. Os dois problemas foram resolvidos com essa perspectiva, e uma exploração de como estender nossos resultados a outras estruturas geométricas, além das G_2-estruturas, foi conduzida.

Keywords: teorias de calibre, Calabi-Yau de contato, instantons, G_2-estruturas, instantons acoplados, equações espinoriais de Killing, G_2-sistema heterótico.

Abstract

This dissertation explores the relationship between solutions for the heterotic G_2 system and the conditions for generalized Ricci flatness and to satisfy coupled instantons equations within the context of transitive Courant algebroids in generalized geometry. Motivated by recent developments in theoretical physics, the study reinterprets the equations of the heterotic G_2-system, examining the interconnections between coupled instantons, generalized Ricci-flat metrics, and Killing spinors on a Courant algebroid.

The research addresses two main problems related to recent developments in both physics, notably by De la Ossa, Larfors, and Svanes [dlOLS18a], and in mathematics for Calabi-Yau manifolds, as seen in recent works by Garcia-Fernandez and González Molina [GFGM23]. The dissertation presents the complete resolution of the first problem and, specifically for G_2 structures in seven dimensions, significant progress is made in addressing the second problem. Additionally, the work examines approximate solutions for the heterotic G_2-system, particularly those developed on 7-dimensional contact Calabi-Yau manifolds by Sá Earp and Lotay [LSE23]. Here, we confirm the existence of approximate coupled G_2 instantons and approximately Ricci-flat generalized metrics.

Finally, the dissertation introduces a non-spinorial approach to defining instantons. Both problems were resolved from this perspective, and an exploration of how to extend our results to other geometric structures beyond G_2 structures was conducted.

Keywords: gauge theory, contact Calabi-Yau, instantons, G_2-structures, coupled instantons, Killing spinor equations, heterotic G_2-system.

Contents

Introduction

This project explores the intersection of two intensely researched areas in differential geometry: *generalized geometry* and the *heterotic* G_2*-system*. Our primary objective is to investigate how solutions for the heterotic G_2-system naturally align with the conditions of *generalized Ricci flatness* and *coupled instanton equations* within the framework of generalized geometry.

Nigel Hitchin's pioneer work on generalized geometry, which suggests a changing on the tangent bundle of a manifold M by the sum bundle $TM \oplus T^*M$ (what we call now *exact Courant algebroids*) with a closed 3-form H (the NS-flux in the physics literature), opened new avenues for studying *type II string theory*. This framework has been extended to *heterotic string theory*, where the NS-flux, still a 3-form, satisfies the *heterotic Bianchi identity*:

$$dH = \frac{\alpha'}{4}\mathrm{tr}\,(F_A \wedge F_A - F_\theta \wedge F_\theta).$$

This formulation integrates the curvatures of the connections A and θ within both the tangent bundle and an additional vector bundle. To align with generalized geometry, we consider a principal bundle $P \to M$ and the *transitive Courant algebroid*, commonly denoted as $E = TM \oplus \mathrm{ad}P \oplus T^*M$. These connections are a central focus of our study.

Generalized geometry extends the study of geometric properties beyond the traditional tangent bundle for more general objects, the *Courant algebroids*. This broader perspective introduces the concept of *generalized connections*, analogous to connections in Riemannian geometry, and allows for the definition of *generalized Ricci curvature*.

On the other hand, the exploration of manifolds endowed with G_2-structures was initially motivated by Berger's list of possible holonomy groups:

$$\mathrm{SO}(n), \quad \mathrm{U}(m), \quad \mathrm{SU}(m), \quad \mathrm{Sp}(k), \quad \mathrm{Sp}(k)\mathrm{Sp}(1), \quad \mathrm{G}_2, \quad \mathrm{Spin}(7).$$

The pioneering constructions in this realm were due to Bryant, Salamon, and Joyce [Bry87, BS89, Joy96]. G_2-structures found their way into theoretical physics due to the Hull-Strominger system which describes the supersymmetric background in heterotic string theories, originally in the context of Calabi-Yau 3-folds [Hul86, Str86] and which lately culminates in the G_2 version, called *heterotic* G_2*-system*.

Solutions to such a system are naturally expressed within the framework of generalized geometry using transitive Courant algebroids. In this context, we can explore generalized Ricci flatness for solutions of the heterotic G_2-system, which is a primary focus of this dissertation.

The second main focus of this dissertation is the concept of the *coupled instanton equation*, which represents the instanton condition for a specific connection D in a part of the Courant algebroid, namely $TM \oplus \mathrm{ad}P$. This connection D is induced by another connection θ in P. The introduction of this concept is based on an important result by de la Ossa, Larfors, and Svanes ([dlOLS18a]), which states that the instanton condition for θ implies the coupled instanton equation, and vice versa, up to (α')-approximations. This approach is commonly used in theoretical physics. From a mathematical perspective, this idea is studied by Garcia-Fernandez and Molina for $\mathrm{U}(m)$-structures [GFGM23]. Here, we revisit this notion of coupled instanton for G_2-structures and its relation to the *Killing spinor equations*.

About this project

This master's project began in the first semester of 2022. During the first year, the author of this dissertation studied the preliminary concepts for the project, including general aspects of geometry using mainly references [KN63, Dar94, KN96, Joy00, Tu10, Tu17, FSE19], an overview of generalized geometry using mainly the references [GFS20, GF14] and the heterotic G_2-system using mainly the references [LSE23] (from the advisor of this master project) and [Bla10].

The master's project included two international research internships. The first one (funded by FAPESP grant nº 2023/00126-6, BEPE - Bolsa de Estágio de Pesquisa no Exterior) took place during the first semester of 2023 at the Instituto de Ciencias Matemáticas (ICMAT), affiliated with the Universidad Autónoma de Madrid (UAM), under the supervision of Professor Mario Garcia-Fernandez. This internship's primary focus was exploring the coupled instanton equations within the framework of G_2-structures, inspired by the recent work of Garcia-Fernandez [GFGM23]. The findings from this research are now discussed in Chapter 4 and are further elaborated in parts of Chapter 2 (as we will describe below). Key references during this period included [GFGM23, dlOLS18a, GF14, GF19, CGFT22].

The second internship (also funded by FAPESP grant nº 2022/13162-8, BEPE - Bolsa de Estágio de Pesquisa no Exterior) took place during the second semester of 2023 at the Mathematical Institute of Oxford University. Professor Jason D. Lotay guided this research internship. The primary objective of this second internship was to leverage the knowledge of generalized geometry, coupled equations and conditions for generalized Ricci flatness the author had acquired during the first BEPE, applying it to the approximate solutions constructed in [LSE23] - a collaborative effort between Jason D. Lotay and the master's project advisor. The advancements from this project are documented in Chapter 5. Additionally, this period marked the beginning of efforts to generalize certain concepts previously performed to G_2-structures. These generalized concepts are thoroughly developed in Chapter 6, with applications to Spin(7)-structures and almost-Hermitian structures in the Chapter 7. Key literature references during this internship include [LSE23, dlOG21, FI02, LM90, Fri00].

After completing both internships, it was determined that the project results, with some additional work, were substantial enough to warrant the writing of a paper authored by the author's dissertation, the advisor, and the advisors in the internships, Mario Garcia-Fernandez and Jason D. Lotay. By April 2024, the paper was completed and submitted for the special issue "At the Interface of Complex Geometry and String Theory" in the International Journal of Mathematics (IJM) (preprint available on arXiv.org and referenced as [dSJGFLSE24]).

Outline of the dissertation

The structure of chapters in this dissertation is heavily influenced by the previously mentioned paper [dSJGFLSE24], co-authored by the author and advisors. In particular, Chapter 2, Chapter 4, and Chapter 5 mirror specific chapters of [dSJGFLSE24], albeit with targeted modifications. Chapter 1 not only introduces the content of Chapter 2 but also substantially expands upon the introductory material and appendix from [dSJGFLSE24], delving into the concepts of generalized geometry and generalized Ricci curvature. Similarly, Chapter 3 serves as a primer for the discussions in Chapter 4 and Chapter 5, expanding the quick review about G_2-structures presented in [dSJGFLSE24]. Both Chapter 1 and Chapter 3 aim to provide a more thorough exposition of the topics introduced at the level of a master's dissertation.

On the other hand, Chapter 6 and Chapter 7 introduce content not covered in [dSJGFLSE24]. These chapters explore an alternative concept of instantons and address the problems previously presented in Chapter 2, now using this novel notion of instantons. Specifically, they apply these alternative instantons to different geometric structures, notably focusing on Spin(7) structures and $U(m)$ (almost-Hermitian structures) in Chapter 7. This part of the dissertation is a current work and aims to become a publication in the future.

In sequence, we will detail how each chapter is organized: outlining its structure, highlighting the main results, and emphasising the unique research contexts in which the author has been involved.

Chapter 1 introduces fundamental concepts in generalized geometry, including Courant algebroids, generalized metrics, generalized connections, and divergence operators. Section 1.5 concludes the chapter with a detailed computation of the generalized Ricci curvature for transitive Courant algebroids, a key focus of our investigation (and which was initially proved in [GF14]); the main result is Theorem 1.5.5, which provides the explicit formula for the Ricci curvature.

Chapter 2 introduces the concept of *Killing spinor equations* in alignment with generalized geometry and explores the *coupled instanton equations* within the framework of transitive Courant algebroids over an oriented spin manifold. This chapter lays the foundation by presenting two pivotal problems that will serve as the cornerstone throughout the dissertation: understanding the conditions required to satisfy the coupled instanton equations (refer to Problem 1), and investigating the criterion for generalized Ricci flatness (refer to Problem 2).

Among the main results in Chapter 2, we have Proposition 2.1.6, which demonstrates that solutions to the Killing spinor equations exhibit generalized Ricci flatness. Additionally, a key highlight is Theorem 2.3.2, establishing that solutions to the gravitino equation inherently satisfy the coupled instanton equations, thereby addressing Problem 1. This theorem represents a critical piece of research developed through the author's collaborative efforts during an internship in Spain, specifically focusing on G_2-structures. The current version of this theorem has been refined and enhanced as documented in the final iteration of the paper [dSJGFLSE24].

Chapter 3 provides a comprehensive overview of G_2-*structures*, starting with the geometric motivation using the cross product in seven dimensions and how this choice defines a G_2-structure punctually on the tangent space of a 7-manifold. The chapter then delves into the decomposition of the spaces of forms into irreducible G_2-modules in the presence of such a structure, followed by a discussion on the torsion of G_2-structures and their connections with skew-symmetric torsion. Finally, the chapter explores the spinorial description of G_2-structures.

Subsequently, Chapter 4, we utilize the theory of Killing spinor equations and coupled equations, along with their connection to generalized Ricci flatness as introduced in Chapter 2, in the context of G_2-*structures* and solutions of the *heterotic* G_2-*system*. We also present an alternative proof of the generalized Ricci flatness for such structures, building on the results from [IS23b]. This approach will be extended to other structures in Chapter 6 and Chapter 7. Additionally, we provide several examples at the end of the chapter to illustrate the theory.

While Chapter 3 comprehensively outlines established knowledge regarding G_2-structures, Chapter 4 presents new findings in this domain. A pivotal development detailed in this chapter is the equivalence between the heterotic G_2-system and the Killing spinor equations for G_2-structures. This equivalence crucially leads to generalized Ricci flatness, as detailed in Proposition 4.1.2 and Theorem 4.2.1. Additionally, the sequence of results culminates in the characterization of the coupled G_2-instanton equations, a line in which the author has worked since his internship in Spain, cf. Proposition 4.3.5 and Theorem 4.3.6.

In Chapter 5, we apply the theory from Chapter 4 to a specific class of solutions of the heterotic G_2-system, which were constructed in [LSE23]. These solutions are actually 'approximate' because the instanton condition $F_\theta \wedge \psi$ is not zero but of class $\mathcal{O}(\alpha')^2$. As a result of the author's internship in Oxford, this chapter examines how the results behave under these approximate solutions and establishes conditions for such solutions to be *'approximate' generalized Ricci flat* and satisfy the *'approximate' coupled instanton equation*.

Notably, being approximate generalized Ricci flat is related to the condition $\nabla^{\theta,+} F_\theta \wedge \psi$ to be also of order $\mathcal{O}(\alpha')^2$. Based on the necessity of this condition, we propose a new definition of what should be *approximate instanton*, cf. Definition 5.4.1. Using this definition, we prove the equivalent statements of Chapter 4: approximate coupled instanton equation and approximate generalized Ricci flatness under some conditions (cf. Theorem 5.4.4 and Theorem 5.4.7).

Chapter 6 introduces an alternative concept of *instanton* based on the existence of a 4-form $\psi \in \Omega^4(M)$, rather than the instanton concept presented in Chapter 2, which relies on the existence of a non-vanishing spinor $\eta \in \Omega^0(S)$. In G_2-structures, these two concepts are equivalent. The primary results in this chapter address the problems introduced in Chapter 2, related to generalized Ricci flatness and coupled equations. For instance, using the spinorial approach, we still have not a general solution for the problem of generalized Ricci flatness. However, we solved it for G_2-structures because the method was based on the equivalence of these two notions.

The main result of this chapter is the introduction of the flux operator $\mathbf{H} : \Omega^3(M) \to \Omega^3(M)$ and a method to calculate the torsion H of compatible connection with skew-symmetric torsion using the eigenvalues of this operator (cf. Theorem B.3.6). In sequence, we derive conditions to rewrite the Yang-Mills equation, cf. Theorem 6.2.3 for an instanton, to make the first term for generalized Ricci curvature in Theorem 1.5.5 be zero. This culminates in finding conditions for generalized Ricci flatness in Theorem 6.3.6 in a string algebroid which pair (H, θ) given by the torsion of the connection compatible with structure induced by ψ and θ an instanton with relation to instanton form $\psi \in \Omega^4(M)$.

Finally, the Chapter 7 presents an overview of Spin(7) and almost-Hermitian structures (U(m)-structures) and applies the results of Chapter 6 to the case of these structures. So, as we have obtained for G_2-structures in Chapter 4, here we have obtained conditions of generalized Ricci flatness and how an instanton satisfies the coupled instanton equations in Spin(7) and almost-Hermitian structures.

1 Introduction to Courant algebroids and generalized geometry

This chapter provides a comprehensive overview of foundational principles in generalized geometry, aimed at familiarizing readers with crucial concepts and examples essential for understanding the subsequent text, such as *Courant algebroids* and *generalized curvature tensors*. The concepts introduced primarily follow the book [GFS20] and draw on ideas from [GF19, GF14]. The concluding section, which focuses on an explicit computation for the generalized Ricci curvature, is a more specific computation.

This chapter is organized as follows: Section 1.1 delves into the fundamental concepts of generalized geometry, focusing on Courant algebroids as outlined in Definition 1.1.1. These structures are a versatile replacement for the traditional tangent bundle, offering a broader perspective on Riemannian geometry. Our exploration encompasses various forms of these algebroids, with particular attention to *transitive* Courant algebroids, exemplified in Example 1.1.5 and Example 1.1.5 respectively.

Moving forward, Section 1.2 will delve into the concept of *generalized metric*, focusing mainly on its canonical manifestation in transitive algebroids when the basis manifold is Riemannian (refer to Example 1.2.4), which will be the primary focus of our discussions in the future.

Following this, Section 1.3 introduces *generalized connections* and discusses the notion of curvature associated with such connections, particularly the specificities of direct generalizations of curvature in the context of generalized geometry. Notably, we introduce the idea of *generalized Ricci curvature* (cf. Definition 1.3.7), an essential concept that reoccurs throughout subsequent chapters.

Section 1.4 is dedicated to introducing *divergence operators* and some examples (as divergence induced by a connection and the Riemannian divergence), pivotal for exploring generalized Ricci flatness within transitive Courant algebroids.

Finally, in Section 1.5, we present an explicit formula for the generalized Ricci curvature for a torsion-free generalized connection compatible with a given generalized metric and a given divergence operator in the context of transitive Courant algebroids (cf. Theorem 1.5.5). These calculations were initially expounded upon in meticulous detail by [GF14] and also revisited in [dSJGFLSE24].

1.1 Courant algebroids

Generalized geometry presents a more general framework than the one typically considered in Riemannian geometry. Instead of the tangent bundle TM of a manifold M, we embrace a more expansive vector bundle $E \to M$, called Courant or Dorfman algebroid. Initially proposed as $TM \oplus T^*M$ by Nigel Hitchin, this notion has since evolved into a more versatile concept. Furthermore, we shift from the familiar Lie bracket on vector fields to the Courant bracket, an operation between sections of the Courant algebroid [Hit10].

The essence of generalized geometry lies in leveraging our understanding of traditional differential geometry and extending it through analogy. These extensions find their roots in Type II string theory, where the NS-flux manifests as a closed 3-form denoted as H. Later, the impetus for further exploration stems from heterotic string theory, when the 3-form H satisfies the anomaly cancellation condition regarding its exterior derivative dH, [Sis19, Sec. 1].

Courant algebroids are like a broader version of the usual tangent bundle TM of a Riemannian manifold. They incorporate similar concepts like Lie brackets and inner products, forming the foundation of generalized geometry [Hit10].

The material of this chapter is based mainly on the book [GFS20] and the papers [GFRT20, GFS20, BH15, GF14]).

Definition 1.1.1. *A* Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ *over a manifold M consists of a vector bundle $E \to M$ endowed with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ and a* Dorfman bracket $[\cdot,\cdot]$ *on $\Omega^0(E)$, and a bundle map $\pi\colon E \to TM$, called an* anchor map *, such that the following axioms are satisfied, for all $a, b, c \in \Omega^0(E)$ and $f \in C^\infty(M)$:*

(1) $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$,

(2) $\pi[a,b] = [\pi(a),\pi(b)]$,

(3) $[a, fb] = f[a,b] + \pi(a)(f)b$,

(4) $\pi(a)\langle b,c\rangle = \langle [a,b],c\rangle + \langle b,[a,c]\rangle$,

(5) $[a,b] + [b,a] = \mathcal{D}\langle a,b\rangle$.

Here, the notation $\mathcal{D} : C^\infty(M) \to \Omega^0(E)$ denotes the composition of the exterior differential $d\colon C^\infty(M) \to \Omega^1(M)$, the dual map $\pi^*\colon T^*M \to E^*$ and the isomorphism $E^* \cong E$ provided by the non-degenerancy of $\langle\cdot,\cdot\rangle$.

Remark 1.1.2 (Non-skew-symmetry of Dorfman Bracket)*.* Note that condition (5) implies that the Dorfman bracket is generally not skew-symmetric, unlike the Lie brackets between vector fields. However, if $a, b \in \Gamma(E)$ are two orthogonal sections, i.e., $\langle a, b\rangle = 0$, then they anti-commute.

This lack of anti-commutation has implications for condition (3). Does $[fa, b]$ satisfy a condition similar to (3)? The answer is *no*. The relation is slightly different:

$$\begin{aligned}
[fa, b] &= -[b, fa] + \pi^* d\langle fa, b\rangle \\
&= -f[b, a] - \pi(b)(f)a + \pi^*\Big(df\ \langle a, b\rangle + fd\langle a, b\rangle\Big) \\
&= f\Big(-[b, a] + \pi^* d\langle a, b\rangle\Big) - \pi(b)(f)a + \langle a, b\rangle \pi^* df \\
&= f[a, b] - \pi(b)(f)a + \langle a, b\rangle \pi^* df
\end{aligned}$$

Note that there is a minus sign with $\pi(b)(f)a$ in this expression (which is different from the case of $[a, fb]$), and there is an additional term $\langle a, b\rangle \pi^* df$ (which is zero in the particular case of orthogonal sections). ◯

Considering the non-skew-symmetric nature of the Dorfman bracket, some notions still need the skew-symmetry of the bracket to be defined. To account for this, we define the skew-symmetrization of the Dorfman bracket as follows:

$$[\![a, b]\!] := \frac{1}{2}[a, b] - \frac{1}{2}[b, a]$$

for $a, b \in \Gamma(E)$. It is evident that $[\![\cdot, \cdot]\!]$ represents a skew-symmetric operation on sections.

We will denote a Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ simply by E. Using the isomorphism $\langle\cdot,\cdot\rangle : E \to E^*$, we obtain a sequence of vector bundles

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0. \tag{1.1.1}$$

We consider three relevant categories of Courant algebroids: exact, transitive, and regular Courant algebroids.

Definition 1.1.3. *A Courant algebroid $E \to M$ is called*

- exact*, if the sequence in* (1.1.1) *is exact;*
- transitive*, (sometimes called* heterotic*) if the anchor map π is subjective and;*
- regular*, if the anchor map π has constant rank.*

For the class of exact Courant algebroids, we have by definition of exact sequence that π^* is injective, π subjective, and

$$\mathrm{Im}(\pi^*) = \ker(\pi)$$

and since the fibers of TM and T^*M are vector spaces with the same dimension n, we have by the range-kernel theorem

$$\operatorname{rank}(E) = \dim(\operatorname{Im}(\pi)) + \dim(\ker(\pi)) = n + n = 2n.$$

In particular, π is surjective. Since π being surjective implies the map's rank is maximum, in particular, constant, we have that:

$$\text{exact} \implies \text{transitive} \implies \text{regular}.$$

Example 1.1.4 (Example of exact Courant algebroid)**.** Let M be a manifold and $H \in \Omega^3(M)$. Consider $E = TM \oplus T^*M$ and the symmetric pairing

$$\langle X + \xi, Y + \eta \rangle \coloneqq \frac{1}{2} \left(i_X\eta + i_Y\xi \right), \tag{1.1.2}$$

the bracket is defined by

$$[X + \xi, Y + \eta] \coloneqq [X, Y] + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H, \tag{1.1.3}$$

with the canonical projection in the first coordinate $\pi : E \to TM$. Then the data $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ is a Courant algebroid (cf. Definition 1.1.1)) if, and only if H is a closed 3-form, i.e., $dH = 0$.

Conversely, if we have an exact Courant algebroid $E \to M$ is possible to construct a closed 3-form $H \in \Omega^3(M)$ such that E is isomorphic (as Courant algebroid) to $TM \oplus T^*M$ (as constructed above), so the example discussed is all exact Courant algebroids, up to isomorphism, cf. [GFS20, Proposition 2.10]. △

Example 1.1.5 (Example of transitive Courant algebroid)**.** Let K be a Lie group, and $P \to M$ be a principal K-bundle over M. We assume a non-degenerate bi-invariant pairing on the Lie algebra $\mathfrak{k}$ of K (which subscript $\mathfrak{k}$ can be omitted by convenience)

$$\langle \cdot, \cdot \rangle_{\mathfrak{k}} : \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}. \tag{1.1.4}$$

Consider the Whitney sum of vector bundles[1]

$$E = TM \oplus \operatorname{ad}P \oplus T^*M \tag{1.1.5}$$

endowed with the symmetric pairing

$$\langle X + r + \xi, Y + t + \eta \rangle = \frac{1}{2} \left(\eta(X) + \xi(Y) \right) + \langle r, t \rangle_{\mathfrak{k}} \tag{1.1.6}$$

and the canonical projection $\pi : X + r + \xi \in E \mapsto X \in TM$ on the first coordinate.

[1] Given a principal G-bundle, we have the adjoint representation $\operatorname{Ad} : K \to \operatorname{Aut}(\mathfrak{k})$ which induces an associated bundle $\operatorname{ad}P \coloneqq P \times_{\operatorname{Ad}} \mathfrak{k}$, called the *adjoint bundle* which fibres are copies of $\mathfrak{k}$.

Given a 3-form $H \in \Omega^3(M)$ and a connection form $\theta \in \Omega^1(P, \mathfrak{k})$ on P, and denote $F_\theta \in \Omega^2(P, \mathfrak{k})$ its curvature 2-form (which can be considered living in $F_\theta \in \Omega^2(M, \mathrm{ad}P)$ canonically, due its invariance). Thus, we can define a bracket on $\Omega^0(E)$ as follows:

$$\begin{aligned}[X + r + \xi, Y + t + \eta] = [X, Y] &+ \mathcal{L}_X \eta - i_Y d\xi + i_Y i_X H \\ &- [r, t] - F_\theta(X, Y) + d^\theta_X t - d^\theta_Y r \\ &+ 2\langle d^\theta r, t\rangle_{\mathfrak{k}} + 2\langle i_X F_\theta, t\rangle_{\mathfrak{k}} - 2\langle i_Y F_\theta, r\rangle_{\mathfrak{k}}.\end{aligned} \tag{1.1.7}$$

Then, the data $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ satisfies the axioms of a Courant algebroid (see Definition 1.1.1) if and only if the so-called *heterotic Bianchi identity*[2] is satisfied:

$$dH = \langle F_\theta \wedge F_\theta\rangle_{\mathfrak{k}}. \tag{1.1.8}$$

In the affirmative case, the data constitutes a transitive Courant Algebroid since $\pi : E \to TM$ is surjective as it represents a projection.

Conversely, if we have a transitive Courant algebroid $E \to M$, is possible to find a 3-form $H \in \Omega^3(M)$, a principal K-bundle $P \to M$ (which Lie algebra $\mathfrak{k}$ is endowed with a symmetric pairing $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$) endowed with a connection form $\theta \in \Omega^1(P, \mathfrak{k})$ satisfying the heterotic Bianchi identity (1.1.8) such that E is isomorphic to the example constructed above, cf. [CSX13]. △

1.2 Generalized metrics

In this section, we recall essential aspects of generalized Riemannian geometry, following [GF19, GFS20]. As we will see later, some concepts in generalized geometry are not so well-behaved as in Riemannian geometry; for instance, curvature operators are not tensors. For this reason, it is interesting to have the Courant algebroid orthogonally split in order to make some of these concepts be well-defined.

Definition 1.2.1. *Let M^n be an oriented manifold endowed with a Courant algebroid E. A* generalized metric *on E is an orthogonal decomposition $E = V_+ \oplus V_-$, such that the restriction of $\langle\cdot,\cdot\rangle$ to V_+ is positive definite, the restriction to V_- is negative definite, and that $\pi_{|V_+} : V_+ \to TM$ is an isomorphism.*

Given a generalized metric, we can naturally define the *projections* into the components $\pi_\pm : a_+ + a_- \in E \mapsto a_\pm \in V_\pm$.

[2] We cannot define a wedge product between such forms if we have E-valued differential forms $\xi, \zeta \in \Omega(M, E)$. However, if there is an operation $\mu \in \Gamma(T^{2,1}(M))$, then the wedge product is defined in the following way: without loss of generality, suppose $\xi = \xi_0 \otimes S$ and $\zeta = \zeta_0 \otimes T$ for $\xi_0, \zeta_0 \in \Omega(M)$ and $S, T \in \Gamma(E)$, we define the wedge product of such forms as

$$\xi \wedge \zeta := (\xi_0 \wedge \zeta_0) \otimes \mu(S, T).$$

In our context, the Lie algebra is endowed with a symmetric pairing $\langle\cdot,\cdot\rangle$, which is μ in this case.

Lemma 1.2.2. *A generalized metric $E = V_+ \oplus V_-$ is equivalent to an endomorphism $\mathbf{G} : E \to E$ satisfying the following conditions:*

(1) $\mathbf{G}$ *is an $\langle \cdot, \cdot \rangle$-isometry, i.e., $\langle \mathbf{G}a, \mathbf{G}b \rangle = \langle a, b \rangle$,*

(2) $\mathbf{G}$ *is $\langle \cdot, \cdot \rangle$-self adjoint, i.e., $\langle \mathbf{G}a, b \rangle = \langle a, \mathbf{G}b \rangle$,*

(3) The bilinear pairing $\langle \mathbf{G}a, b \rangle$ is symmetric and positive definite,

(4) The restricted anchor map $\pi|_{V_+}$ is an isomorphism.

for all sections $a, b \in \Gamma(E)$. In this case, the projections are given by

$$\pi_\pm = \frac{1}{2}(\mathbf{G} \pm \mathrm{Id}) \tag{1.2.1}$$

Proof. ($\Rightarrow$) Given $V_+ \oplus V_-$ a generalized metric, we can define the endomorphism $\mathbf{G}$ by the formula

$$\mathbf{G}(e_+ + e_-) = e_+ - e_-.$$

Note that conditions (1), (2), and (3) are immediately satisfied.

($\Leftarrow$) Conversely, if we have $\mathbf{G}$ the endomorphism satisfying the conditions in the statement, then $\mathbf{G}^2 = \mathrm{Id}$:

$$\langle a, b \rangle \overset{(1)}{=} \langle \mathbf{G}a, \mathbf{G}b \rangle \overset{(2)}{=} \langle \mathbf{G}^2 a, b \rangle,$$

and we conclude $\mathbf{G}^2 a = a$ by the non-degeneracy of $\langle \cdot, \cdot \rangle$. The fact that $\mathbf{G}$ squares the identity implies the eigenvalues of $\mathbf{G}$ are ± 1. Define $V_\pm \leq E$ as the respective eigen-bundles. Then the decomposition $E = V_+ \oplus V_-$ into eigen-bundles is orthogonal. The restriction of $\langle \cdot, \cdot \rangle$ in $V_\pm$ is positive/negative definite by condition (3). $\square$

Corollary 1.2.3. *A generalized metric $E = V_+ \oplus V_-$ on a Courant algebroid $E \to M$ induces canonically a Riemannian metric in the tangent bundle TM.*

Proof. In fact, the item (3) in Lemma 1.2.2 guarantees $\langle \cdot, \cdot \rangle$ is positive definite in V_+ and the isomorphism $\pi|_{V_+} : V_+ \to TM$ guarantees the Riemannian metric in TM. $\square$

For our purposes on transitive Courant algebroids in the standard form $TM \oplus \mathrm{ad}P \oplus T^*M$ as in Example 1.1.5, there is a natural choice of generalized metric, which we discuss below.

Example 1.2.4 (Transitive Courant algebroid)**.** In Example 1.1.5 of transitive Courant algebroids (in particular for transitive Courant algebroids in Example 1.1.5), where

$$E \cong TM \oplus \mathrm{ad}P \oplus T^*M,$$

for a uniquely determined $H \in \Omega^3(M)$ and principal connection θ on P satisfying the heterotic Bianchi identity (1.1.8). There is a natural choice of generalized metric given by

$$\begin{aligned} V_+ &= \{X + gX : X \in TM\}, \\ V_- &= \{X + r - gX : X \in TM, r \in \mathrm{ad}P\}, \end{aligned} \tag{1.2.2}$$

when M is endowed with the Riemannian metric g. In this case, the induced endomorphism is given by

$$\mathbf{G} = \begin{pmatrix} 0 & 0 & g^{-1} \\ 0 & -\mathrm{Id} & 0 \\ g & 0 & 0 \end{pmatrix},$$

with orthogonal projections

$$\begin{aligned} \pi_+(X + r + \zeta) &= \frac{1}{2}(X + gX + g^{-1}\zeta + \zeta), \\ \pi_-(X + r + \zeta) &= \frac{1}{2}(X - gX - g^{-1}\zeta + \zeta) + r. \end{aligned} \tag{1.2.3}$$

Note that a metric in M induces a generalized metric in the transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ via (1.2.2). Analogously, if we have a generalized metric $\mathbf{G}$ in $E = TM \oplus \mathrm{ad}P \oplus T^*M$, then it induces a Riemannian metric g in TM (cf. Corollary 1.2.3). △

Remark 1.2.5. In the context of transitive Courant algebroids and the canonical generalized metric, let's introduce the notation

$$\sigma_\pm \colon X \in TM \mapsto X \pm gX \in V_\pm, \tag{1.2.4}$$

which will be useful for us later. ◯

1.3 Generalized connections and related curvature operators

The concept of a generalized connection resembles affine connections on vector bundles. However, rather than deriving sections in the direction of a vector field, we differentiate sections from other sections. This aligns with our understanding of generalized geometry, which substitutes vector fields with sections of a more general vector bundle, namely the Courant algebroid. A more detailed definition is presented below.

Definition 1.3.1. *A generalized connection D (or simply, a connection in the context of generalized geometry) on a Courant algebroid $E \to M$ is a first-order differential operator*

$$D : \Gamma(E) \to \Gamma(E^* \otimes E) \tag{1.3.1}$$

satisfying the following Leibniz-type rule

$$D_a(fb) = fD_ab + \pi(a)(f)b \tag{1.3.2}$$

for $a, b \in \Gamma(E)$ *and* $f \in \mathcal{C}^\infty(M)$.

Example 1.3.2. Every Courant algebroid $E \to M$ admits a generalized connection. This is because every vector bundle admits an affine connection ∇, so we define D by:

$$D_a e := \nabla_{\pi(a)} e,$$

where $\pi : E \to TM$ is the anchor map. △

The generalized connections we will care about are the $\langle\cdot,\cdot\rangle$-compatible ones.

Definition 1.3.3. *A generalized connection D on a Courant algebroid E is said to be compatible with the inner product $\langle\cdot,\cdot\rangle$ (or simply $\langle\cdot,\cdot\rangle$-compatible) on E if it satisfies*

$$\pi(e)\langle a, b\rangle = \langle D_e a, b\rangle + \langle a, D_e b\rangle \tag{1.3.3}$$

the set of all $\langle\cdot,\cdot\rangle$-compatible connections is denoted by $\mathcal{D}$.

We also have the notion of compatibility with a generalized metric $\mathbf{G}$ as:

Definition 1.3.4. *Let E be Courant algebroid endowed with a generalized metric V_+ and D a generalized connection $\langle\cdot,\cdot\rangle$-compatible on E. We say that D is* V_+-compatible *(or* compatible with the generalized metric V_+*) if*

$$D(\Gamma(V_\pm)) \subset \Gamma(E^* \otimes V_\pm). \tag{1.3.4}$$

The set of all V_+-compatible generalized connections will be denoted by $\mathcal{D}^{V_+}$ or $\mathcal{D}^{\mathbf{G}}$, where $\mathbf{G}$ is the induced endomorphism of V_+.

For generalized connections, we encounter analogous concepts to torsion, curvature, and Ricci curvature, albeit with some distinctive characteristics. Given a generalized connection D, it seems natural to define the *generalized curvature* of D by

$$\mathrm{GR}_D(e_1, e_2) = D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[e_1,e_2]},$$

based on the usual definition of *Riemannian curvature* for ordinary connections in vector bundles. However, the curvature defined as above is not skew-symmetric, and not even a tensor. The skew-symmetric property could be fixed using the skew-symmetrization for the Dorfman bracket:

$$\mathrm{GR}_D(e_1, e_2) = D_{e_1}D_{e_2} - D_{e_2}D_{e_1} - D_{[\![e_1,e_2]\!]},$$

but this definition is still not tensorial. This condition is a problem because the tensorial properties of curvature are useful in many concepts. So, we will change its definition to make the pair of sections to which it applies more restrictive. Before that, we must introduce the compatibility of a generalized connection and a generalized metric.

Lemma 1.3.5 (Generalized Curvature)**.** *Let E be a Courant algebroid endowed with a generalized metric $\mathbf{G}$ and $D \in \mathcal{D}^{\mathbf{G}}$ a $\mathbf{G}$-compatible generalized connection. The generalized curvatures $\mathrm{GR}_D^{\pm}$ defined by*

$$\mathrm{GR}_D^{\pm}(e_1^{\pm}, e_2^{\mp}) := D_{e_1^{\pm}} D_{e_2^{\mp}} - D_{e_2^{\mp}} D_{e_1^{\pm}} - D_{[e_1^{\pm}, e_2^{\mp}]}, \tag{1.3.5}$$

are tensors $\mathrm{GR}_D^{\pm} \in \Gamma(V_{\pm}^* \otimes V_{\mp}^* \otimes \mathrm{End}(V_{\pm}))$.

Remark 1.3.6 (Skew-symmetry for the generalized curvature)*.* Note that the generalized curvature operators $\mathrm{GR}_D^{\pm}$ are skew-symmetric in the sense that:

$$\mathrm{GR}_D^{\pm}(e_1^{\pm}, e_2^{\mp}) = -\mathrm{GR}_D^{\mp}(e_2^{\mp}, e_1^{\pm}).$$

Thus, even though we have two curvature operators $\mathrm{GR}^{\pm}$, each of them contains all the information about the curvature, due to the skew-symmetry property. So, from now on, we will refer to the generalized curvature operator as the operator $\mathrm{GR}_D \in \Gamma(V_+^* \otimes V_-^* \otimes \mathrm{End}(V_+))$.. ◯

Definition 1.3.7 (Generalized Ricci curvature)**.** *Given a generalized connection $D \in \mathcal{D}^{\mathbf{G}}$, we define the generalized Ricci curvatures* $\mathrm{Ric}^{\pm} \in \Gamma(V_{\mp}^* \otimes V_{\pm}^*)$ *as the operators*

$$\mathrm{GRic}^{\pm}(e_2^{\mp}, e_3^{\pm}) = \mathrm{tr}\left(e_1^{\pm} \mapsto \mathrm{GR}_D^{\pm}(e_1^{\pm}, e_2^{\mp}) e_3^{\pm}\right). \tag{1.3.6}$$

The generalized Ricci curvature emerges as a central concept in our discussion, especially concerning the condition of generalized Ricci flatness on transitive Courant algebroids constructed via G_2-structures on 7-manifolds. We will revisit the concept of generalized Ricci curvature to derive an explicit formula for this quantity, underscoring its significance in our analysis.

Remark 1.3.8 (Equivalent definitions of generalized Ricci curvature)*.* Different definitions of generalized Ricci curvature were proposed in the literature, and they were proved to be equivalent, cf. [CPR24]. The first one, which is the one we are considering here (cf. Definition 1.3.7), was given in [GF14] by Mario Garcia-Fernandez. For the purpose of comparison in this remark, let's denote it as below:

$$\mathrm{GRic}_{\mathrm{GF}}^{\pm}(e_2^{\mp}, e_3^{\pm}) := \mathrm{tr}\left(e_1^{\pm} \mapsto \mathrm{GR}_D^{\pm}(e_1^{\pm}, e_2^{\mp}) e_3^{\pm}\right).$$

A second definition, due to B. Jurco and J. Vysoky (cf. [JV16]), defines the generalized Ricci curvature $\mathrm{GRic}_{\mathrm{JV}} \in \Gamma(E^* \otimes E^*)$ as

$$\mathrm{GRic}_{\mathrm{JV}}(e_2, e_3) := \mathrm{tr}\left(e_1 \mapsto \mathrm{GR}_{\mathrm{JV}}(e_1, e_2) e_3\right),$$

where the curvature tensor $\mathrm{GR}_{\mathrm{JV}} \in \Gamma(E^* \otimes E^* \otimes E^* \otimes \mathfrak{so}(E))$ in their perspective is related by our definition of curvature GR_D (cf. Definition 1.3.5) by the expression:

$$\langle \mathrm{GR}_{\mathrm{JV}}(a,b)c,e\rangle := \frac{1}{2}\left(\langle \mathrm{GR}_D(a,b)c,e\rangle + \langle \mathrm{GR}_D(c,e)a,b\rangle + \langle (Da)^*b,(Dc)^*e\rangle\right).$$

A third definition, proposed by Cavalcanti, Pedregal and Rubio (cf. [CPR24]), is given in terms of the generalized Ricci curvature as proposed by Jurco and Vysoky as

$$\mathrm{GRic}_{\mathrm{SSCV}}(a,b) := \mathrm{GRic}_{\mathrm{JV}}(a,b) - \mathrm{GRic}_{\mathrm{JV}}(\mathbf{G}a,\mathbf{G}b).$$

The equivalence between them is given by

$$\mathrm{GRic}_{\mathrm{SSCV}}(a_\mp,b_\pm) = 2\mathrm{GRic}_{\mathrm{JV}}(a_\mp,b_\pm) = \mathrm{GRic}^{\pm}_{\mathrm{GF}}(a_\mp,b_\pm) + \mathrm{GRic}^{\mp}_{\mathrm{GF}}(b_\pm,a_\mp).$$

cf. [CPR24, Theorem 8] for the proof of this equivalence and further discussions. ◯

We finish defining the torsion of a generalized connection:

Lemma 1.3.9 ([GF19])**.** *Given a generalized connection D on a Courant algebroid E, we define the generalized torsion by*

$$T_D(a,b,c) = \langle D_a b - D_b a - [a,b], c\rangle + \langle D_c a, b\rangle. \tag{1.3.7}$$

Then $T_D \in \Gamma(\Lambda^3(E^*))$.

In addition, we also have the concept of generalized scalar curvature for generalized connections. However, defining it for generalized connections is more delicate than for ordinary connections (which is just the trace of the Ricci curvature), to be explored in Section 2.4.

1.4 Divergence operators

Later, we will define the generalized Ricci curvature and show that, under certain circumstances, it depends not on the connection itself, but only on the generalized metric and an additional parameter related to the connection, the *divergence operator*, which we will introduce now.

Definition 1.4.1. *A* divergence operator *on a Courant algebroid E is a bundle map* $\mathrm{div}\colon \Gamma(E) \to C^\infty(M)$ *satisfying the Leibniz-type rule*

$$\mathrm{div}(fa) = f\,\mathrm{div}(a) + \pi(a)(f), \tag{1.4.1}$$

for $f \in C^\infty(M)$ and $a \in \Gamma(E)$.

Example 1.4.2 (Divergence of a generalized connection)**.** Let E a Courant algebroid and D a generalized connection, then the induced operator

$$\mathrm{div}^D(e) := \mathrm{tr}(De)$$

defines a divergence operator on E. In fact, the Leibniz rule for the connection D gives us that $D(fe) = \pi(\cdot)f \cdot e + fDe$, consequently

$$\begin{aligned}\mathrm{div}^D(fe) &= \mathrm{tr}(D(fe)) = \mathrm{tr}\,(\pi(*)(f) \cdot e + f\ De)\\ &= \mathrm{tr}(\pi(\cdot)f \cdot e) + f\,\mathrm{tr}(De)\\ &= \pi(e)(f) + f\,\mathrm{div}^D(e),\end{aligned}$$

as we claimed. Note that the last line is obtained writing $e = \zeta^k e_k$ in a frame, so the operator on the second line is, in particular, the projection on the j-th coordinate given by $\zeta^j \pi(e_j)f \cdot e_j$, and we have the trace given by

$$\mathrm{tr}(\pi(\cdot)f \cdot e) := \sum_j \zeta^j \pi(e_j)f = \sum_j \pi(\zeta^j e_j)f = \pi(e)f$$

This shows, in particular, that a divergence operator always exists for a Courant algebroid since a Courant algebroid always admits a generalized connection. △

Definition 1.4.3. *Let $E \to M$ be a Courant algebroid endowed with a generalized metric $\mathbf{G}$ and let* div *a divergence operator on E. We define the space of $\mathbf{G}$-compatible generalized connections with divergence* div *by*

$$\mathcal{D}(\mathbf{G}, \mathrm{div}) = \{D \in \mathcal{D}^{\mathbf{G}} : \mathrm{div}^D = \mathrm{div}\}. \tag{1.4.2}$$

We also define the subspace $\mathcal{D}^0(\mathbf{G}, \mathrm{div}) \subset \mathcal{D}(\mathbf{G}, \mathrm{div})$ of the torsion-free connections.

$$\mathcal{D}^0(\mathbf{G}, \mathrm{div}) = \{D \in \mathcal{D}^{\mathbf{G}} : \mathrm{div}^D = \mathrm{div}, T_D = 0\}. \tag{1.4.3}$$

Lemma 1.4.4 ([GFS20])**.** *The space of divergence operators on a Courant algebroid E forms an affine space modelled on $\Gamma(E)$. That is, if we fix a divergence operator* div*, any other divergence operator* div′ *is given by*

$$\mathrm{div}' = \mathrm{div} + \langle e, \cdot\rangle \tag{1.4.4}$$

for a unique section $e \in \Gamma(E)$.

In the context of transitive Courant algebroids $E = TM \oplus \mathrm{ad}P \oplus T^*M$ (cf. Example 1.1.5), we have an important divergence operator to discuss:

Example 1.4.5 (Riemannian divergence)**.** Consider an oriented Riemannian manifold (M, g, vol_M) and let $E = TM \oplus \mathrm{ad}P \oplus T^*M$ be the standard transitive Courant algebroid over M (which is endowed with a generalized metric $\mathbf{G}$ induced by g) as in Examples 1.1.5 and 1.2.4, then we can define a divergence operator on E by

$$\operatorname{div}^{\mathbf{G}}(e) := \frac{\mathcal{L}_{\pi(e)}\mathrm{vol}_M}{\mathrm{vol}_M}, \tag{1.4.5}$$

which is called the *Riemannian divergence* of $\mathbf{G}$. Using that $\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge i_X\omega$, we can check that this defines a divergence operator on the Courant algebroid. In fact,

$$\operatorname{div}^{\mathbf{G}}(fe) = \frac{\mathcal{L}_{\pi(fe)}\mathrm{vol}}{\mathrm{vol}} = \frac{f\mathcal{L}_{\pi(e)}\mathrm{vol} + df \wedge i_{\pi(e)}\mathrm{vol}}{\mathrm{vol}}.$$

N+ow, using that $df \wedge \mathrm{vol} = 0$ since vol is top-form, we have by the product rule for contraction under wedge

$$df \wedge i_{\pi(e)}\mathrm{vol} = i_{\pi(e)}(\underbrace{df \wedge \mathrm{vol}}_{=0}) + i_{\pi(e)}df \wedge \mathrm{vol} = \pi(e)(f)\mathrm{vol}$$

and the result follows. As a result of the Lemma 1.4.4, any divergence div on E can be expressed in the form

$$\operatorname{div} = \operatorname{div}^{\mathbf{G}} - \langle e, \cdot \rangle,$$

for a uniquely determined section $e \in \Gamma(E)$. △

1.5 Generalized Ricci curvature on transitive Courant algebroids

In our context, we are interested in transitive Courant algebroids $E = TM \oplus \mathrm{ad}P \oplus T^*M$ with structures given by a pair

$$H \in \Omega^3(M), \qquad \theta \in \Omega^1(P, \mathfrak{k}),$$

satisfying the heterotic Bianchi identity $dH = \langle F_\theta \wedge F_\theta \rangle$ (where $\langle \cdot, \cdot \rangle$ is the symmetric and non-degenerate pairing in the Lie algebra $\mathfrak{k}$ for the principal K-bundle $P \to M$), cf. Example 1.1.5.

Our goal here is providing an explicit formula for the generalized Ricci curvature (cf. Definition 1.3.7) for a generalized connection $D \in \mathcal{D}^0(\mathbf{G}, \operatorname{div})$.We will perform this computation in steps.

We start establishing the independence of the generalized Ricci curvature from the choice of connection by demonstrating that for any $D_1, D_2 \in \mathcal{D}^0(\mathcal{G}, \operatorname{div})$ (as outlined in Lemma 1.5.1), it holds that $\mathrm{GRic}^{\pm}_{D_1} = \mathrm{GRic}^{\pm}_{D_2}$. Consequently, the generalized Ricci curvature associated with connections in $\mathcal{D}^0(\mathcal{G}, \operatorname{div})$ is uniquely determined by the pair $(\mathcal{G}, \operatorname{div})$. Thus, our focus narrows to computing the Ricci curvature within a specific connection $D \in \mathcal{D}^0(\mathcal{G}, \operatorname{div})$.

In sequence, we construct an explicit connection $\tilde{D} \in \mathcal{D}^0(\mathbf{G}, \mathrm{div}^{\mathbf{G}})$ and provide an explicit expression for its generalized Ricci curvature (see Lemma 1.5.2). Following this, we introduce a novel connection, denoted by D, which includes terms $\chi^{\pm}$ (which we will specify later)

$$D = \tilde{D} + \frac{1}{\dim V_+ - 1}\chi^+ + \frac{1}{\dim V_- - 1}\chi^-,$$

where the section $e \in \Gamma(E)$ is defined by $\langle e, \cdot \rangle = \mathrm{div} - \mathrm{div}^{\mathbf{G}}$. This construction guarantees $D \in \mathcal{D}^0(\mathbf{G}, \mathrm{div})$, as outlined in Lemma 1.5.3, thus providing a specific connection within the space $\mathcal{D}^0(\mathbf{G}, \mathrm{div})$, so the generalized Ricci curvature to any other connection $D \in \mathcal{D}^0(\mathbf{G}, \mathrm{div})$ has the same expression.

After this, we will compute the generalized Ricci curvature for the connection $\tilde{D}$, discussed in Lemma 1.5.3. For this, we will utilize the transformation behaviour of the Ricci curvature under the transition from $\tilde{D}$ to D as above (cf. Proposition 1.5.4) and derive the Ricci curvature for $\mathcal{D}^0(\mathbf{G}, \mathrm{div})$. This culminates in Theorem 1.5.5, wherein we present the explicit expression for the Ricci curvature within $\mathcal{D}^0(\mathbf{G}, \mathrm{div})$, as desired.

Lemma 1.5.1. *If $D_1, D_2 \in \mathcal{D}^0(\mathbf{G}, \mathrm{div})$ are generalized connections on the transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ as in Example 1.1.5 and standard generalized metric $\mathbf{G}$ as in Example 1.2.4, we have*

$$\mathrm{GRic}_{D_1} = \mathrm{GRic}_{D_2},$$

i.e., the generalized Ricci curvature doesn't depend on the connection in $\mathcal{D}^0(\mathbf{G}, \mathrm{div})$.

This lemma tells us that if M has a Riemannian metric g, then torsion-free $\mathbf{G}$-compatible generalized connections (for $\mathbf{G}$ induced by g) on the standard transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ has the generalized Ricci curvature *depending uniquely on the divergence* div^D, or equivalently only on the section e defined by $\mathrm{div}^D = \mathrm{div}^{\mathbf{G}} - \langle e, \cdot \rangle$. Because of this, we can denote

$$\mathrm{GRic}(\mathbf{G}, \mathrm{div}) = \mathrm{GRic}_D, \qquad \forall D \in \mathcal{D}^0(\mathbf{G}, \mathrm{div}).$$

We define the connection $\tilde{D} \in \mathcal{D}^0(\mathbf{G}, \mathrm{div}^{\mathbf{G}})$ and provide an explicit formula for its generalized Ricci curvature. For this, let's introduce some connections by convenience

$$\nabla^{\pm 1/3} := \nabla^g \pm \frac{1}{6} g^{-1} H, \qquad \nabla^{\pm} := \nabla^g \pm \frac{1}{2} g^{-1} H,$$

where ∇^g is the Levi-Civita connection for g in M. These connections are metric connections and have skew-symmetric torsion, respectively, given by $\pm\frac{1}{3}H$ and $\pm H$. The connection ∇^+ is called *Bismut connection* and ∇^-, the *Hull connection*, and they will play a distinguished role in this study, especially the Bismut connection.

Lemma 1.5.2. *Let $E = TM \oplus \mathrm{ad}P \oplus T^*M$ transitive Courant algebroid as in Example 1.1.5 endowed with the standard generalized metric $\mathbf{G}$ as in Example 1.2.4, and consider the generalized connection defined by*[3]

$$\begin{aligned}\tilde{D}_{a_-}b_- &= \sigma_-\left(\nabla_V^{-1/3}W - \tfrac{2}{3}g^{-1}\langle i_V F_\theta, t\rangle - \tfrac{1}{3}g^{-1}\langle i_W F_\theta, r\rangle\right)\\ &\qquad + d_V^\theta t - \tfrac{2}{3}F_\theta(V,W) + \tfrac{1}{3}[t,r]\\ \tilde{D}_{a_+}b_+ &= \sigma_+\left(\nabla_V^{+1/3}W\right)\end{aligned}$$

where $a_+ = V + gV$, $a_- = V + r - gV$, $b_+ = W - gW$ and $b_- = W + t - gW$. Then $\tilde{D}$ is in $\mathcal{D}^0(\mathbf{G}, \mathrm{div}^{\mathbf{G}})$, i.e., it is torsion-free, compatible with the generalized metric and its divergence is the Riemannian divergence (1.4.5). Furthermore, the generalized Ricci curvature is given by [4]

$$\mathrm{GRic}^+_{\tilde{D}}(a_-, b_+) = \left(\mathrm{Ric}^+ + F_\theta \circ F_\theta\right)(V,W) - i_W\left\langle d^{\theta *}F_\theta - F_\theta \lrcorner H, r\right\rangle, \tag{1.5.1}$$

where $\mathrm{Ric}^+ = \mathrm{Ric}_{\nabla^+}$ is the Ricci curvature for the Bismut connection ∇^+ and $F_\theta \circ F_\theta = \langle i_{e_j}F_\theta, i_{e_j}F_\theta\rangle$ for some orthonormal frame $\{e_j\}$ on TM.

In sequence, we consider an arbitrary divergence operator div on $E = TM \oplus \mathrm{ad}P \oplus T^*M$, then by the Lemma 1.4.4, we have the existence of a unique section $e \in \Gamma(E)$ such that

$$\mathrm{div} = \mathrm{div}^{\mathbf{G}} - \langle e, \cdot\rangle,$$

and decompose it $e = e_+ + e_-$, for $e_\pm \in V_\pm$. Define the operators $\chi^{e_\pm} \in \Gamma(V_\pm^* \otimes V_\pm^* \otimes V_\pm)$ by the formula

$$\chi^{e_\pm}(a_\pm, b_\pm) \equiv \chi^{e_\pm}_{a_\pm} b_\pm := \langle a_\pm, b_\pm\rangle e_\pm - \langle e_\pm, b_\pm\rangle a_\pm. \tag{1.5.2}$$

With these notations, we can define a new connection D:

Lemma 1.5.3. *Let $E = TM \oplus \mathrm{ad}P \oplus T^*M$ be the transitive Courant algebroid endowed with the standard generalized metric $\mathbf{G}$ as in Example 1.2.4 and div a fixed divergence operator on E, Consider*

$$D := \tilde{D} + \frac{1}{\dim V_+ - 1}\chi^{e_+} + \frac{1}{\dim V_- - 1}\chi^{e_-}, \tag{1.5.3}$$

where $\tilde{D} \in \mathcal{D}^0(\mathbf{G}, \mathrm{div}^{\mathbf{G}})$ (defined in Lemma 1.5.2) and $\chi^{e_\pm}$ given by (1.5.2), for $e = e_+ + e_-$ such that $\mathrm{div} = \mathrm{div}^{\mathbf{G}} - \langle e, \cdot\rangle$. Then $D \in \mathcal{D}^0(\mathbf{G}, \mathrm{div})$.

[3] The co-differential $d^{\theta *} := (-1)^{n(k+1)+1} * d^\theta *$ acting on k-forms in an n-dimensional manifold, where d^θ is the covariant exterior derivative induced by the connection θ.

[4] The contraction of forms

$$F_\theta \lrcorner H := \frac{1}{2!1!}F^{ij}H_{ijk}e^k = (-1)^{n+1} * (F_\theta \wedge *H),$$

where $F_\theta = \frac{1}{2!}F_{ij}e^{ij}$ and $H = \frac{1}{3!}H_{ijk}e^{ijk}$. For more details about the contraction of forms, cf. Appendix A.

Finally, in order to compute the generalized Ricci curvature for D, we need to understand how the generalized Ricci curvature of $\tilde{D}$ behaves when we add the terms $\chi^{e\pm}$ as above and use the result in Lemma 1.5.2 to find an explicit expression for GRic_D.

Proposition 1.5.4. *On the transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ (cf. Example 1.1.5), the generalized Ricci curvature for a pair $(\mathbf{G}, \mathrm{div})$ satisfies*

$$\mathrm{GRic}^{\pm}(\mathbf{G}, \mathrm{div})(a_{\mp}, b_{\pm}) = \mathrm{GRic}^{\pm}(\mathbf{G}, \mathrm{div}^{\mathbf{G}})(a_{\mp}, b_{\pm}) - \langle [e_{\pm}, a_{\mp}], b_{\pm} \rangle, \tag{1.5.4}$$

where $e \in \Gamma(E)$ defined by $\mathrm{div} = \mathrm{div}^{\mathbf{G}} - \langle e, \cdot \rangle$.

An immediate corollary is the following result, which is the main result of the section, and gives us the final explicit formula for the generalized Ricci curvature for generalized connections in $\mathcal{D}^0(\mathbf{G}, \mathrm{div})$.

Theorem 1.5.5. *Let $(\mathbf{G}, \mathrm{div})$ be a pair given by a generalized metric $\mathbf{G}$ and a divergence operator div on a transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ as in Example 1.2.4 and consider $e \in \Gamma(E)$ defined by*

$$\mathrm{div} = \mathrm{div}^{\mathbf{G}} - \langle e, \cdot \rangle, \qquad e = Z + s + \zeta \in \Gamma(E).$$

Set $\zeta_+ = \frac{1}{2}(Z^\flat + \zeta) \in \Omega^1(M)$, then the generalized Ricci curvature for every connection $D \in \mathcal{D}^0(\mathbf{G}, \mathrm{div})$ is given by

$$\begin{aligned} \mathrm{GRic}_D^+(a_-, b_+) = i_Y i_X \left(\mathrm{Ric}^+ + F_\theta \circ F_\theta + \nabla^+ \zeta_+ \right) \\ - i_Y \left\langle d^{\theta *} F_\theta - F_\theta \lrcorner H + i_{\zeta_+^\#} F_\theta, r \right\rangle, \end{aligned} \tag{1.5.5}$$

where $a_- = X + r - X^\flat \in V_-$, $b_+ = Y + Y^\flat \in V_+$, Ric^+ is the Ricci curvature of the Bismut connection ∇^+ and $F_\theta \circ F_\theta = \langle i_{e_j} F_\theta, i_{e_j} F_\theta \rangle$ for some orthonormal frame $\{e_j\}$ of TM.

Proof. By Lemma 1.5.1, we need to compute the generalized Ricci curvature for the connection $D \in \mathcal{D}^0(\mathbf{G}, \mathrm{div})$ constructed in Lemma 1.5.3. By the Proposition 1.5.4 and Lemma 1.5.2, we have that

$$\begin{aligned} \mathrm{GRic}_D^+(a_-, b_+) &= \mathrm{GRic}_{\tilde{D}}^+(a_-, b_+) - \langle [e_+, a_-], b_+ \rangle \\ &= \left(\mathrm{Ric}^+ + F_\theta \circ F_\theta \right)(V, W) - i_W \left\langle d^{\theta *} F_\theta - F_\theta \lrcorner H, r \right\rangle - \langle [e_+, a_-], b_+ \rangle \end{aligned}$$

we need to prove that

$$\langle [e_+, a_-], b_+ \rangle = \langle F_\theta(\zeta_+^\#, W), r \rangle - \nabla_V^+ \zeta_+(W)$$

by direct computation using the Dorfman bracket$[\cdot, \cdot]$ defined in the Example 1.1.5, using $\langle e_+, a_- \rangle = 0 \Rightarrow [e_+, a_-] = -[a_-, e_+]$ (cf. Remark 1.1.2) and $e_+ = \frac{1}{2}(Z + gZ + \zeta + g^{-1}\zeta) =$

$\zeta_+ + \zeta_+^\#$ (cf. Example 1.2.4), we then have

$$\begin{aligned}[a_-, e_+]_+ &= [V - V^\flat + r, \zeta_+ + \zeta_+^\#]_+ = [V - V^\flat, \zeta_+ + \zeta_+^\#]_+ + [r, \zeta_+ + \zeta_+^\#]_+ \\ &= \sigma_+(\nabla_V^+ \zeta_+) + \pi_+ \left(-d^\theta_{\zeta_+^\#} r - 2\langle i_{\zeta_+^\#} F_\theta, r\rangle \right) \\ &= \sigma_+ \left(\nabla_V^+ \zeta_+ - g^{-1} \langle i_{\zeta_+^\#} F_\theta, r\rangle \right).\end{aligned}$$

Now, take the inner product with b_+. Since the term $[a_-, e_+]_-$ does not interfere due to the orthogonal decomposition of $V_\pm$, we have

$$\langle [a_-, e_+], b_+\rangle = \nabla_V^+ \zeta_+(W) - \langle F_\theta(\zeta_+^\#, W), r\rangle$$

as desired. □

A particular case which we will be interested is when the section which determines the divergence satisfies $e \in T^*M \subset E = TM \oplus \mathrm{ad}P \oplus T^*M$, in this case, $\zeta_+ = \frac{1}{2}e$ and we have

$$\mathrm{div} = \mathrm{div}^{\mathbf{G}} - 2\langle \zeta_+, \cdot\rangle, \tag{1.5.6}$$

and the generalized Ricci curvature given explicitly in terms of ζ_+ as in (1.5.5).

2 Killing spinor equations and coupled instantons

In this chapter, we present two essential concepts in this dissertation: the *Killing spinor equations* within generalized geometry [GFRT16, GF19] and the *coupled instanton equations* [GFGM23, dlOLS18a].

Notably, the Killing spinor equations encompass the Hull–Strominger system and the heterotic G_2-system as specific instances. The significance of Killing spinor equations in generalized geometry lies in their ability to yield unique instances of generalized Ricci-flat metrics. For the coupled instanton equations, they were inspired by recent developments on *coupled instantons*, both in the physics [dlOLS18a, dlOLS18b] and mathematical literature [GFGM23, GFJS23]. These equations are closely related to Killing spinors and generalized Ricci-flat metrics and play an important role in recent developments around the Hull–Strominger system and non-Kähler mirror symmetry [ACDAdLHGF24].

This chapter is organized as follows: In Section 2.1, we introduce the *Killing spinor equations*, encompassing the *gravitino* and *dilatino* equations, historically rooted in physics, (cf. Definition 2.1.2), within the framework of a transitive Courant algebroid over a spin manifold. Additionally, we establish a crucial result: solutions to the Killing spinor equations yield *generalized Ricci flatness* through a direct spinor calculation (cf. Proposition 2.1.6).

In the following, Section 2.2, we introduce the *coupled instantons equations* within the framework of transitive Courant algebroids over spin manifolds. In this context, is presented the connection D in the auxiliary bundle $TM \oplus \mathrm{ad}P$ and we propose two problems intended for ongoing exploration throughout the text (cf. Problem 1 and Problem 2).

Continuing our investigation, Section 2.3 thoroughly explores Problem 1 and presents a complete solution for it for arbitrary dimensions (cf. Theorem 2.3.2). Problem 1 elucidates the connection between the coupled instanton condition and the gravitino equation, while Problem 2 delves into the relationship between coupled instantons and generalized Ricci flatness, still open. We solve Problem 2 in Chapter 4.3 for the case of seven dimensions with the spinor field defining a G_2-structure.

In Section 2.4, we delve into the concept of *generalized scalar curvature*, a term we avoid defining in Chapter 1 due to its nuanced nature. Here, we discuss these intricacies, addressing them through discussions outlined in Lemma 2.4.1 and Definition 2.4.3. Moreover, we explore its intricate relationship with the Killing spinor equations (cf. Proposition 2.4.7).

Lastly, in Section 2.5, we unveil an algorithm aimed at constructing iteratively instantons of increasing rank from solutions of the gravitino equation, focusing on the scenario of $dH = 0$; these solutions are called “instanton towers” (cf. Proposition 2.5.1).

2.1 Killing spinors with parameter λ

For the purposes of this section, we fix M^n to be a spin-oriented manifold M, which spin structure $\tilde{P}$ induces the bundle of spinors S via spin representation $\kappa : \mathrm{Spin}(n) \to \mathrm{SO}(\Delta_n)$ (where $\Delta_n \cong \mathbb{R}^{2^{\lfloor n/2 \rfloor}}$), i.e., $S = \tilde{P} \times_\kappa \Delta_n$, cf. [LM90].

One of the main challenges to introducing natural curvature quantities associated with a generalized metric $\mathbf{G}$ is the absence of a uniquely determined analogue of the Levi-Civita connection [CSCW11, GF19]. Instead, we consider a pair of differential operators

$$D_-^+ \colon \Gamma(V_+) \to \Gamma(V_-^* \otimes V_+) \qquad \text{and} \qquad D_+^- \colon \Gamma(V_-) \to \Gamma(V_+^* \otimes V_-), \tag{2.1.1}$$

defined on sections $a_- \in \Gamma(V_-)$ and $b_+ \in \Gamma(V_+)$ respectively by

$$\langle a_-, D_-^+ b_+ \rangle = \pi_+[a_-, b_+] \qquad \text{and} \qquad \langle b_+, D_+^- a_- \rangle = \pi_-[b_+, a_-].$$

In the sequel, we will simply write $D_{a_-} b_+ := \langle a_-, D_-^+ b_+ \rangle$, and similarly for D_+^-. They satisfy natural Leibniz rules concerning the anchor map for any smooth function $f \in C^\infty(M)$:

$$\begin{aligned} D_{a_-}(f b_+) &= \pi(a_-)(f) b_+ + f D_{a_-} b_+, \\ D_{b_+}(f a_-) &= \pi(b_+)(f) a_- + f D_{a_+} a_-. \end{aligned}$$

Associated with the pair $(\mathbf{G}, \mathrm{div})$, there are canonical first-order differential operators defined via the operators $D_\pm^\mp$ as mentioned above [GFRT16, GF19] (see also [CSCW11]):

$$D_-^S \colon \Omega^0(S) \to \Omega^0(V_-^* \otimes S), \qquad \text{and} \qquad \not{D}^+ \colon \Omega^0(S) \to \Omega^0(S). \tag{2.1.2}$$

The operator D_-^S corresponds to the unique lift to the spinor bundle S of the metric-preserving operator D_-^+ in (2.1.1). The Dirac-type operator $\not{D}^+$ is defined by the formula

$$\not{D}^+ \eta := \mu^j \cdot D_{\mu_j} \eta,$$

where $\{\mu_j\}$ is an orthonormal basis for V_+ and $D \in \mathcal{D}^0(\mathbf{G}, \mathrm{div})$ the torsion-free generalized connection constructed in Section 1.5 (cf. Lemma 1.5.3). Here we are using the isometry $X + gX \in V_+ \cong X \in (TM, g)$ to consider the connection D induced on spinors. These two operators can written in terms of more elementary concepts.

Lemma 2.1.1. *Let* $(\mathbf{G}, \mathrm{div})$ *be a generalized metric and a divergence operator on a transitive Courant algebroid* E*, and let* (H, θ) *be the unique pair satisfying* (1.1.8) *determined by* $\mathbf{G}$*, where* $H \in \Omega^3(M)$ *and* θ *is a principal connection on* P*. Denote* $\mathrm{div}^{\mathbf{G}} - \mathrm{div} = \langle e, \cdot\rangle$*, set* $\zeta = g(\pi e_+, \cdot) \in T^*M$*. Then, for any spinor* $\eta \in \Omega^0(S)$ *and* $a_- = X + r - g(X) \in \Gamma(V_-)$*, one has*

$$D^S_{a_-}\eta \coloneqq \nabla^+_X \eta - \langle F_\theta, r\rangle \cdot \eta, \tag{2.1.3}$$

$$\not{D}^+ \eta \coloneqq \not{\nabla}^{1/3}\eta - \tfrac{1}{2}\zeta \cdot \eta, \tag{2.1.4}$$

where $\nabla^{1/3}_X Y = \nabla^g_X Y + \frac{1}{6} g^{-1} H(X, Y, \cdot)$ *and* $\not{\nabla}^{1/3}$ *is the associated Dirac operator.*

Proof. [GFRT16, Lemma 5.2]. □

Once the relevant operators are introduced, we can proceed to discuss our primary system of equations of interest, which are the *Killing spinor equations* in the context of generalized geometry.

Definition 2.1.2. *Let* E *be a transitive Courant algebroid over a spin manifold* M*, and fix a constant* $\lambda \in \mathbb{R}$*. A triple* $(\mathbf{G}, \mathrm{div}, \eta)$*, given by a generalized metric* $\mathbf{G}$*, a divergence operator* div*, and a spinor* $\eta \in \Omega^0(S)$*, is a solution of the* Killing spinor equations with parameter λ*, if*

$$D^S_- \eta = 0, \tag{2.1.5}$$

$$\not{D}^+ \eta = \lambda \eta. \tag{2.1.6}$$

From their origins in theoretical physics, we will refer to (2.1.5) *as the* gravitino equation*, and* (2.1.6) *as the* dilatino equation *[dSJGFLSE24].*

By the Lemma 2.1.1, where we have expression for the operators D^S_- and $\not{D}^+$, the gravitino and dilatino equations are:

$$\nabla^+_X \eta - \langle F_\theta, r\rangle \cdot \eta = 0, \qquad \left(\not{\nabla}^{1/3} - \frac{1}{2}\zeta\right) \cdot \eta = \lambda\eta.$$

Since r is arbitrary in the expression above, we can also rewrite the first equation (gravitino) in two parts. The Killing spinor equations then become:

$$\nabla^+ \eta = 0, \qquad F_\theta \cdot \eta = 0, \qquad \left(\not{\nabla}^{1/3} - \frac{1}{2}\zeta\right) \cdot \eta = \lambda\eta. \tag{2.1.7}$$

Note that these equations don't depend on the Courant algebroid structure, so normally, they are stated together with the heterotic Bianchi identity (1.1.8): $dH = \langle F_\theta \wedge F_\theta\rangle$.

Any solution $(g, H, \theta, \eta, \zeta)$ of (2.1.7) satisfying the heterotic Bianchi identity (1.1.8) determines a transitive Courant algebroid as in Definition 1.1.5, endowed with a solution of the gravitino and dilatino equation in Definition (2.1.2). The proof is in [GFRT20, Proposition A.6] and an almost identical to the proof of [ACDAdLHGF24, Lemma 3.8].

Remark 2.1.3 (Even dimensions). In even dimensions, the system (2.1.5) and (2.1.6) for a spinor η inducing a $\mathrm{SU}(m)$-structure forces $\lambda = 0$, cf. [GFRT16, GF19]. ○

Remark 2.1.4. In the mathematical physics literature, the name *gravitino equation* is often reserved only for the part $\nabla^+\eta = 0$, while the second receives the name of *gaugino equation*, referring to the superpartners of the graviton field and the gauge field, respectively (see, e.g. [II05]). The unified treatment of the first two equations (2.1.7) is motivated by the way they appear in generalized geometry (2.1.5). ○

We next present the first structural property of the Killing spinor equations (2.1.5) and (2.1.6) with parameter λ about generalized Ricci-flat metrics, which motivates Definition 2.1.2. This result is based on an exciting formula for the generalized Ricci tensor in terms of operators D_-^S and $\not{D}^+$, discovered in the physics literature [CSCW11] (without proof) and first established in [GF19, Lemma 4.7].

Lemma 2.1.5. *Let* $(\mathbf{G}, \mathrm{div})$ *be a pair given by a generalized metric and a divergence operator on a transitive Courant algebroid* E *over a spin manifold* M*. Then, for any* $a_- \in \Omega^0(V_-)$ *and any spinor* $\eta \in \Omega^0(S)$*, the generalized Ricci tensor* $\mathrm{GRic}^+_{\mathbf{G},\mathrm{div}} \in \Gamma(V_-^* \otimes V_+^*)$ *satisfies*

$$\langle a_-, \mathrm{GRic}^+_{\mathbf{G},\mathrm{div}}\rangle \cdot \eta = 4\Big(\not{D}^+ D^S_{a_-} - D^S_{a_-}\not{D}^+ - \sum_{j=1}^n e_j \cdot D^S_{\pi_-[e_j,a_-]}\Big)\eta,$$

where $\{e_j\}$ *is any choice of local orthonormal frame for* V_+*.*

Proof. Cf. [GF19, Lemma 4.7] □

As a direct consequence of the previous formula, any solution of the Killing spinor equations with parameter λ is generalized Ricci-flat.

Proposition 2.1.6. *Let* $(\mathbf{G}, \mathrm{div}, \eta)$ *be a solution of the Killing spinor equations as in Definition* (2.1.2) *with parameter* $\lambda \in \mathbb{R}$*, on a transitive Courant algebroid* E *over a spin* n*-manifold* M*. Provided that* η *is nowhere-vanishing on* M*, the pair* $(\mathbf{G}, \mathrm{div})$ *satisfies*

$$\mathrm{GRic}^+_{\mathbf{G},\mathrm{div}} = 0.$$

More explicitly, in terms of the tuple $(g, H, \theta, \eta, \zeta)$ determined by $(\mathbf{G}, \mathrm{div})$, cf. Proposition 1.5.5 one has[1]

$$\begin{aligned} \mathrm{Ric}^g - \tfrac{1}{4}H^2 + F_\theta \circ F_\theta + \tfrac{1}{2}\mathcal{L}_{\zeta^\#}g = 0, \\ d^*H - d\zeta + i_{\zeta^\#}H = 0, \\ d^{\theta *}F_\theta - F_\theta \lrcorner H + i_{\zeta^\#}F_\theta = 0. \end{aligned} \tag{2.1.8}$$

where $F_\theta \circ F_\theta = \sum_j \langle i_{e_j}F_\theta, i_{e_j}F_\theta\rangle$, for some local orthonormal frame $\{e_j\}$ for g.

Proof. Applying Lemma 2.1.5 to a solution of the Killing spinor equations in Definition (2.1.2), we have

$$\langle a_-, \mathrm{GRic}^+_{\mathbf{G},\mathrm{div}}\rangle \cdot \eta = -4D^S_{a_-}\not{D}^+\eta = -4\lambda D^S_{a_-}\eta = 0,$$

for every $a_- \in \Omega^0(V_-)$. Consequently,

$$|\langle a_-, \mathrm{GRic}^+_{\mathbf{G},\mathrm{div}}\rangle|^2\eta = \langle a_-, \mathrm{GRic}^+_{\mathbf{G},\mathrm{div}}\rangle \cdot \langle a_-, \mathrm{GRic}^+_{\mathbf{G},\mathrm{div}}\rangle \cdot \eta = 0,$$

and therefore $|\langle a_-, \mathrm{GRic}^+_{\mathbf{G},\mathrm{div}}\rangle|^2 = 0$ for every section a_-, since η is nowhere-vanishing. The first part of the proof follows now from the fact that the pairing on V_+ is positive-definite.

As for the second part of the statement, equations (2.1.8) follow from the explicit formula for the generalized Ricci tensor (1.5.5), together with the unique decomposition of $\mathrm{Ric}^+ = \mathrm{Ric}_{\nabla^+}$ and $\nabla^+\zeta$ into symmetric and skew-symmetric 2-tensors:

$$\begin{aligned} \mathrm{Ric}^+ &= \mathrm{Ric}^g - \frac{1}{4}H^2 - \frac{1}{2}d^*H, \\ \nabla^+\zeta &= \frac{1}{2}\mathcal{L}_{\zeta^\#}g + \frac{1}{2}d\zeta - \frac{1}{2}i_{\zeta^\#}H. \end{aligned} \tag{2.1.9}$$

see e.g. [GFS20, IP01]. □

Remark 2.1.7. When $\zeta = d\phi$, for a smooth function ϕ, equations (2.1.8) match the heterotic supergravity equations of motion for the metric, the 3-form flux, and the gauge field, in the mathematical physics literature, see e.g. [GF14, Mol24]. This fact suggests that solutions of the Killing spinor equations (cf. Definition (2.1.2)) with closed, or even exact, one-form ζ, play a special role; we explore this aspect in Section 2.4. ○

1 The notation $\alpha \lrcorner^1 \beta$ means for a 2-form $\alpha = \frac{1}{2!}\alpha_{ij}e^{ij}$ and a 3-form $\beta = \frac{1}{3!}\beta_{ijk}e^{ijk}$, we have:

$$\alpha \lrcorner^1 \beta := \frac{1}{2}\alpha^\mu{}_i\beta_{\mu jk}e^{ijk} \in \Omega^3(M).$$

For details about partial contractions, cf. Appendix A.2. In the same way, $\beta^2 = \beta \lrcorner \beta = \langle \beta, \beta\rangle$.

2.2 Coupled instantons

In order to introduce our equations of interest, we first fix a transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ over an oriented spin manifold M and which algebroid structure is determined by a pair (H, θ). Considering the generalized metric as in Example 1.2.4, we can identify V_- isometrically with $TM \oplus \mathrm{ad}P$ via

$$\begin{array}{rccc} \sigma_- : & (TM \oplus \mathrm{ad}P, \langle\cdot,\cdot\rangle^0) & \longrightarrow & V_- \\ & X + r & \longmapsto & X + r - gX \end{array} \tag{2.2.1}$$

where $\langle X + r, X + r\rangle^0 = g(X, X) - \langle r, r\rangle$. With this, we now introduce the connection D on the vector bundle $TM \oplus \mathrm{ad}P$ [dlOLS18a, GFGM23]:

$$D = \begin{pmatrix} \nabla^- & \mathbb{F}^\dagger \\ -\mathbb{F} & d^\theta \end{pmatrix}, \tag{2.2.2}$$

where $\nabla^- = \nabla^g - \frac{1}{2} g^{-1} H$ is the Hull connections and $\mathbb{F} \in \Omega^1(\mathrm{Hom}(TM, \mathrm{ad}P))$ is the $\mathrm{Hom}(TM, \mathrm{ad}P)$-valued 1-form

$$(i_X \mathbb{F})(Y) := F_\theta(X, Y) \tag{2.2.3}$$

and $\mathbb{F}^\dagger \in \Omega^1(\mathrm{Hom}(\mathrm{ad}P, TM))$ is the corresponding $\langle\cdot,\cdot\rangle^0$-adjoint

$$(i_X \mathbb{F}^\dagger)(r) = -g^{-1}\langle i_X F_\theta, r\rangle^0.$$

We will use the standard notation $R_{\nabla^\pm}$ for the curvature of $\nabla^\pm$ and also $\nabla^{\theta,\pm}$ for the covariant derivative induced by θ and $\nabla^\pm$ on $\Lambda^2 T^*M \otimes \mathrm{ad}P$. In particular,

$$(\nabla^{\theta,\pm}_Z F_\theta)(X, Y) = d^\theta_Z(F_\theta(X, Y)) - F_\theta(\nabla^\pm_Z X, Y) - F_\theta(X, \nabla^\pm_Z Y) \tag{2.2.4}$$

An explicit formula for the curvature of D is given in the lemma below.

Lemma 2.2.1. *The curvature of D is given by*

$$F_D = \begin{pmatrix} R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} & -(\nabla^{\theta,+} F_\theta)^\dagger \\ \nabla^{\theta,+} F_\theta & [F_\theta, \cdot] - \mathbb{F} \wedge \mathbb{F}^\dagger \end{pmatrix}, \tag{2.2.5}$$

where

$$\begin{aligned} i_Y i_X \mathbb{F}^\dagger \wedge \mathbb{F}(Z) &= g^{-1}\langle i_Y F_\theta, F_\theta(X, Z)\rangle - g^{-1}\langle i_X F_\theta, F_\theta(Y, Z)\rangle, \\ i_Y i_X \mathbb{F} \wedge \mathbb{F}^\dagger(r) &= F_\theta(Y, g^{-1}\langle i_X F_\theta, r\rangle) - F_\theta(X, g^{-1}\langle i_Y F_\theta, r\rangle). \end{aligned}$$

Proof. Following [GFGM23, Lemma 4.7], we conclude the two terms on the diagonal, and we have that the non-diagonal terms of the matrix of F_D are given by $\mathbb{I}$ and $\mathbb{I}^\dagger$, where $\mathbb{I}$ is given by

$$i_Y i_X \mathbb{I}(Z) = (\nabla^{\theta,-}_Z F_\theta)(X, Y) + F_\theta(X, g^{-1} i_Z i_Y H) - F_\theta(Y, g^{-1} i_Z i_X H)$$

and our result is obtained computing directly:

$$\begin{aligned}
i_W i_V \mathbb{I}(Z) &= (\nabla^{\theta,-}_Z F_\theta)(V,W) - F_\theta(V,H(Z,W)) + F_\theta(W,H(Z,V)) \\
&= d^\theta_Z(F_\theta(V,W)) - F_\theta(V,\nabla^-_Z W) - F_\theta(\nabla^-_Z V,W) - F_\theta(V,H(Z,W)) \\
&\qquad + F_\theta(W,H(Z,V)) \\
&= d^\theta_Z(F_\theta(V,W)) - F_\theta\left(V,\nabla^g_Z W + \tfrac{1}{2}H(Z,W)\right) - F_\theta\left(\nabla^g_Z V + \tfrac{1}{2}H(Z,V),W\right) \\
&= (\nabla^{\theta,+}_Z F_\theta)(V,W).
\end{aligned}$$

and the result follows. □

We can finally define the notion of coupled G-instantons using the connection D in (2.2.2).

Definition 2.2.2. *Let E be a transitive Courant algebroid over an oriented spin manifold M^n. A pair $(\mathbf{G},\eta)$, given by a generalized metric $\mathbf{G}$ and a spinor $\eta \in \Omega^0(S)$, is a solution of the* coupled instanton equation*, if*

$$F_D \cdot \eta = 0, \tag{2.2.6}$$

where the connection D in $TM \oplus \mathrm{ad}P$ is defined in (2.2.2). When η is nowhere-vanishing, denoting by G the stabilizer of η in Spin(n)*, we will refer to a solution of (2.2.6) as a* coupled G-instanton*.*

Examples of solutions for the coupled instanton equations will be discussed in Section 4.5 in the context of G_2-structures.

Remark 2.2.3. Due to the Lemma 2.2.1, the coupled G-instanton equation is equivalent to the following system of equations

$$\begin{aligned}
(R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F}) \cdot \eta &= 0, \\
\nabla^{\theta,+} F_\theta \cdot \eta &= 0, \\
[F_\theta \cdot \eta, \cdot] - \mathbb{F} \wedge \mathbb{F}^\dagger \cdot \eta &= 0, \\
dH - \langle F_\theta \wedge F_\theta \rangle &= 0.
\end{aligned} \tag{2.2.7}$$

Conversely, any solution (g,H,θ,η) of (2.2.7) determines a transitive Courant Courant algebroid as in Example 1.2.4, endowed with a solution of the coupled instanton equation (2.2.6). ○

We introduce in sequence two problems that link the coupled instanton equation (2.2.6) to the Killing spinor equations in Definition 2.1.2 and generalized Ricci-flat metrics, providing essential motivation for their study. We first observe that, if M^n is even-dimensional with $n = 2m$, then any solution of the gravitino equation in (2.1.5), with complex pure spinor η and integrable complex structure is, in fact, a coupled $\mathrm{SU}(m)$-instanton, in the sense of Definition 2.2.2, cf. [GFGM23, Lemma 5.4]. In this case, solutions of the Killing spinor equations with η pure and ζ exact are in correspondence with solutions of the Hull–Strominger system on complex Calabi-Yau manifolds, with the Hermitian Yang-Mills Ansatz, see [GFGM23]. This correspondence motivates the following.

Problem 1. *Let E be a transitive Courant Courant algebroid over an oriented spin manifold M^n. Let $(\mathbf{G}, \eta)$ be a solution of the gravitino equation in* (2.1.5)*, i.e.*

$$D_-^S \eta = 0.$$

Then, $(\mathbf{G}, \eta)$ satisfies the coupled instanton equation (2.2.6)*.*

More explicitly, the data $(E, \mathbf{G}, \eta)$ in the hypothesis of the previous Problem is equivalent to a tuple (g, H, θ, η) solving the equations:

$$\nabla^+ \eta = 0, \qquad F_\theta \cdot \eta = 0, \qquad dH - \langle F_\theta \wedge F_\theta \rangle = 0. \tag{2.2.8}$$

Furthermore, coupled $\mathrm{SU}(m)$-instantons which satisfy the gravitino equation in (2.1.5) with integrable complex structure are generalized Ricci-flat by [GFGM23, Proposition 5.6], for a suitable choice of divergence operator canonically determined by the Lee form of the $\mathrm{SU}(m)$-structure, which is the motivation for our second problem. This result is recovered with the theory in Chapter 6, cf. Theorem 6.3.6.

Problem 2. *Let E be a transitive Courant algebroid over an oriented spin manifold M^n. Let $(\mathbf{G}, \eta)$ be a solution of the coupled instanton equation* (2.2.6)*. Find the precise conditions, in terms of the G-structure determined by η, which imply that*

$$\mathrm{GRic}^+_{\mathbf{G},\mathrm{div}_0} = 0,$$

for a canonical choice of divergence operator $\mathrm{div}_0 = \mathrm{div}(\mathbf{G}, \eta)$ *uniquely determined by the pair* $(\mathbf{G}, \eta)$*.*

Remark 2.2.4. Notice that the generalized Ricci tensor $\mathrm{Ric}^+_{\mathbf{G},\mathrm{div}}$ depends only on $\zeta_+ = \frac{1}{2}(Z^\flat + \zeta)$ in the decomposition $e = Z + z + \zeta$ where $\langle e, \cdot \rangle := \mathrm{div}^{\mathbf{G}} - \mathrm{div}$, for $\zeta_\pm \in \Gamma(T^*M)$ and $z \in \Gamma(\mathrm{ad}P)$, see Theorem 1.5.5. We expect a solution of (2.1.5) to determine the divergence uniquely by imposing the constraint $e \in T^*M$. ○

In subsequent Section 2.3, we present a complete answer to Problem 1 in arbitrary dimensions via a spinorial proof, cf. Theorem 2.3.2. Section 4.2 and Section 4.3 are devoted to studying those two Problems in the 7-dimensional case, where any nowhere-vanishing spinor determines a G_2-structure. In particular, by direct application of Theorem 2.3.2, in Theorem 4.3.6, we extend a result by de la Ossa, Larfors and Svanes in seven dimensions in the physics literature [dlOLS18a, dlOLS18b].

On the other hand, we propose an approach to Problem 1 and Problem 2 in Chapter 6 not using spinors, but an alternative notion of instantons which will agree for G_2, Spin(7) and SU(m)-instantons. In this alternative, we will solve both problems.

2.3 Generalized Ricci flatness for gravitino solutions

This section presents a complete solution to Problem 1 in arbitrary dimensions. We will need the following essential technical lemma, which generalises [dlOLS18a, Lemma 5], initially proved in dimension 7, to arbitrary dimensions.

Lemma 2.3.1. *Let (M^n, g) be an oriented spin manifold Riemannian manifold. Let $\alpha, \beta \in \Omega^2(M)$ a pair of 2-forms on M and $\eta \in \Gamma(S)$ an arbitrary spinor. Then, we have*[2]

$$(\alpha \cdot \beta - \beta \cdot \alpha) \cdot \eta = \left(\alpha \lrcorner^1 \beta\right) \cdot \eta \tag{2.3.1}$$

Proof. Writing the 2-forms in the standard way $\alpha = \frac{1}{2!}\alpha_{ij}e^{ij}$ and $\beta = \frac{1}{2!}\beta_{ij}e^{ij}$ and using the canonical embedding of Λ^2 into the Clifford algebra $e^j \wedge e^k = e^{jk} \mapsto \frac{1}{2}e_je_k$, we have

$$\alpha \cdot (\beta \cdot \eta) = \frac{1}{16}\alpha_{ij}\beta_{kl}e_ie_je_ke_l \cdot \eta, \qquad \beta \cdot (\alpha \cdot \eta) = \frac{1}{16}\alpha_{ij}\beta_{kl}e_ke_le_ie_j \cdot \eta. \tag{2.3.2}$$

Furthermore, $\gamma = \alpha \lrcorner^1 \beta \in \Omega^2$ is given by $\gamma = \frac{1}{2}\left(\alpha_{jk}\beta_{jl} - \alpha_{jl}\beta_{jk}\right)e^{kl}$ and, consequently:

$$\gamma \cdot \eta = \frac{1}{4}\left(\alpha_{jk}\beta_{jl} - \alpha_{jl}\beta_{jk}\right)e_ke_l \cdot \eta. \tag{2.3.3}$$

The basic Clifford identity $e_ie_j = -e_je_i - 2\delta_{ij}$ implies

$$\begin{aligned}
e_ke_le_ie_j &= -e_ke_ie_le_j - 2\delta_{il}e_ke_j = e_ke_ie_je_l + 2\delta_{jl}e_ke_i - 2\delta_{il}e_ke_j \\
&= -e_ie_ke_je_l - 2\delta_{ik}e_je_l + 2\delta_{jl}e_ke_i - 2\delta_{il}e_ke_j \\
&= e_ie_je_ke_l + 2\delta_{jk}e_ie_l - 2\delta_{ik}e_je_l + 2\delta_{jl}e_ke_i - 2\delta_{il}e_ke_j.
\end{aligned}$$

Using the previous expression and substituting in the first equation in (2.3.2), we then have

$$\begin{aligned}
\beta \cdot (\alpha \cdot \eta) &= \frac{1}{16}\alpha_{ij}\beta_{kl}\left(e_ie_je_ke_l + 2\delta_{jk}e_ie_l - 2\delta_{ik}e_je_l + 2\delta_{jl}e_ke_i - 2\delta_{il}e_ke_j\right) \cdot \eta \\
&= \alpha \cdot (\beta \cdot \eta) + \left(\frac{1}{8}\alpha_{ij}\beta_{jl}e_ie_l - \frac{1}{8}\alpha_{ij}\beta_{il}e_je_l + \frac{1}{8}\alpha_{ij}\beta_{kj}\ e_ke_i - \frac{1}{8}\alpha_{ij}\beta_{ki}e_ke_j\right) \cdot \eta \\
&= \alpha \cdot (\beta \cdot \eta) + \left(-\frac{1}{4}\alpha_{ij}\beta_{il}e_je_l + \frac{1}{4}\beta_{jk}\alpha_{ji}e_ke_i\right) \cdot \eta \\
&= \alpha \cdot (\beta \cdot \eta) - (\alpha \lrcorner^1 \beta) \cdot \eta,
\end{aligned}$$

as desired. □

[2] The notation $\alpha \lrcorner^1 \beta$ means a 'partial contraction' in the sense that for 2-forms $\alpha = \frac{1}{2!}\alpha_{ij}e^{ij}$ and $\beta = \frac{1}{2!}\beta_{ij}e^{ij}$, we have:

$$\alpha \lrcorner^1 \beta := \alpha^{\mu}{}_{i}\beta_{\mu j}e^{ij} \in \Omega^2(M).$$

For details about partial contractions, cf. Appendix A.2.

As in the previous section, we consider a transitive Courant algebroid E over an oriented spin manifold M^n. A solution $(\mathbf{G}, \eta)$ of the gravitino equation in Definition (2.1.2) is equivalent to a tuple (g, H, θ, η) solving the equations (2.1.7) and the heterotic Bianchi identity (1.1.8). Similarly, the coupled instanton equation (2.2.6) on E is equivalent to a tuple (g, H, θ, η) solving the coupled instanton system (2.2.7). With these preliminaries, the following result provides a complete solution to Problem 1 in arbitrary dimensions.

Theorem 2.3.2. *Let $P \to M$ be a principal K-bundle over an oriented spin manifold of arbitrary dimension. Then any solution (g, H, θ, η) of the equations*

$$\nabla^+ \eta = 0, \qquad F_\theta \cdot \eta = 0, \qquad dH - \langle F_\theta \wedge F_\theta \rangle = 0 \tag{2.3.4}$$

solves the coupled instanton system (2.2.7), and consequently the connection on $T \oplus \mathrm{ad}P$ defined in (2.2.2) is an instanton with respect to η, i.e.

$$F_D \cdot \eta = 0.$$

In particular, given any solution $(g, H, \theta, \eta, \zeta)$ of the Killing spinor equations (2.1.7), satisfying the heterotic Bianchi identity (1.1.8), the tuple (g, H, θ, η) solves the coupled instanton system (2.2.7).

Proof. To prove that the first equation in (2.2.7) is satisfied with our hypothesis, we start by writing (using the summation convention, as we shall use throughout the proof):

$$\mathbb{F}^\dagger \wedge \mathbb{F} := \frac{1}{2} f^l{}_{kij} e^{ij} \otimes e^k \otimes e_l \in \Omega^2(M, \mathrm{End}(TM))$$

in a local orthonormal frame $\{e_j\}$ of TM. The coefficients are computed as follows:

$$\begin{aligned} f^l{}_{kij} &:= e^l(i_{e_j} i_{e_i} \mathbb{F}^\dagger \wedge \mathbb{F}(e_k)) = e^l \left(g^{-1} \langle i_{e_j} F_\theta, F_\theta(e_i, e_k) \rangle - g^{-1} \langle i_{e_i} F_\theta, F_\theta(e_j, e_k) \rangle \right) \\ &= e^l \Big(g^{-1} \langle F_{jp} e^p, F_{ik} \rangle - g^{-1} \langle F_{ip} e^p, F_{jk} \rangle \Big) = e^l \left(\Big(\langle F_j{}^p, F_{ik} \rangle - \langle F_i{}^p, F_{jk} \rangle e_p \Big) \right) \\ &= \langle F_j{}^l, F_{ik} \rangle - \langle F_i{}^l, F_{jk} \rangle, \end{aligned}$$

where $F_\theta := \frac{1}{2} F_{\mu\nu} e^{\mu\nu}$ and $F_{\mu\nu} := F_\theta(e_\mu, e_\nu) \in \Omega^0(\mathrm{ad}P)$. Note that

$$\langle F^l \wedge F_k \rangle = \langle F^l{}_i, F_{kj} \rangle e^{ij} = \frac{1}{2} \Big(\langle F^l{}_i, F_{kj} \rangle - \langle F^l{}_j, F_{ki} \rangle \Big) e^{ij},$$

i.e.

$$\langle F^l{}_j, F_{ki} \rangle - \langle F^l{}_i, F_{kj} \rangle = f^l{}_{kij} = -\langle F^l \wedge F_k \rangle_{ij}.$$

On the other hand,

$$\begin{aligned} \langle F_\theta \wedge F_\theta \rangle &= \frac{1}{4} \langle F_{li}, F_{kj} \rangle e^{likj} = \frac{1}{12} \left(\langle F_{li}, F_{kj} \rangle - \langle F_{lk}, F_{ij} \rangle + \langle F_{lj}, F_{ik} \rangle \right) e^{likj} \\ &= \frac{1}{12} \Big(\langle F_{li}, F_{kj} \rangle - \langle F_{lj}, F_{ki} \rangle - \langle F_{lk}, F_{ij} \rangle \Big) e^{likj} \\ &= -\frac{1}{12} \left(-\langle F_l \wedge F_k \rangle_{ij} + \langle F_{lk}, F_{ij} \rangle \right) e^{likj}, \end{aligned}$$

which gives the components

$$-\langle F_l \wedge F_k \rangle_{ij} + \langle F_{lk}, F_{ij} \rangle = -\frac{1}{2} \langle F_\theta \wedge F_\theta \rangle_{likj}.$$

Combining the above and introducing the heterotic Bianchi identity $dH = \langle F_\theta \wedge F_\theta \rangle$, we therefore have

$$\begin{aligned} f^l{}_{kij} = -\langle F^l \wedge F_k \rangle_{ij} &= -\frac{1}{2} \langle F_\theta \wedge F_\theta \rangle^l{}_{kij} - \langle F_\theta, F^l{}_k \rangle_{ij} \\ &= -\frac{1}{2} (dH)^l{}_{kij} - \langle F_\theta, F^l{}_k \rangle_{ij}. \end{aligned}$$

From the relation between the curvatures of the connections $\nabla^\pm = \nabla^g \pm \frac{1}{2} g^{-1} H$:

$$g(R_{\nabla^+}(X,Y)Z,W) = g(R_{\nabla^-}(Z,W)X,Y) + \frac{1}{2} dH(X,Y,Z,W), \tag{2.3.5}$$

we deduce:

$$\begin{aligned} R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} &= -\frac{1}{2} \left(f^l{}_{kij} - (R_{\nabla^-})^l{}_{kij} \right) e^{ij} \otimes e^k \otimes e_l \\ &= -\frac{1}{2} \left(-\cancel{\frac{1}{2} (dH)^l}{}_{kij} - \langle F_\theta, F^l{}_k \rangle_{ij} + \cancel{\frac{1}{2} (dH)^l}{}_{kij} - (R_{\nabla^+})_{ij}{}^l{}_k \right) e^{ij} \otimes e^k \otimes e_l \\ &= \frac{1}{2} \left(\langle F_\theta, F^l{}_k \rangle_{ij} + (R_{\nabla^+})_{ij}{}^l{}_k \right) e^{ij} \otimes e^k \otimes e_l. \end{aligned} \tag{2.3.6}$$

Applying now the first equation in (2.3.4), we also have

$$g(R_{\nabla^+}(X,Y)e_i, e_j) e_i e_j \cdot \eta = \nabla^+_X \nabla^+_Y \eta - \nabla^+_Y \nabla^+_X \eta - \nabla^+_{[X,Y]} \eta = 0,$$

for arbitrary vector field $X, Y \in \Gamma(TM)$. Hence, the first equation in (2.2.7) follows from

$$\left(R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} \right) \cdot \eta = \frac{1}{4} \left(g(R_{\nabla^+}(e_l, e_k)e_i, e_j) e_i e_j \cdot \eta + 2 \langle F_\theta \cdot \eta, F^l{}_k \rangle \right) \otimes e^k \otimes e_l = 0.$$

We next prove that the second and third equations in (2.2.7) are satisfied. Since $\nabla^+ \eta = 0$ and $F_\theta \cdot \eta = 0$ by hypothesis, we have

$$(\nabla^{\theta,+} F_\theta) \cdot \eta = \nabla^{\theta,+} (F_\theta \cdot \eta) = 0, \qquad \text{and} \qquad [F_\theta \cdot \eta, \cdot] = 0,$$

so, it only remains to show that $\mathbb{F} \wedge \mathbb{F}^\dagger \cdot \eta = 0$. To see this, taking an orthonormal basis $\{\zeta_j\}$ for the Lie algebra $\mathfrak{k}$, write

$$\mathbb{F} \wedge \mathbb{F}^\dagger = \frac{1}{2} h^l{}_{kij} e^{ij} \otimes \zeta^k \otimes \zeta_l \in \Omega^2(M, \mathrm{End}(\mathrm{ad}P)),$$

where the coefficients are given by

$$\begin{aligned} h^l{}_{kij} &= \zeta^l \Big(i_{e_j} i_{e_i} \mathbb{F} \wedge \mathbb{F}^\dagger (\zeta_k) \Big) = \zeta^l \Big(F_\theta(e_j, g^{-1} \langle i_{e_i} F_\theta, \zeta_k \rangle) - F_\theta(e_i, g^{-1} \langle i_{e_j} F_\theta, \zeta_k \rangle) \Big) \\ &= \zeta^l \Big(F_\theta(e_j, \langle F_i{}^a e_a, \zeta_k \rangle) - F_\theta(e_i, \langle F_j{}^a e_a, \zeta_k \rangle) \Big) \\ &= \zeta^l \Big(F_\theta(e_j, \langle F^\alpha{}_i{}^a e_a \otimes \zeta_\alpha, \zeta_k \rangle) - F_\theta(e_i, \langle F^\alpha{}_j{}^a e_a \otimes \zeta_\alpha, \zeta_k \rangle) \Big) \\ &= \zeta^l \Big(F_\theta(e_j, F_{ki}{}^a e_a) - F_\theta(e_i, F_{kj}{}^a e_a) \Big) = \zeta^l \Big(F^\alpha{}_{ja} F_{ki}{}^a \zeta_\alpha - F^\alpha{}_{ia} F_{kj}{}^a \zeta_\alpha \Big) \\ &= F^l{}_{ja} F_{ki}{}^a - F^l{}_{ia} F_{kj}{}^a. \end{aligned} \tag{2.3.7}$$

Those are precisely the coefficients of the 2-form $F^l \lrcorner^1 F_k$, and hence the proof follows from $F_\theta \cdot \eta = 0$ by direct application of Lemma 2.3.1. $\square$

2.4 Generalized scalar curvature

A comparison with the physics setup (see Remark 2.1.7) leads naturally to asking whether the Killing spinor equations (cf. Definition 2.1.2) imply the analogue of the equation of motion for the *dilaton field*, a scalar equation given by the vanishing of the function

$$R_g - \frac{1}{2}|H|^2 + |F_\theta|^2 - 2d^*\zeta - |\zeta|^2 \in C^\infty(M), \tag{2.4.1}$$

where $(g, H, \theta, \eta, \zeta)$ is the associated data. This scalar quantity plays an important role in the theory of generalized Ricci flow, being closely related to the volume density of the generalized Perelman energy functional, see [GFS20, GFGMS24]. In (2.4.1), and the sequel, we use the Hodge norm on differential forms $|\beta|^2 = \beta \lrcorner \beta$ for $\beta \in \Omega^k$. Note further that the summand $|F_\theta|^2$ in (2.4.1) is computed using the bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{k} \otimes \mathfrak{k} \longrightarrow \mathbb{R}$, via $|F_\theta|^2 = \langle F_\theta \lrcorner F_\theta \rangle$, Hence, it might not be non-negative as a function of M.

To describe the dynamics of the dilaton field and provide an answer to the above question, we start by giving an interpretation of the scalar (2.4.1) in generalized geometry, using the operators (2.1.2). We build on a Lichnerowicz-type formula for the *cubic Dirac operator* $\not{\nabla}^{1/3}$ due to Bismut [Bis89], see also [AF03, Theorem 6.2]. We follow closely [GFS20, Proposition 3.39], see also [CSCW11] (alternative approaches can be found in [AMP24, ŠV20, SSCV24]). Given a pair $(\mathbf{G}, \mathrm{div})$, we can define a *rough Laplacian* operator

$$\Delta_-^S : \Omega^0(S) \to \Omega^0(S)$$

by the formula

$$\Delta_-^S \eta := \mathrm{tr}_{V_-}(D_-^- \otimes D_-^S)(D_-^S \eta),$$

where we recall that $D_-^S \eta \in \Omega^0(V_-^* \otimes S)$ and D_-^- is the operator defined in Lemmas 1.5.2 and 1.5.3 and Lemma 1.5.2. It is not difficult to see that Δ_-^S is independent of the choice of $\mathbf{G}$-compatible torsion-free generalized connection with divergence div (see [GF19, Lemma 3.4]). Hence, it is a natural quantity associated canonically with the pair $(\mathbf{G}, \mathrm{div})$. We give below an explicit formula for Δ_-^S.

Lemma 2.4.1. *Let $E = TM \oplus \mathrm{ad}P \oplus T^*M$ transitive Courant algebroid as in Example 1.2.4. Define $e \in \Omega^0(E)$ by $\langle e, \cdot \rangle := \mathrm{div}^{\mathbf{G}} - \mathrm{div}$ where $e = Z + z + \zeta$ and consider $\zeta_\pm = \frac{1}{2}(Z^\flat + \zeta)$. Then, for any spinor $\eta \in \Omega^0(S)$, one has*

$$\Delta_-^S \eta = (\nabla^+)^* \nabla^+ \eta + \frac{1}{4}\langle F_\theta \wedge F_\theta \rangle \cdot \eta - \frac{1}{4}|F_\theta|^2 \eta - \nabla^+_{\zeta_-^\#} \eta + \langle F_\theta, z \rangle \cdot \eta, \tag{2.4.2}$$

where $\nabla^+ = \nabla^g + \frac{1}{2} g^{-1} H$.

Proof. Taking $a_-, c_- \in \Omega^0(V_-)$, we have

$$\langle (D_-^- \otimes D_-^S)(D_-^S \eta), a_- \otimes c_- \rangle := D_{a_-}^S D_{c_-}^S \eta - D_{D_{a_-} c_-}^S \eta.$$

To calculate the different elements in this formula explicitly, consider the natural isometries, cf. (1.2.2),

$$\begin{array}{ccc} (TM,g) & \to & V_+ \\ X & \mapsto & X+gX \end{array} \quad \text{and} \quad \begin{array}{ccc} (TM,-g)\oplus(\mathrm{ad}P,\langle\cdot,\cdot\rangle) & \to & V_- \\ X+r & \mapsto & X+r-gX. \end{array} \tag{2.4.3}$$

Via the identification $TM \oplus \mathrm{ad}P \cong V_-$, we have, cf. Example 1.2.4:

$$\begin{aligned} D^-_{X+r}(Z+t) &= \nabla^{-1/3}_X Z - \tfrac{2}{3}g^{-1}\langle i_X F_\theta, t\rangle) - \tfrac{1}{3}g^{-1}\langle i_Z F_\theta, r\rangle \\ &\quad + d^\theta_X t - \tfrac{2}{3}F_\theta(X,Z) - \tfrac{1}{3}[r,t] \\ &\quad + \tfrac{1}{\dim\mathfrak{k}+n-1}\Big((\langle r,t\rangle - g(X,Z))(z+\zeta_-^\#) - (\langle z,t\rangle - \zeta_-(Z))(X+r)\Big). \end{aligned}$$

Since S is the spinor bundle for (TM,g), the Clifford bundle $\mathrm{Cl}(TM)$ is defined via the relation (we follow [LM90])

$$X \cdot X = -g(X,X)$$

and, consequently, in a local orthonormal frame $\{e^j\}$ of T, the 2-forms $e^i \wedge e^j \in \mathfrak{so}(T) = \Lambda^2 T^*M$ embed as $\frac{1}{2}e^i \cdot e^j$ into $\mathrm{Cl}(T)$, cf. [LM90, Proposition 6.2]. Hence, we have an identification of D^S_- in (2.1.3) with the operator:

$$D^S_{X+r}\eta = \nabla^+_X\eta - \langle F_\theta, r\rangle \cdot \eta,$$

for any local spinor η. We choose a local orthogonal frame $\{v_\mu\}$ of V_- and let v^μ denote the corresponding metric dual frame so that $\langle v_\mu, v^\nu\rangle = \delta_{\mu\nu}$, which we assume without loss of generality to be of the form:

$$v_\mu = \begin{cases} X_\mu & \text{if } 1 \le \mu \le n, \\ r_{\mu-n} & \text{if } n < \mu \le \dim\mathfrak{k}+n, \end{cases} \qquad v^\mu = \begin{cases} -X_\mu & \text{if } 1 \le \mu \le n, \\ r^{\mu-n} & \text{if } n < \mu \le \dim\mathfrak{k}+n, \end{cases}$$

where X_μ lie in TM and r_μ lie in $\mathrm{ad}P$, and analogously, for their metric duals. Using this, we calculate

$$\begin{aligned} \sum_{\mu=1}^n D^S_{v_\mu} D^S_{v^\mu}\eta &= \sum_{\mu=1}^n D^S_{X_\mu}\left(-\nabla^+_{X_\mu}\eta\right) + \sum_{\mu=1}^{\dim\mathfrak{k}} D^S_{r_\mu}\left(-\langle F_\theta, r^\mu\rangle\cdot\eta\right) \\ &= -\sum_{\mu=1}^n \nabla^+_{X_\mu}\nabla^+_{X_\mu}\eta + \sum_{\mu=1}^{\dim\mathfrak{k}} \langle F_\theta, r_\mu\rangle\langle F_\theta, r^\mu\rangle\cdot\eta \\ &= -\sum_{\mu=1}^n \nabla^+_{X_\mu}\nabla^+_{X_\mu}\eta + \frac{1}{16}\sum_{i,j,k,l=1}^n \langle F_{ij}, F_{kl}\rangle X_i X_j X_k X_l \cdot \eta \\ &= -\sum_{\mu=1}^n \nabla^+_{X_\mu}\nabla^+_{X_\mu}\eta + \frac{1}{4}\langle F_\theta \wedge F_\theta\rangle\cdot\eta - \frac{1}{4}|F_\theta|^2\eta. \end{aligned}$$

Since the element

$$\Omega_\mathfrak{k} = \sum_{\mu=1}^{\dim\mathfrak{k}} [r_\mu, r^\mu] \in \mathfrak{k}$$

is independent of the choice of local frame $\{r_\mu\}$ and therefore

$$\Omega_{\mathfrak{k}} = \sum_{\mu=1}^{\dim \mathfrak{k}} [r^\mu, r_\mu] = -\Omega_{\mathfrak{k}} = 0.$$

Using this fact, we also have

$$\begin{aligned}
\sum_{\mu=1}^{\dim \mathfrak{k}+n} D^-_{v_\mu} v_\mu &= \sum_{\mu=1}^{n} D_{X_\mu}(-X_\mu) + \sum_{\mu=1}^{\dim \mathfrak{k}} D_{r_\mu} r^\mu \\
&= \sum_{\mu=1}^{n} \Big(-\nabla^{-1/3}_{X_\mu} X_\mu + \tfrac{2}{3} F_\theta(X_\mu, X_\mu) + \tfrac{1}{\dim \mathfrak{k}+n-1} (\delta_{\mu\mu}(z+\zeta_-^\#) - (\zeta_-(X_\mu))X_\mu) \Big) \\
&\quad + \sum_{\mu=1}^{\dim \mathfrak{k}} \Big(-\tfrac{1}{3}[r_\mu, r^\mu] + \tfrac{1}{\dim \mathfrak{k}+n-1} (\delta_{\mu\mu}(z+\zeta_-^\#) - (\langle z, r^\mu\rangle) r_\mu) \Big) \\
&= -\sum_{\mu=1}^{n} \Big(\nabla^g_{X_\mu} X_\mu \Big) + \tfrac{1}{\dim \mathfrak{k}+n-1} (nz + (n-1)\zeta_-) \\
&\quad + \tfrac{1}{\dim \mathfrak{k}+n-1} ((\dim \mathfrak{k} - 1)z + \dim \mathfrak{k} \zeta_-^\#) \\
&= -\sum_{\mu=1}^{n} \Big(\nabla^g_{X_\mu} X_\mu \Big) + \zeta_-^\# + z.
\end{aligned}$$

From the last formula

$$\sum_{\mu=1}^{n} D^S_{D_{v_\mu} v_\mu} \eta = -\sum_{\mu=1}^{n} \nabla^+_{\nabla^g_{X_\mu} X_\mu} \eta + \nabla^+_{\zeta_-^\#} \eta - \langle F_\theta, z\rangle \cdot \eta.$$

Using now [AF03, Theorem 6.1], which states

$$(\nabla^+)^* \nabla^+ \eta = -\sum_{\mu=1}^{n} \Big(\nabla^+_{X_\mu} \nabla^+_{X_\mu} \eta + \nabla^+_{\nabla^g_{X_\mu} X_\mu} \eta \Big),$$

consequently

$$\Delta^S_- \eta = (\nabla^+)^* \nabla^+ \eta + \frac{1}{4} \langle F_\theta \wedge F_\theta \rangle \cdot \eta - \frac{1}{4} |F_\theta|^2 \eta - \nabla^+_{\zeta_-^\#} \eta + \langle F_\theta, z\rangle \cdot \eta.$$

as desired. □

We are ready to provide the technical results of this section.

Proposition 2.4.2. *Let* $(\mathbf{G}, \mathrm{div})$ *be given by a generalized metric and a divergence operator on a transitive Courant algebroid* E*. Consider the pair* (H, θ) *satisfying* (1.1.8) *uniquely determined by* $\mathbf{G}$*. Define* $e = Z + z + \tilde{\zeta} \in \Omega^0(E)$ *by* $\langle e, \cdot \rangle := \mathrm{div}^{\mathbf{G}} - \mathrm{div}$ *and set* $\zeta_\pm = \frac{1}{2}(Z^\flat + \tilde{\zeta}) \in \Omega^0(T^*M)$ *and* $\zeta = \zeta_+$*. Then, for any spinor* $\eta \in \Omega^0(S)$*, one has*

$$\Big(\Big(\not{D}^+ \Big)^2 - \Delta^S_- - D^S_{\tilde{e}_-} \Big) \eta = \frac{1}{4} (\mathcal{S}^+ - 2d\zeta) \cdot \eta, \tag{2.4.4}$$

$\tilde{e}_- = \sigma_-(\zeta^\# + \zeta_-^\#) + z$ *and*

$$\mathcal{S}^+ = R_g - \frac{1}{2} |H|^2 + |F_\theta|^2 - 2d^*\zeta - |\zeta|^2. \tag{2.4.5}$$

Proof. Let's use the same notation as in Lemma 2.4.1 and its proof. Then, the anchor map π applied on the operator D_+^+ is given by the following affine metric connection in the tangent bundle [GFS20, p. 44]

$$\tilde{\nabla}_X Y = \nabla_X^{1/3} Y + \frac{1}{n-1}(g(X,Y)\zeta^\# - \zeta(Y)X) = \nabla_X^{1/3} Y + \frac{1}{n-1} g^{-1}(X^\flat \wedge \zeta)(Y)$$

and the operator $\not{D}^+$, defined as the Dirac operator for D_+^+ (see [GF19, Lemma 3.4]), is therefore $\not{D}^+\eta = \tilde{\not{\nabla}}\eta$. Hence, given a local spinor η, we have

$$\tilde{\nabla}_X \eta = \nabla_X^{1/3}\eta + \frac{1}{4(n-1)}(-\zeta\cdot X + X\cdot\zeta)\cdot\eta.$$

Moreover, writing $\zeta = \sum_k \zeta_k e^k$, we have

$$\begin{aligned}\tilde{\not{\nabla}}\eta &= \not{\nabla}^{1/3}\eta + \frac{1}{4(n-1)}\sum_{j,k}\zeta_k e^j\cdot(-e^k\cdot e^j + e^j\cdot e^k)\cdot\eta \\ &= \not{\nabla}^{1/3}\eta + \frac{1}{4(n-1)}\sum_{j,k}\zeta_k(2\delta_{jk}e^j - 2e^k)\cdot\eta = \not{\nabla}^{1/3}\eta - \frac{1}{2}\zeta\cdot\eta.\end{aligned}$$

With these preliminaries, following the proof of [GFS20, Proposition 3.39], we compute

$$\tilde{\not{\nabla}}^2\eta = \left(\not{\nabla}^{1/3}\right)^2\eta - \frac{1}{2}\left(e_j\cdot\nabla^g_{e_j}\zeta + \frac{1}{6}e_j\cdot H(e_j,\zeta^\#,\cdot)\right)\cdot\eta + \nabla^+_{\zeta^\#}\eta - \frac{1}{3}i_{\zeta^\#}H\cdot\eta - \frac{1}{4}|\zeta|^2\eta,$$

where we have used that $e^j\cdot\alpha + \alpha\cdot e^j = -2\alpha_j$, for any $\alpha = \alpha_j e^j \in T^*M$. Now, for any $\alpha,\beta\in T^*M$, we have (cf. [LM90, Proposition 3.9]) $(\alpha\wedge\beta)\cdot\eta = (\alpha\cdot\beta)\cdot\eta + (\alpha \lrcorner \beta)\cdot\eta$, and hence

$$\sum_j e_j\cdot\nabla^g_{e_j}\zeta\cdot\eta = \sum_j(e_j\wedge\nabla^g_{e_j}\zeta)\cdot\eta - \sum_j(e_j \lrcorner \nabla^g_{e_j}\zeta)\cdot\eta = (d\zeta + d^*\zeta)\cdot\eta.$$

Moreover,

$$\frac{1}{3}i_{\zeta^\#}H\cdot\eta = \frac{1}{6}\sum_{j,k,l}\zeta_l H_{jkl}e^j\wedge e^k\cdot\eta = \frac{1}{12}\sum_{j,k,l}\zeta_l H_{jkl}e^j\cdot e^k\cdot\eta - \frac{1}{12}\sum_j e_j\cdot H(e_j,\zeta,\cdot)\cdot\eta.$$

We deduce that

$$\tilde{\not{\nabla}}^2\eta = \left(\not{\nabla}^{1/3}\right)^2\eta - \frac{1}{2}d\zeta\cdot\eta - \frac{1}{2}d^*\zeta\eta + \nabla^+_{\zeta^\#}\eta - \frac{1}{4}|\zeta|^2\eta.$$

Applying now Lemma 2.4.1 and using the Lichnerowicz-type formula [Bis89], cf. [AF03, Theorem 6.2],

$$\left(\not{\nabla}^{1/3}\right)^2 - (\nabla^+)^*\nabla^+ = \frac{1}{4}R_g - \frac{1}{8}|H|^2 + \frac{1}{4}dH$$

we conclude, applying the Bianchi identity (1.1.8),

$$\begin{aligned}((\not{D}^+)^2 - \Delta^S_-)\cdot\eta &= \frac{1}{4}\left(R_g - \frac{1}{2}|H|^2 + |F_\theta|^2 - 2d^*\zeta - |\zeta|^2\right)\eta + \frac{1}{4}dH\cdot\eta \\ &\quad - \frac{1}{2}d\zeta\cdot\eta + \nabla^+_{\zeta^\#}\eta - \frac{1}{4}\langle F_\theta\wedge F_\theta\rangle\cdot\eta + \nabla^+_{\zeta^\#_-}\eta - \langle F_\theta, z)\rangle\cdot\eta \\ &= \frac{1}{4}\left(R_g - \frac{1}{2}|H|^2 + |F_\theta|^2 - 2d^*\zeta - |\zeta|^2\right)\eta - \frac{1}{2}d\zeta\cdot\eta + \nabla^+_{\zeta^\#+\zeta^\#_-}\eta - \langle F_\theta, z)\rangle\cdot\eta.\end{aligned}$$

as desired □

The previous result motivates the following definition:

Definition 2.4.3. *Let* $(\mathbf{G}, \mathrm{div})$ *be given by a generalized metric and a divergence operator on a transitive Courant algebroid* E. *Define* $e = Z + z + \tilde{\zeta} \in \Omega^0(E)$ *by* $\langle e, \cdot\rangle := \mathrm{div}^{\mathbf{G}} - \mathrm{div}$ *and set* $\zeta_\pm = \frac{1}{2}(Z^\flat + \tilde{\zeta}) \in \Omega^0(T^*M)$ *and* $\zeta = \zeta_+$. *The* generalized scalar curvature

$$\mathcal{S}^+ = \mathcal{S}^+_{\mathbf{G},\mathrm{div}} \in C^\infty(M)$$

of the pair $(\mathbf{G}, \mathrm{div})$ *is defined by*

$$\mathcal{S}^+\eta = 4\left(\left(\not{D}^+\right)^2 - \Delta^S_- - D^S_{\tilde{e}_-} + \frac{1}{2}d\zeta\right)\cdot\eta, \tag{2.4.6}$$

for any spinor $\eta \in \Omega^0(S)$, *where* $\tilde{e}_- = \sigma_-(\zeta^\# + \zeta^\#_-) + z$. *The generalized scalar curvature is well-defined, and explicitly given by* (2.4.5) *in Proposition 2.4.2.*

Remark 2.4.4. As we will see shortly, the generalized scalar curvature plays a distinguished role when $e \in \Omega^0(\ker\pi)$ and $[e, \cdot] = 0$; in other words, when e gives a symmetry of the Dorfman bracket lying in the kernel of the anchor map. In this case, one has $\zeta = -\zeta_-$ and consequently $\tilde{e}_- = z$. Furthermore, the condition $[e, \cdot] = 0$ implies

$$d\zeta + 2\langle F_\theta, z\rangle = 0, \qquad d^\theta z = 0, \qquad [z, \cdot] = 0.$$

A fascinating instance arises when $z = 0$, that is, when e lies on the cotangent subbundle $T^*M \subset E$, as in this case $d\zeta = 0$ and $\tilde{e}_- = 0$. An essential fact about $\mathcal{S}^+_{\mathbf{G},\mathrm{div}}$, which we will see in the proof of Proposition 2.4.7, is that it does *not* always coincide with the trace of the symmetric part of the generalized Ricci tensor $\mathrm{Ric}^+_{\mathbf{G},\mathrm{div}}$ (cf. [GFS20, Remark 3.42]). ○

Remark 2.4.5. We can regard (2.4.6) as a local formula on M so that there is no obstruction for the existence of the spinor bundle. Therefore, we can define the generalized scalar curvature of a pair $(\mathbf{G}, \mathrm{div})$ by (2.4.5) for a transitive Courant algebroid over an arbitrary smooth manifold. ○

We finish this section by establishing the desired relation between the generalized Ricci-flat condition, the Killing spinor equations in Definition 2.1.2, and the generalized scalar curvature.

Proposition 2.4.6. *Let* $(\mathbf{G}, \mathrm{div})$ *be a pair given by a generalized metric and a divergence operator on a transitive Courant algebroid* E *over an* n*-manifold* M. *Define* $e = Z + z + \tilde{\zeta} \in \Omega^0(E)$ *by* $\langle e, \cdot\rangle := \mathrm{div}^{\mathbf{G}} - \mathrm{div}$ *and set* $\zeta = \frac{1}{2}(Z + \tilde{\zeta})$. *Then, assuming that* $\mathrm{GRic}^+_{\mathbf{G},\mathrm{div}} = 0$, *cf.* (2.1.8), *one has*

$$d\mathcal{S}^+ = (-1)^n * (d\zeta \wedge *H). \tag{2.4.7}$$

In particular, if $d\zeta = 0$, *the generalized scalar curvature of* $(\mathbf{G}, \mathrm{div})$ *is constant and furthermore one has*

$$d\left(|H|^2 - |F_\theta|^2 - d^*\zeta - |\zeta|^2\right) = 0 \tag{2.4.8}$$

Proof. [Mol24, Proposition 6.4.5]. □

The formula (2.4.7) follows from an explicit calculation in local coordinates while the proof of (2.4.8) follows by subtracting $\mathcal{S}^+$ minus the trace of the symmetric tensor in the generalized Ricci-flat equations (2.1.8), cf. Remark 2.4.4. In the last result of this section, we establish that solutions for the killing spinor equations have constant generalized scalar curvature.

Proposition 2.4.7. *Provided that* $(\mathbf{G}, \mathrm{div}, \eta)$ *is a solution of the Killing spinor equations (cf. Definition 2.1.2) with parameter* $\lambda \in \mathbb{R}$*, then the generalized scalar curvature satisfies*

$$(\mathcal{S}^+ - 4\lambda^2 - 2d\zeta) \cdot \eta = 0,$$

where $\mathrm{div}^{\mathbf{G}} - \mathrm{div} = \langle e, \cdot \rangle$ *and* $\zeta = g(\pi e_+, \cdot) \in T^*M$*. In particular, if* $d\zeta = 0$ *and* η *is nowhere-vanishing, one has*

$$\mathcal{S}^+ = |H|^2 - |F_\theta|^2 - d^*\zeta - |\zeta|^2 = 4\lambda^2.$$

Proof. Combining the Killing spinor equations (2.1.5) and (2.1.6) with (2.4.4), we conclude immediately

$$(\mathcal{S}^+ - 2d\zeta)\eta = 4\Big(\Big(\not{D}^+\Big)^2 - \Delta^S_- - D^S_{\tilde{\varepsilon}_-}\Big)\eta = 4\lambda^2\eta.$$

The last part of the statement follows, as in the proof of Proposition 2.4.6, by subtracting $\mathcal{S}^+$ minus the trace of the first equation in (2.1.8). □

Remark 2.4.8. Equation (2.4.7) implies that, for a Ricci-flat pair $(\mathbf{G}, \mathrm{div})$ with $d\zeta = 0$, the *dilaton equation of motion* $\mathcal{S}^+ = 0$ is satisfied, up to an overall constant on the manifold. Furthermore, for any solution of the Killing spinor equations in Definition 2.1.2 with $d\zeta = 0$ and parameter $\lambda \neq 0$, one has $\mathcal{S}^+ > 0$. ○

Remark 2.4.9. For a solution $(\mathbf{G}, \mathrm{div}, \eta)$ of the Killing spinor equations in dimension 6 with $\eta \neq 0$, one has that $\lambda = 0$ (because η is pure). Imposing further that $\zeta = d\phi$, one has that $(\mathbf{G}, \mathrm{div}, \eta)$ is equivalent to a solution of the Hull-Strominger system [GFRT16] and the previous result implies that $\mathcal{S}^+ = 0$ in this case. ○

2.5 Instanton towers

We will now discuss a curious phenomenon which creates infinite numbers of instantons, with increasing rank, from solutions of the gravitino equation (2.1.5) with $dH = 0$. Concrete examples of these *instanton towers* are discussed in Section 4.5 in the seven-dimensional case by application of Theorem 4.3.6. Fundamental to our development is the following symmetry, originally due to Bismut [Bis89] (see also [GFS20, Proposition 3.21]), between the curvatures of the metric connections $\nabla^{\pm} = \nabla^g \pm \frac{1}{2} g^{-1} H$ with totally skew-symmetric torsion $\pm H \in \Omega^3$:

$$g(R_{\nabla^+}(X,Y)Z,W) = g(R_{\nabla^-}(Z,W)X,Y) + \frac{1}{2} dH(X,Y,Z,W). \tag{2.5.1}$$

We discuss next the salient implications of the previous result for *instanton engineering* on the oriented spin n-manifold M. To start the iteration scheme, consider M endowed with a metric g, a spinor η, a three-form $H \in \Omega^3$, and a connection θ on a principal K-bundle $P \to M$, satisfying

$$\nabla^+ \eta = 0, \qquad F_\theta \cdot \eta = 0, \qquad dH = 0.$$

We consider the Lie algebra $\mathfrak{k}$ of the structure group K of P endowed with a bi-invariant pairing $\langle , \rangle$. Using that $(d^\theta)^2 = [F_\theta, \cdot]$, it follows that the covariant derivative $\nabla^1 = d^\theta$ on the orthogonal bundle $V_1 = (\mathrm{ad}P, \langle , \rangle)$ is an instanton with respect to η. Let P_1 be the principal bundle of split orthogonal frames of $V_1 \oplus V_1$, with structure group $K_1 = \mathrm{SO}(r_1) \times \mathrm{SO}(r_1)$, for $r_1 = \dim \mathfrak{k}$ (here we abuse notation for the special orthogonal group, since $\langle , \rangle$ may have arbitrary signature), and Lie algebra $\mathfrak{k}_1$ endowed with the neutral pairing

$$\langle \cdot, \cdot \rangle_1 = \mathrm{tr}_{\mathfrak{so}(r_1)} - \mathrm{tr}_{\mathfrak{so}(r_1)} .$$

The product connection $D^1 = \nabla^1 \times \nabla^1$ provides a new solution (g, H, D^1, η) of the gravitino equation (2.1.5),

$$\nabla^+ \eta = 0, \qquad F_{D^1} \cdot \eta = 0,$$

together with a trivial split solution of the Bianchi identity (cf. (1.1.8)):

$$dH = 0, \qquad \langle F_{D^1} \wedge F_{D^1} \rangle_1 = \mathrm{tr}_{\mathfrak{so}(r_1)}(F_{\nabla^1} \wedge F_{\nabla^1}) - \mathrm{tr}_{\mathfrak{so}(r_1)}(F_{\nabla^1} \wedge F_{\nabla^1}) = 0. \tag{2.5.2}$$

This type of ansatz for solving the (supersymmetry) equations is known in the supergravity literature as the *standard embedding*, cf. [GFRST22].

Applying now Theorem 2.3.2, from the data (g, H, D^1, η) we can construct an instanton ∇^2 on the bundle

$$V_2 = T \oplus \mathrm{ad}P_1$$

for the same metric g and spinor η, explicitly given by (2.2.2) (with θ replaced by D^1). Note that V_2 is an orthogonal bundle with metric

$$\langle X + r, X + r \rangle_2 = g(X,X) - \langle r, r \rangle_1,$$

and the connection ∇^2 is compatible with this metric. As before, let P_2 be the principal bundle of split orthogonal frames of $V_2 \oplus V_2$, with structure group $K_2 = \mathrm{SO}(r_2) \times \mathrm{SO}(r_2)$, for $r_2 = n + r_1(r_1 - 1)$, and Lie algebra $\mathfrak{k}_2$ endowed with the neutral pairing

$$\langle \cdot, \cdot \rangle_2 = \mathrm{tr}_{\mathfrak{so}(r_2)} - \mathrm{tr}_{\mathfrak{so}(r_2)} .$$

The product connection $D^2 = \nabla^2 \times \nabla^2$ provides a new solution (g, H, D^2, η) of the gravitino equation (2.1.5) and a trivial split solution of the Bianchi identity (2.5.2). Iterating this scheme, we obtain, by induction, an infinite tower of instantons with rank going to infinity on a fixed manifold. We will summarise the construction using the result below.

Proposition 2.5.1. *Let M be an oriented spin manifold of dimension n endowed with a metric g, a three-form $H \in \Omega^3$, a spinor η, and a connection θ on a principal K-bundle $P \to M$, solving the equations*

$$\nabla^+ \eta = 0, \qquad F_\theta \cdot \eta = 0, \qquad dH = 0.$$

Then, there exists an infinite sequence of instantons $\{(V_k, \nabla^k)\}_{k \in \mathbb{N}}$ on M, for the same metric g and spinor η, where V_k is a real orthogonal bundle of rank

$$r_k = n + r_{k-1}(r_{k-1} - 1), \qquad r_1 = \dim K,$$

and ∇^k is a linear orthogonal connection on V_k.

Remark 2.5.2. Given a solution (g, H, η) of the equations,

$$\nabla^+ \eta = 0, \qquad dH = 0,$$

the proof of Theorem 2.3.2 implies that the connection ∇^- on the orthogonal vector bundle (TM, g) is an instanton. This is a direct consequence of the identity (2.3.5). Hence, in this setup, one can choose $\theta = \nabla^-$ to start the iteration scheme. ◯

3 G_2-Structures

This chapter provides an introduction to the geometry of G_2-structures, which are defined as a reduction of the frame bundle of a seven-dimensional manifold M^7 from $\mathrm{Gl}(7)$ to G_2. In Chapter 4, we will use the theory developed in Chapter 2 applied to manifolds endowed with G_2-structures. Key references for this chapter include [SW17] for the introductory geometric explanation of the cross product in seven dimensions, [Bry05, Kar20, Kar08a, dlOLS18a] for the general theory of G_2-structures in manifolds, encompassing their decomposition of forms and torsion forms. Additionally, we will draw insights from [FI02] to study the intrinsic torsion of G_2-structures. Furthermore, references such as [FKMS97, ACFH15] were consulted for the spinorial description of G_2-structures.

This chapter is structured as follows: in the first Section 3.1, we explore the concept of the cross product in seven dimensions, extending its traditional cross product in three dimensions. By drawing parallels between the classical cross product in three dimensions and the quaternion algebra $\mathbb{H}$, we establish connections with the octonions $\mathbb{O}$. In the following, we introduce the group G_2 in this context with the fundamental 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$. Furthermore, we elucidate its correlation with the metric structure of $\mathbb{R}^7$ (cf. Lemma 3.1.5). We finish the section with the identities for the coefficients of φ_0 and its dual $\psi_0 = *\varphi_0$ which will be useful throughout all the text (cf. Proposition 3.1.6). This discussion follows mainly [SW17].

In Section 3.2, we present the existence of the 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ (fundamental G_2-structure in $\mathbb{R}^7$) induces a decomposition of the space of forms $\Lambda^k(\mathbb{R}^7)^*$ into irreducible G_2-representations. Through a detailed exploration of this decomposition and the characterization of each component, we gain insight into the structure's impact on form spaces for general structure, which will be essential in Chapter 6, where we generalize some results of the Chapter 4 for other geometrical structures. This section is mainly based on [Bry05, Kar08a] and follows the conventions in [dlOLS18a].

Continuing our investigation, in Section 3.3, we present the G_2-structures on manifolds. This can be understood as the existence of the cross product in each tangent space of a 7-dimensional manifold M. This is equivalent to the existence of a 3-form $\varphi \in \Omega^3(M)$, referred to as a G_2-structure on M and locally modelled by φ_0. Throughout this chapter, we meticulously explore the geometry inherent in such structures, emphasising the torsion they exhibit and detailing the torsion forms (cf. Propositions 3.3.2) and the intrinsic torsion of a G_2-structure (cf. Proposition 3.3.4). We follow mainly [Kar08a] for the description of torsion forms and [FI02, Fri02] for the description of the intrinsic torsion, included in Appendix B for general structures.

In Section 3.4, we study the existence of compatible (with the G_2-structure) connections with totally skew-symmetric torsion. Connections with skew-symmetric torsion have appeared in Chapter 1 and Chapter 2 (connections $\nabla^{\pm}, \nabla^{\pm 1/3}$). Within the realm of G_2-structures, the significance of such connections lies in their uniqueness when they exist. Moreover, their torsion matches the flux H elucidated in the first two chapters. This marks the start point of the selection process for the pair (H, θ) (cf. Example 1.1.5) within the context of G_2: H represents the torsion of this compatible connection, and θ assumes the role of the so-called G_2-instantons, defined via the form φ. To find the expression of the torsion H of the compatible connection (cf. Theorem 3.4.4), we use an original approach (which works for other geometrical structures), which is discussed in Appendix B.

Finally, in Section 3.5, we present a characterization of G_2-structures utilizing spinors, leveraging the canonical isomorphism between G_2 and a subgroup of Spin(7). This spinorial perspective proves invaluable, especially in light of the spinorial treatment explored in Chapter 2. The insights gleaned from this section will serve as a foundation for the subsequent Chapter 4. This section follows mainly [FKMS97, ACFH15].

3.1 Geometrical motivation: the cross product

In elementary analytical geometry of the three-dimensional space $\mathbb{R}^3$, two operations are essential: the scalar product (also called the inner product) and the cross product (also called vector product)

$$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \qquad \text{and} \qquad \times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3.$$

These operations on vectors can measure angles and distances; the cross product is a skew-symmetric operation which results in a vector, such vector is perpendicular to its entries

$$(u \times v) \cdot u = (u \times v) \cdot v = 0, \tag{3.1.1}$$

and the norm of scalar and cross product are "circular complementary" in the following sense:

$$|u \times v|^2 + (u \cdot v)^2 = |u|^2 |v|^2 \tag{3.1.2}$$

The inner product is naturally generalized to any dimension and allows the same study of angles and distance in higher dimensions.

The natural question is: "What about the cross-product in higher dimensions?" To answer this question, we have to understand what is, actually, the cross product in $\mathbb{R}^3$. For this, we introduce the notion of *quaternions*. The quaternions, denoted by $\mathbb{H}$, generalise complex numbers but with three imaginary identities. Recall that the complex numbers are the vector space $\mathbb{C} = \mathbb{R}^2$, where the standard basis is denoted by $(e_0, e_1) =: (1, \mathbf{i})$, and the algebra product is defined by relation $\mathbf{i}^2 = -1$. In the same way:

Definition 3.1.1 (Quaternionic Algebra)**.** *The quaternions $\mathbb{H} = \mathbb{R}^4$ is a vector space with a product defined in the standard basis $(e_0, e_1, e_2, e_3) =: (1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ by the relations*

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

this operation defines an algebra structure (associative, anti-commutative and a normed algebra, that is, $|uv| = |u| \cdot |v|$).

The most general form of a quaternion is $\mathbf{v} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $a, b, c, d \in \mathbb{R}$. The real part is $a = \mathrm{Re}(\mathbf{v})$ and the imaginary part is a vector part $\mathrm{Im}(\mathbf{v}) = (b, c, d) \in \mathbb{R}^3$. The natural identification $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$ is crucial for understanding the essence of the cross-product.

Lemma 3.1.2. *Let $u, v \in \mathbb{R}^3$ and identify $\mathbb{R}^3$ with the purely imaginary quaternions $\mathrm{Im}(\mathbb{H}) \subset \mathbb{H}$. Then, the cross product of u and v can be expressed, using the product of quaternions, as*

$$u \times v = \mathrm{Im}(uv),$$

where the product on the right-hand side is the quaternionic multiplication of u and v.

With this identification, the properties of the cross product (3.1.1) and (3.1.2) are immediate consequences of $\mathbb{H}$ being a *normed division algebra*. With this, we obtain a way to find a cross product in other dimensions: if there is a normed algebra in $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$, we can define a cross product as the imaginary projection in $\mathbb{R}^n = \mathrm{Im}(\mathbb{R} \oplus \mathbb{R}^n)$. In fact, cf. [SW17], the cross-product exists only in these cases.

The 'problem' here is that the division normed algebra doesn't exist in every dimension. Surprisingly, it exists only in dimensions $n = 1, 2, 4, 8$. In dimension $n = 1$ is the algebra of real numbers $\mathbb{R}$; for $n = 2$, we have the algebra of complex numbers; in $n = 4$, the algebra of quaternionic numbers $\mathbb{H}$, as we have discussed above; and for $n = 8$, we have the algebra of octonions $\mathbb{O}$.

Definition 3.1.3 (Octonionic Algebra)**.** *The octonions $\mathbb{O} = \mathbb{R}^8$ is a vector space with a product defined in the standard basis $(e_0, \ldots, e_7)$ by the relations*

$$e_i \cdot e_j = -\delta_{ij} + \varepsilon_{ijk} e_k; \qquad i, j \in \{1, \cdots, 7\},$$

where δ_{ij} is the Kronecker delta and ε_{ijk} is totally skew-symmetric with value 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$. This operation defines an algebra structure (non-associative and non-commutative).

Lemma 3.1.4. *Let $u, v \in \mathbb{R}^3$ and identify $\mathbb{R}^3$ with the purely imaginary quaternions $\mathrm{Im}(\mathbb{H}) \subset \mathbb{H}$. Then, the cross product of u and v defined, using the product of quaternions, as*

$$u \times v = \mathrm{Im}(uv),$$

where the product on the right-hand side is the octonionic multiplication of u and v. Then this expression defines a cross product in the sense it satisfies (3.1.1)*,* (3.1.2) *and is skew-symmetric.*

On the cross product in $\mathbb{R}^3$, we can contract the operation using the metric (standard in $\mathbb{R}^3$) and define the (3,0)-tensor φ_0:

$$\varphi_0(u,v,w) = g(u\times v, w) = \det(u,v,w),$$

which contains all the information about the cross-product and has the property of being completely skew-symmetric. We have the cross product on $\mathbb{R}^7$, and we can define the 3-form φ_0 similarly. But it will not be the volume in seven dimensions as in $\mathbb{R}^3$ because we have more dimensions.

Lemma 3.1.5 ([SW17])**.** *Let $\mathbb{R}^7$ with standard metric $g_0 = e^1\otimes e^1 + \cdots + e^7\otimes e^7$ and the cross product as the imaginary projection in the octonions. Define the (3,0)-tensor*

$$\varphi_0(u,v,w) = g_0(u\times v, w)$$

then $\varphi_0 \in \Lambda^3(\mathbb{R}^{7*})$, i.e., it is a totally skew-symmetric tensor called fundamental G_2-structure on $\mathbb{R}^7$*, which recovers the inner product via the expression:*

$$g_0(u,v) = \frac{1}{6\mathrm{vol}} i_u\varphi_0 \wedge i_v\varphi_0 \wedge \varphi_0. \tag{3.1.3}$$

Using the standard dual basis $\{e^1,\cdots,e^7\}$ of $\mathbb{R}^{7*}$, φ_0 and its dual $\psi_0 = *\varphi_0$ are given by[1]

$$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \tag{3.1.4}$$
$$\psi_0 = e^{3456} + e^{1256} + e^{1234} - e^{2467} + e^{2357} + e^{1457} + e^{1367}. \tag{3.1.5}$$

In particular, $|\varphi_0|^2 = |\psi_0|^2 = 7$.

Using the standard way to write the forms (cf. Appendix A), we will denote the coordinates of φ_0 and ψ_0 (using the standard basis of $\mathbb{R}^7$) as[2]

$$\varphi_0 = \frac{1}{3!}\varphi_{ijk}e^{ijk}, \qquad \text{and} \qquad \psi_0 = \frac{1}{4!}\psi_{ijkl}e^{ijkl}.$$

With these coefficients, the cross product in coordinates is given by

$$e_i \times e_j = \varphi_{ij}{}^{k} e_k \tag{3.1.6}$$

These coefficients satisfy some relations under contractions:

1 Here $e^{ijk} := e^i\wedge e^j\wedge e^k$.

2 where $\varphi_{ijk} := \varphi(e_i,e_j,e_k)$ and $\psi_{ijkl} := \psi(e_i,e_j,e_k,e_l)$.

Proposition 3.1.6. *The fundamental G_2-structure on $\mathbb{R}^7$ (in the standard basis as in (3.1.4) and (3.1.5)) satisfies the following relations between the coefficients:*

$$\begin{aligned}
\varphi^{\mu\nu\rho}\varphi_{\mu\nu\rho} &= 42\\
\varphi^{\mu\nu i}\varphi_{\mu\nu a} &= 6\delta^i{}_a\\
\varphi^{\mu ij}\varphi_{\mu ab} &= \delta^i{}_a\delta^j{}_b - \delta^j{}_a\delta^i{}_b + \psi^{ij}{}_{ab}\\
\varphi^{\mu\nu\rho}\psi_{\mu\nu\rho i} &= 0\\
\varphi^{\mu\nu i}\psi_{\mu\nu ab} &= 4\varphi^i{}_{ab}\\
\varphi^{\mu ij}\psi_{\mu abc} &= \delta^i{}_a\varphi^j{}_{bc} + \delta^i{}_b\varphi_a{}^j{}_c + \delta^i{}_c\varphi_{ab}{}^j - \delta^j{}_a\varphi_i{}^{bc} - \delta^j{}_b\varphi_a{}^i{}_c - \delta^j{}_c\varphi_{ab}{}^i\\
\psi^{\mu\nu\rho\eta}\psi_{\mu\nu\rho\eta} &= 168\\
\psi^{\mu\nu\rho i}\psi_{\mu\nu\rho a} &= 24\delta^i{}_a\\
\psi^{\mu\nu ij}\psi_{\mu\nu ab} &= 4\delta^i{}_a\delta^j{}_b - 4\delta^i{}_b\delta^j{}_a + 2\psi^{ij}{}_{ab}\\
\psi^{\mu ijk}\psi_{\mu abc} &= -\varphi_a{}^{jk}\varphi^i{}_{bc} - \varphi^i{}_a{}^k\varphi^j{}_{bc} - \varphi^{ij}{}_a\varphi^k{}_{bc} + \delta^i{}_a\delta^j{}_b\delta^k{}_c + \delta^i{}_b\delta^j{}_c\delta^k{}_a\\
&\quad + \delta^i{}_c\delta^j{}_a\delta^k{}_b - \delta^i{}_a\delta^j{}_c\delta^k{}_b - \delta^i{}_b\delta^j{}_a\delta^k{}_c - \delta^i{}_c\delta^j{}_b\delta^k{}_a\\
&\quad + \delta^i{}_a\psi^{jk}{}_{bc} + \delta^j{}_a\psi^{ki}{}_{bc} + \delta^k{}_a\psi^{ij}{}_{bc} - \delta_{ab}\psi^{ijk}{}_c + \delta_{ac}\psi^{ijk}{}_b
\end{aligned}$$

Proof. This proof can be found in [Kar08a], but our conventions follow [dlOLS18a]. They can also be obtained via direct computation using (3.1.4) and (3.1.5). □

The designation of 'G_2-structure' for the form φ_0 may appear arbitrary at first glance, but it holds significant mathematical relevance. The name stems from the algebraic structure known as G_2, one of the five exceptional simple simply-connected Lie groups, and one of the two exceptional holonomy groups listed in Berger's classification alongside Spin(7). Furthermore, the connection between G_2 and the form φ_0 is direct: G_2 is the stabilizer group of such a form, meaning it is the set of transformations that preserve the form:

$$G_2 := \Big\{ g \in \mathrm{GL}(7) : g^*\varphi_0 = \varphi_0 \Big\}. \tag{3.1.7}$$

The group G_2 is compact, simple, simply-connected and a subgroup of SO(7) and, as a manifold, $\dim G_2 = 14$ [Joy00]. Another way to characterize the group G_2 is as the group of algebra automorphisms of the octonions, i.e., $G_2 = \mathrm{Aut}(\mathbb{O})$ [SW17].

3.2 Decomposition of the space of forms

The group G_2 acts on $\mathbb{R}^7$ via matrix multiplication and consequently on all space of forms $\Lambda^k(\mathbb{R}^7)^*$. Consequently, they can be decomposed into G_2-irreducible representations. Details are given below and based on [Bry05, Kar20, FI02]. The list of irreducible G_2-representations can be found in [FKS20].

There is a standard representation of the group G_2 in a seven-dimensional vector space, $\rho : \mathrm{G}_2 \to \mathrm{Aut}(\mathbb{R}^7)$, given by matrix multiplication since we have defined $\mathrm{G}_2 \subset \mathrm{GL}(7)$. It is irreducible and extends naturally in tensor product spaces, particularly within forms. In particular, the space of 1-forms is irreducible:

$$\Lambda^1 \cong \Lambda^1_7$$

In general, the subscript indicates the dimension of the space, so $\Lambda^1_7 \cong \mathbb{R}^7$. For the space of 0-forms, we have the 1-dimensional trivial representation:

$$\Lambda^0 = \Lambda^0_1 \cong \mathbb{R}.$$

Since the Hodge star operator induces isomorphisms $* : \Lambda^k \cong \Lambda^{7-k}$, we have to study the decomposition of 2-forms and 3-forms.

The space of 2-forms has $\binom{7}{2} = 21$ dimensions and contains the adjoint 14-dimensional representation via identification $\mathfrak{g}_2 < \mathfrak{so}(7) = \Lambda^2(\mathbb{R}^7)^*$ which is irreducible since G_2 is simple. Furthermore, the space of 2-forms also contains the vector representation $\mathbb{R}^7$ because the map

$$X \in \Lambda^1 \mapsto X \lrcorner \varphi_0 \in \Lambda^2$$

is a G_2-invariant map and an embedding of Λ^1 into Λ^2 since φ_0 is non-degenerate. Now, since $14 + 7 = 21$, we have the decomposition of the space of 2-forms into irreducible modules:

$$\Lambda^2(\mathbb{R}^7)^* = \Lambda^2_7 \oplus \Lambda^2_{14} \cong \Lambda^1 \oplus \mathfrak{g}_2$$

What we have to do now is characterize these spaces. We have already known that $\Lambda^2_7 = \{X \lrcorner \varphi_0 : X \in \Lambda^1\}$ by the embedding $\Lambda^1 \to \Lambda^2$. Let's find a more complete characterization of these spaces.

Proposition 3.2.1 (Decomposition of 2-forms)**.** *The space $\Lambda^2(\mathbb{R}^7)^*$ of 2-forms decomposes into G_2-irreducible representations as*

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \cong \Lambda^1 \oplus \mathfrak{g}_2 \tag{3.2.1}$$

where

$$\Lambda^2_7 = \{X \lrcorner \varphi_0 : X \in \mathbb{R}^7\} = \{\beta \in \Lambda^2 : (\beta \lrcorner \varphi_0) \lrcorner \varphi_0 = 3\beta\} = \{\beta \in \Lambda^2 : \beta \lrcorner \psi_0 = 2\beta\} \tag{3.2.2}$$

$$\Lambda^2_{14} = \{\beta \in \Lambda^2 : \beta \wedge \psi_0 = 0\} = \{\beta \in \Lambda^2 : \beta \lrcorner \varphi_0 = 0\} = \{\beta \in \Lambda^2 : \beta \lrcorner \psi_0 = -\beta\} \tag{3.2.3}$$

furthermore, we have the projection formulas for these spaces

$$\pi_7(\beta) = \frac{1}{3}(\beta \lrcorner \varphi_0) \lrcorner \varphi_0 = \frac{1}{3}(\beta + \beta \lrcorner \psi_0) \tag{3.2.4}$$

$$\pi_{14}(\beta) = \frac{1}{3}(2\beta - \beta \lrcorner \psi_0) \tag{3.2.5}$$

Proof. The first characterization of Λ^2_7 we have already proved. Now let's consider another G_2-invariant map:

$$\beta \in \Lambda^2 \mapsto \beta \lrcorner \varphi_0 \in \Lambda^1.$$

Given $\Lambda^1 = \mathbb{R}^7$ and $\Lambda^2 = \mathbb{R}^7 \oplus \mathfrak{g}_2$, then $\mathfrak{g}_2$ is in the kernel of this map since it doesn't appear in the decomposition of Λ^1 and invariant maps preserve the irreducible components [FI02]. About Λ^2_7, we can compute directly how it behaves under this map using the fact that $\Lambda^2_7 = \{X \lrcorner \varphi_0 : X \in \mathbb{R}^7\}$ and the contraction identities proved in Proposition 3.1.6:

$$(X \lrcorner \varphi_0) \lrcorner \varphi_0 = \frac{1}{1!2!1!} X^\mu {\varphi_\mu}^{ij} \varphi_{ijk} e^k = \frac{1}{2} X^\mu \cdot 6\delta_{\mu k} e^k = 3X_k e^k = 3X$$

consequently, we have $((X \lrcorner \varphi_0) \lrcorner \varphi_0) \lrcorner \varphi_0 = 3X \lrcorner \varphi_0$ and since $\beta \lrcorner \varphi_0 = 0 \iff \beta \wedge \psi_0 = 0$, we have proved the following characterizations

$$\Lambda^2_7 = \{X \lrcorner \varphi_0 : X \in \Lambda^1\} = \{\beta \in \Lambda^2 : (\beta \lrcorner \varphi_0) \lrcorner \varphi_0 = 3\beta\}$$
$$\Lambda^2_{14} = \{\beta \in \Lambda^2 : \beta \lrcorner \varphi_0 = 0\} = \{\beta \in \Lambda^2 : \beta \wedge \psi_0 = 0\}.$$

Finally, let's investigate the G_2-invariant map between 2-forms

$$\beta \in \Lambda^2 \mapsto \beta \lrcorner \psi_0 \in \Lambda^2$$

since it is an invariant endomorphism between 2-forms, it decomposes into eigenspaces, and the irreducible components are contained in a unique eigenspace. Let's compute the eigenvalues explicitly. Firstly, for the component Λ^2_7:

$$(X \lrcorner \varphi_0) \lrcorner \psi_0 = \frac{1}{1!2!2!} X^\mu {\varphi_\mu}^{ij} \psi_{ijab} e^{ab} = \frac{1}{4} X^\mu \cdot 4\varphi_{\mu ab} e^{ab} = 2X \lrcorner \varphi_0.$$

Now, to compute the eigenvalue for the space $\mathfrak{g}_2 = \Lambda^2_{14}$, we have no general form of an element in it, but since the map is invariant, it is enough to compute for a specific element in $\mathfrak{g}_2$. Let β_0 any element not completely in Λ^2_7, and consider $\beta_1 = 2\beta_0 - \beta_0 \lrcorner \psi_0 \in \Lambda^2_{14}$. Performing:

$$\begin{aligned}(\beta_0 \lrcorner \psi_0) \lrcorner \psi_0 &= \frac{1}{2!2!2!} {\beta_0}^{ab} {\psi_{ab}}^{\mu\nu} \psi_{\mu\nu ij} e^{ij} = \frac{1}{8} {\beta_0}^{ab} (4\delta_{ai}\delta_{bj} - 4\delta_{aj}\delta_{bi} + 2\psi_{abij}) e^{ij} \\ &= \frac{1}{8}(4{\beta_0}_{ij} - 4{\beta_0}_{ji} + 2{\beta_0}^{ab}\psi_{abij}) e^{ij} = 2\beta_0 + \beta_0 \lrcorner \psi_0\end{aligned}$$

then $\beta_1 \lrcorner \psi_0 = 2\beta_0 \lrcorner \psi_0 - 2\beta_0 - \beta_0 \lrcorner \psi_0 = -\beta_1$, therefore the eigenvalue is -1 in $\mathfrak{g}_2$, so we have provided the following characterizations:

$$\Lambda^2_7 = \{\beta \in \Lambda^2 : \beta \lrcorner \psi_0 = 2\beta\} \quad \text{and} \quad \Lambda^2_{14} = \{\beta \in \Lambda^2 : \beta \lrcorner \psi_0 = -\beta\}.$$

The formulas from the projections are straightforward to obtain since we have already computed the eigenvalues of the invariant maps for the irreducible components. $\square$

The space of 3-forms has $\binom{7}{3} = 35$ dimensions and contains trivial 1-dimensional representation since the invariant map

$$f \in \Lambda^0 \mapsto f\varphi_0 \in \Lambda^3$$

is an embedding because $\varphi_0 \neq 0$. Furthermore, we have the vectorial representation $\mathbb{R}^7$ inside Λ^3 via the embedding invariant map

$$X \in \Lambda^1 \mapsto X \lrcorner \psi_0 = - * (X \wedge \varphi_0) \in \Lambda^3$$

this gives $\mathbb{R} \oplus \mathbb{R}^7 < \Lambda^3$ and still left 27-dimenions to describe. It happens that G_2 has a 27-dimensional irreducible representation given by $S^2_0(\mathbb{R}^7)$ the space of symmetric trace-free matrices. This representation can be embedded within 3-forms via the map [Bry05, Eq. 2.15]:

$$i_{\varphi_0} : \alpha \otimes \beta \in S^2_0(\mathbb{R}^7) \mapsto \alpha \wedge i_\beta \varphi_0 + \beta \wedge i_\alpha \varphi_0 \in \Lambda^3 \tag{3.2.6}$$

if $\alpha \wedge i_\beta \varphi + \beta \wedge i_\alpha \varphi = 0 \Rightarrow \alpha^a \beta^b \varphi_{bcd} + \beta^a \alpha^b \varphi_{bcd} = 0$ and since φ is non-degenerate we have $\alpha^a \beta^b + \beta^a \alpha^b = 0$, but the map was defined on symmetric matrices, so $\alpha^a \beta^b = \beta^a \alpha^b$ and we have $\alpha^a \beta^b = 0$ for all a, b and this implies $\alpha \otimes \beta = 0$ and the map is injective. So, we have concluded the decomposition of the space of 3-forms into irreducible modules:

$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27} \cong \Lambda^0 \oplus \Lambda^1 \oplus S^2_0(\mathbb{R}^7)$$

As we have done for 2-forms, we will now characterize these spaces. We have already known that $\Lambda^3_1 = \{f\varphi_0 : f \in \mathbb{R}\}$ and $\Lambda^3_7 = \{X \lrcorner \psi_0 : X \in \mathbb{R}^7\}$ by the embeddings described above.

Proposition 3.2.2 (Decomposition of 3-forms)**.** *The space* $\Lambda^3(\mathbb{R}^7)^*$ *of 3-forms decomposes into* G_2*-irreducible representations as*

$$\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27} \cong \Lambda^0 \oplus \Lambda^1 \oplus S^2_0(\mathbb{R}^7) \tag{3.2.7}$$

where

$$\Lambda^3_1 = \{f\varphi_0 : f \in \mathbb{R}\} \tag{3.2.8}$$

$$\Lambda^3_7 = \{X \lrcorner \psi_0 : X \in \mathbb{R}^7\} \tag{3.2.9}$$

$$\Lambda^3_{27} = \{\gamma \in \Lambda^3 : \gamma \wedge \varphi_0 = 0, \gamma \wedge \psi_0 = 0\}. \tag{3.2.10}$$

Proof. We just have to prove the characterization of $\Lambda^3_{27} = (\Lambda^3_1 \oplus \Lambda^3_{27})^\perp$. In fact, we have:

$$\Lambda^3_1 \perp \Lambda^3_{27} \ \Rightarrow \varphi_0 \perp \gamma \Rightarrow \langle \varphi_0, \gamma \rangle = 0 \Rightarrow \gamma \lrcorner \varphi_0 = 0$$
$$\Lambda^3_7 \perp \Lambda^3_{27} \ \Rightarrow \langle X \lrcorner \psi_0, \gamma \rangle = 0, \forall X \Rightarrow \psi_a{}^{bcd} \gamma_{bcd} = 0 \Rightarrow \gamma \lrcorner \psi_0 = 0$$

so $\gamma \in \Lambda^3_{27} \iff \gamma \perp \Lambda^3_1 \oplus \Lambda^3_7 \iff \gamma \lrcorner \varphi_0 = 0, \gamma \lrcorner \psi_0 = 0 \iff \gamma \wedge \varphi_0 = 0, \gamma \wedge \psi_0 = 0$ and we have proved the characterization of Λ^3_{27}. $\square$

3.3 G_2-structures on manifolds, their torsion forms and intrinsic torsion

In general, a G-structure on a manifold M^n (with $G \leq GL(n)$ a closed subgroup) is a reduction of the frame bundle $Fr(M)$ of $GL(n)$ to G, meaning it is a principal G-subbundle of $Fr(M)$ (cf. [KN63, Tu17, Joy00]). In this context, a G_2-structure on a 7-manifold M is a reduction of the frame bundle of $GL(7)$ to G_2. Particularly, since $G_2 \subset SO(7)$, a G_2-structure implies a Riemannian metric and orientation, as stated in Lemma 3.1.5.

Since G_2 is the stabilizer of the 3-form φ_0 in $\Lambda^3(\mathbb{R}^7)^*$ (cf. Equation (3.1.7)), a G_2-structure on M^7 is equivalent to a 3-form φ in $\Omega^3(M^7)$, which can be punctually expressed in the standard G_2-structure (3.1.4) in $\mathbb{R}^7$. We denote $\psi = *\varphi \in \Omega^4(M)$, where $*$ is considered in relation to the induced metric via φ, as described in Lemma 3.1.5.

As pointed out by [Joy00, §10.1], every non-degenerate 3-form on a seven manifold can be put punctually in the form (3.1.4) by dimensional comparison of the space of forms[3] (what, in [Joy00] nomenclature is called *positive* 3-form and the comparison of forms being the same means the bundle of positive 3-forms is an open topological subbundle of $\Lambda^3(T^*M)$).

Not every 7-manifold admits a G_2-structure, i.e., the cross product punctually modelled in φ_0 which varies smoothly in the manifold. We have:

Theorem 3.3.1 ([Gra69])**.** *A seven-dimensional manifold admits a G_2-structure if, and only if, it is orientable and admits spin structure.*

When calculations involving tensorial operations in $\varphi \in \Omega^3(M)$ and $\psi \in \Omega^4(M)$ need to be carried out, they can all be conducted using the standard forms outlined in Lemma 3.1.5 and Proposition 3.1.6, in particular,

$$|\varphi|^2 = |\psi|^2 = 7.$$

The space of differential forms follows the decomposition discussed in Section 3.2: the space of 2-forms decomposes as $\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$, where

$$\Omega^2_7 = \{X \lrcorner \varphi : X \in TM\} = \{\beta \in \Omega^2 : (\beta \lrcorner \varphi) \lrcorner \varphi = 3\beta\} = \{\beta \in \Omega^2 : \beta \lrcorner \psi = 2\beta\}, \tag{3.3.1}$$

$$\Omega^2_{14} = \{\beta \in \Omega^2 : \beta \wedge \psi = 0\} = \{\beta \in \Omega^2 : \beta \lrcorner \varphi = 0\} = \{\beta \in \Omega^2 : \beta \lrcorner \psi = -\beta\}. \tag{3.3.2}$$

[3] This is something special for G_2-structures. For example, in the $Spin(7)$ case, not every arbitrary non-degenerate 4-form can be put in the standard form of a $Spin(7)$-structure.

analogously, the space of 3-forms decomposes as $\Omega^3(M) = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$ where

$$\begin{aligned} \Omega^3_1 &= \{f\varphi : f \in C^\infty(M)\}, \\ \Omega^3_7 &= \{X \lrcorner \psi : X \in \Omega^1\}, \\ \Omega^3_{27} &= \{\gamma \in \Omega^3 : \gamma \wedge \varphi = 0, \gamma \wedge \psi = 0\}. \end{aligned}$$

As the form varies across different manifolds, we must discuss its derivatives, particularly to describe its non-closed and non-co-closed nature. We introduce the notion of torsion forms.

Proposition 3.3.2 (Torsion Forms)**.** *Let (M^7, φ) be a manifold with G_2-structure, then there are unique differential forms $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^3_{27}$ (called the torsion forms) satisfying*

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3 \tag{3.3.3}$$

$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi \tag{3.3.4}$$

Proof. Since $\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$, we have immediately that there are forms τ_0, τ_1, τ_3 as in the statement satisfying

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3$$

the constants are just for convenience. Analogously since $\Omega^5 = \Omega^5_7 \oplus \Omega^5_{14}$, we have that there are forms $\tilde{\tau}_1, \tau_2$ such that

$$d\psi = 4\tilde{\tau}_1 \wedge \psi + \tau_2 \wedge \varphi$$

We need to prove that $\tau_1 = \tilde{\tau}_1$. The justifying for it is [Kar08a, Theorem 2.23]. □

Definition 3.3.3. *A G_2-structure (M, φ) is said to be* integrable *if $\tau_2 = 0$.*

In the present work, integrable G_2-structures play a distinguished role, as they admit compatible connections with totally skew-symmetric torsion, as discussed in the next section. The name '*integrable*' can be 'questioned' because some authors use *integrable* in the sense that the G_2-structure has reduced holonomy (as we will discuss later), but in this sense, being 'integrable' is equivalent to having all torsion forms vanishing.

To finalize, we will discuss the *intrinsic torsion* of a G_2-structure which follows the discussion of intrinsic torsion in Appendix B and in the case of G_2-structures (to be obtained as the stabilizer of some differential form, cf. Proposition B.1.2) it can be defined as the 1-form $\Gamma \in \Omega^1(M, \mathfrak{g}_2^\perp)$ (where $\mathfrak{g}_2^\perp = \Lambda^2_7(\mathbb{R}^7)^*$) via the expression

$$\nabla^g_X \varphi := \Gamma(X) \lrcorner^1 \varphi, \tag{3.3.5}$$

where ∇^g is the Levi-Civita connection and $\lrcorner^1$ the partial contraction as in Appendix A.2. In the case of G_2-structures, we have

$$\Gamma \in \Omega^1(M, \mathfrak{g}_2^\perp) := \Gamma(TM \otimes \mathfrak{g}_2^\perp) = \Gamma(TM \otimes {}^2_7) \cong \Gamma(\mathrm{End}(TM)),$$

so, punctually, Γ lives in the space $\mathbb{R}^7 \otimes \mathbb{R}^7 \cong \mathrm{End}(\mathbb{R}^7)$ and we have

$$\mathrm{End}(\mathbb{R}^7) = \Lambda^2(\mathbb{R}^7)^* \oplus S^2(\mathbb{R}^7)^* = \Lambda^2_7 \oplus \Lambda^2_{14} \oplus \Lambda^1 \oplus \Lambda^3_{27},$$

so we can consider that the intrinsic torsion lives in $\Omega^0 \oplus \Omega^1 \oplus \Omega^2_{14} \oplus \Omega^3_{27}$. The interesting about this 1-form is that it contains every information of the torsion forms, cf. [MCMS94, p. 5].

Proposition 3.3.4. *The intrinsic torsion $\Gamma \in \Omega^1(M, \mathfrak{g}_2^\perp)$ of a G_2-structure (M, φ), up to identification is precisely the sum of torsion forms*

$$\Gamma = \tau_0 + \tau_1 + \tau_2 + \tau_3 \in \Omega^0 \oplus \Omega^1 \oplus \Omega^2_{14} \oplus \Omega^3_{27} \tag{3.3.6}$$

In particular, $\Gamma \equiv 0$ if, and only if $d\varphi = 0$ and $d\psi = 0$.

Proof. To see the correspondence between Γ and the torsion forms $\tau_p \in \Omega^p$, we will use the result in Proposition B.1.2 (for an orthonormal basis $\{e_j\}$ of $\mathbb{R}^7$) where $V = \Lambda^3(\mathbb{R}^7)^*$:

$$\begin{aligned} \nabla^g_X \varphi = \Gamma(X) \lrcorner^1 \varphi &\Rightarrow d\varphi = e^j \wedge \nabla^g_{e_j} \varphi = e^j \wedge (\Gamma(e_j) \lrcorner^1 \varphi) \\ &\Rightarrow \delta\varphi = -e^j \lrcorner \nabla^g_{e_j} \varphi = -e_j \lrcorner (\Gamma(e_j) \lrcorner^1 \varphi) \end{aligned}$$

so, if $\Gamma \equiv 0$, then $d\varphi = 0$ and $d\psi = 0$. Now, if $d\varphi = 0$ and $d\psi = 0$, we have by the irreducibility of each representation to conclude that each component of Γ is zero. $\square$

Remark 3.3.5 (G_2-manifolds). Let (M, φ) be a G_2-structure. If the intrinsic torsion vanishes, i.e., $\Gamma \equiv 0$, we have the principle of holonomy that the Riemannian holonomy of M is in G_2. By the results above, the holonomy of the Levi-Civita connection of M is in G_2 if, and only if $d\varphi = 0$ and $d\psi = 0$ (cf. [Joy00]). $\bigcirc$

Remark 3.3.6. In the classical text by Fernández and Gray [FG82], the G_2-structures are classified into 16 classes, and each class is a presence or not of the so-called $\mathcal{W}$ spaces:

$$\mathcal{W}_1, \qquad \mathcal{W}_2, \qquad \mathcal{W}_3, \qquad \mathcal{W}_4.$$

Cabrera and Swann [MCMS94] have proved that the presence of these $\mathcal{W}$-spaces is actually whether some torsion form is zero or not, in the following correspondence:

$$\tau_0 \in \mathcal{W}_1, \qquad \tau_2 \in \mathcal{W}_2, \qquad \tau_3 \in \mathcal{W}_3, \qquad \tau_1 \in \mathcal{W}_4.$$

Now, using the approach of Friedrich and Ivanov [FI02] and the result above, these $\mathcal{W}$-spaces are the irreducible components of the intrinsic torsion:

$$\Omega^0 \cong \mathcal{W}_1, \qquad \Omega^2_{14} \cong \mathcal{W}_2, \qquad \Omega^3_{27} \cong \mathcal{W}_3, \qquad \Omega^1 \cong \mathcal{W}_4.$$

$\bigcirc$

3.4 The characteristic connection of integrable G_2-structures

This discussion regarding connections with skew-symmetric torsion follows mainly [Agr06], with additional insights provided in Appendix B. Furthermore, the derivation of the formula for the characteristic is influenced by [FI02], albeit reformulated for clarity and applicability to our context.

Definition 3.4.1. *Let (M^7, φ) be a G_2-structure. An affine connection ∇ is said to be compatible with the G_2-structure if*

$$\nabla\varphi = 0.$$

The condition $\nabla\varphi = 0$ implies $\nabla\psi = 0$ immediately since G_2 is a subgroup of $\mathrm{O}(7)$. Therefore, any ∇ compatible with φ is metric compatible and commutes with the Hodge star operator.

Remark 3.4.2. In particular, the compatibility of a connection ∇ with the G_2-structure implies that the endomorphism part of its curvature R_∇ lives in $\Omega^2_{14} = \mathfrak{g}_2 \subset \Omega^2$, that is, for any pair of vector fields X, Y on M,

$$g(R_\nabla(X,Y)\cdot,\cdot) \in \mathfrak{g}_2 = \Omega^2_{14}. \tag{3.4.1}$$

This is because $\Omega^2_{14} = \mathfrak{g}_2 \subset \mathfrak{so}(7) = \Omega^2$ and the compatibility of the connection. In particular, the endomorphism part is skew-symmetric because it is metric compatible since $\mathrm{G}_2 \subset \mathrm{O}(7)$. ◯

As discussed in Appendix B, a metric connection with totally skew-symmetric torsion is entirely defined by its torsion $T \in \Omega^3(M)$. This kind of connection is important in the context of G_2-structures because it is unique and then defines a natural geometry on the manifold. We have the following result using the theory in Appendix B.

Proposition 3.4.3 ([FI02])**.** *A G_2-structure (M, g, φ) admits a compatible connection with skew-symmetric torsion if and only if $\tau_2 = 0$. In this case, the connection is unique and called the* characteristic connection *or* Bismut connection*.*

Proof. The existence of a connection with totally skew-symmetric torsion hinges upon the condition that the intrinsic torsion Γ resides in $\Omega^3(M)$, as stated by the operator Θ in Theorem B.3.1. According to Proposition 3.3.4, this criterion is satisfied if and only if $\tau_2 = 0$. This is because Ω^3 decomposes into $\Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27} \cong \Omega^0 \oplus \Omega^1 \oplus \Omega^3_{27}$, and a connection is uniquely determined when $\tau_2 = 0$ because the intrinsic torsion inhabits a space isomorphic to Ω^3, rendering Θ injective and thereby uniquely defining the torsion. □

Given the uniqueness of the characteristic connection when it exists, to define it completely, we need to find its torsion $T \in \Omega^3(M)$, as expounded in Appendix B. Therefore, our task primarily entails computing this torsion. This method is meticulously outlined in Theorem B.3.6, an original method from the author's dissertation.

Theorem 3.4.4. *Let (M^7,φ) be an integrable G_2-structure and denote $\nabla^+ = \nabla^g + \frac{1}{2}g^{-1}T$ the characteristic connection. Then, its torsion is given by*

$$T = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3. \tag{3.4.2}$$

This quantity is sometimes called flux *and sometimes denoted by $T = H$.*

Proof. The proof of this theorem follows the notation introduced in Appendix B and use the partial contractions $\lrcorner^q$ in Appendix A.2. Using the Flux's theorem B.3.6, we just have to compute the eigenvalues of the operator $\mathbf{H}$ given by

$$\mathbf{H} : \gamma \in \Omega^3 \mapsto \gamma \lrcorner^2 \psi = *(\gamma \lrcorner^1 \varphi) \in \Omega^3.$$

For this, we must compute the specific elements of each irreducible component. For Ω^3_1, which elements are multiple of φ, we have

$$\begin{aligned}\mathbf{H}(\varphi) &= \frac{1}{2!2!} * (\varphi_{ijk}\varphi_{abk}e^{ijab}) = \frac{1}{4} * ((\cancel{\delta_{ia}\delta_{jb}} - \cancel{\delta_{ib}\delta_{ja}} + \psi_{ijab})e^{ijab}) \\ &= \frac{6}{4!} * (\psi_{ijab}e^{ijab}) = 6 * \psi = 6\varphi\end{aligned}$$

consequently, we have the eigenvalue $k_1 = 6$. Now, for the space Ω^3_7, where a general form has the form $X \lrcorner \psi$, then we have

$$\begin{aligned}\mathbf{H}(X \lrcorner \psi) &= \frac{1}{2!2!} * (X^a\psi_{akbc}\varphi_{kij}e^{bcij}) = \frac{1}{4} * (X^a\psi_{abck}\varphi_{ijk}e^{bcij}) \\ &= \frac{1}{4} * (X^a(-\delta_{ia}\varphi_{jbc} - \cancel{\delta_{ib}\varphi_{ajc}} - \cancel{\delta_{ic}\varphi_{abj}} + \delta_{aj}\varphi_{ibc} + \cancel{\delta_{bj}\varphi_{aic}} + \cancel{\delta_{cj}\varphi_{abi}})e^{bcij}) \\ &= \frac{1}{4} * (-X_i\varphi_{jbc}e^{bcij} + X_j\varphi_{ibc}e^{bcij}) = \frac{1}{4} * (-X_i\varphi_{jbc}e^{ijbc} - X_j\varphi_{ibc}e^{jibc}) \\ &= -3 \cdot \frac{1}{1!3!} * (X_ie^i \wedge \varphi_{jbc}e^{jbc}) = -3 * (X \wedge \varphi) = 3X \lrcorner \psi\end{aligned}$$

and we conclude that the eigenvalue $k_7 = 3$.

For the last component, Ω^3_{27}, we have no general form for their elements, so we must find some specific element in this space and apply $\mathbf{H}$. This eigenvalue is $k_{27} = -1$ as computed in [dlOLS18a, Lemma 3, Appendix A].

We can compute directly this eigenvalue, just considering an explicit element in Ω^3_{27}: $\gamma = e^{127} - e^{135}$ (which we can immediately check that $\gamma \wedge \varphi = \gamma \wedge \psi = 0$ using the expression in Lemma 3.1.5).

Performing the necessary calculations, we have (using the notation $\varphi_k = e_k \lrcorner \varphi$):

$$
\begin{array}{lll}
e_1 \lrcorner \varphi = e^{27} + e^{35} - e^{46} & e_1 \lrcorner \gamma = e^{27} - e^{35} & \gamma_1 \wedge \varphi_1 = -e^{2467} - e^{3456} \\
e_2 \lrcorner \varphi = -e^{17} - e^{36} - e^{45} & e_2 \lrcorner \gamma = -e^{17} & \gamma_2 \wedge \varphi_2 = e^{1367} + e^{1457} \\
e_3 \lrcorner \varphi = e^{47} - e^{15} + e^{26} & e_3 \lrcorner \gamma = e^{15} & \gamma_3 \wedge \varphi_3 = -e^{1457} - e^{1256} \\
e_4 \lrcorner \varphi = -e^{37} + e^{16} + e^{25} & e_4 \lrcorner \gamma = 0 & \gamma_4 \wedge \varphi_4 = 0 \\
e_5 \lrcorner \varphi = e^{67} + e^{13} - e^{24} & e_5 \lrcorner \gamma = -e^{13} & \gamma_5 \wedge \varphi_5 = -e^{1367} - e^{1234} \\
e_6 \lrcorner \varphi = -e^{57} - e^{14} - e^{23} & e_6 \lrcorner \gamma = 0 & \gamma_6 \wedge \varphi_6 = 0 \\
e_7 \lrcorner \varphi = e^{34} + e^{56} - e^{24} & e_7 \lrcorner \gamma = e^{12} & \gamma_7 \wedge \varphi_7 = e^{1234} + e^{1256}
\end{array}
$$

consequently, we have

$$\mathbf{H}(\gamma) = * \sum_{j=1}^{7} (e_j \lrcorner \gamma) \wedge (e_j \lrcorner \varphi) = - * \left(e^{2467} + e^{3456}\right) = e^{135} - e^{127} = -\gamma.$$

We have concluded directly $k_{27} = -1$. Consequently, the expression for the torsion T has been obtained

$$H = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3 \tag{3.4.3}$$

which gives us the expression for the torsion of the characteristic connection of a G_2-structure. □

3.5 The spinorial description of G_2-structures

As we have defined G_2 in (3.1.7), it is a subgroup of $\mathrm{SO}(7)$, functioning as the stabilizer of a particular 3-form φ_0 as in (3.1.4). An intriguing observation is that G_2 is isomorphic to a subgroup of $\mathrm{Spin}(7)$, serving as the stabilizer of a specific spinor $\eta_0 \in \Delta_7$. Thus, what we're poised to explore is the equivalence between the G_2-structure represented by a 3-form $\varphi \in \Omega^3(M)$ and its counterpart as a spinor $\eta \in \Gamma(S)$. To get deeper into this equivalence, first, we will point out the basic concepts for Clifford algebras and spinors in dimension seven. For this, we will reference [LM90] for general aspects of spin geometry and [FKMS97, ACFH15] for spinorial description of G_2-structures.

Let $\mathrm{Cl}(\mathbb{R}^7)$ denote the Clifford algebra of $(\mathbb{R}^7, g_0)$, given by the tensorial algebra of $\mathbb{R}^7$ modulo the relation

$$X \cdot X = -g_0(X, X),$$

for $X \in \mathbb{R}^7$. The algebra $\mathrm{Cl}(\mathbb{R}^7)$ is isomorphic to $\mathrm{End}(\mathbb{R}^8) \oplus \mathrm{End}(\mathbb{R}^8)$, and it admits a real representation $\Delta_7 \cong \mathbb{R}^8$ with generators (see [FKMS97, p. 261]):

$$
\begin{array}{rclrcl}
e_1 & = & E_{18} + E_{27} - E_{36} - E_{45}, & e_2 & = & -E_{17} + E_{28} + E_{35} - E_{46}, \\
e_3 & = & -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 & = & -E_{15} - E_{26} - E_{37} - E_{48}, \\
e_5 & = & -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 & = & E_{14} - E_{23} - E_{58} + E_{67}, \\
e_7 & = & E_{12} - E_{34} - E_{56} + E_{78}, & & &
\end{array} \tag{3.5.1}
$$

where E_{ij} is the standard basis of the Lie algebra $\mathfrak{so}(8)$: it is -1 in the position i, j and skew-symmetric. Upon restriction of this representation to $\mathrm{Spin}(7) \subset \mathrm{Cl}(\mathbb{R}^7)$ we obtain the (irreducible) real spin representation

$$\kappa : \mathrm{Spin}(7) \to \mathrm{SO}(\Delta_7).$$

The group Spin(7) acts transitively on the sphere and G_2 can be identified with the subgroup of Spin(7) preserving a spinor (see e.g. [FKMS97]).

Proposition 3.5.1. *The Lie group* G_2 *is canonically isomorphic to the subgroup of* Spin(7) *preserving the spinor* $\eta_0 := (1, 0, \cdots, 0) \in \Delta_7$*:*

$$G_2 \cong \Big\{ g \in \mathrm{Spin}(7) : g \cdot \eta_0 = \eta_0 \Big\}. \tag{3.5.2}$$

The space of spinors Δ_7 decomposes into irreducible G_2-representations components as $\Delta_7 \cong \Lambda^0 \oplus \Lambda^1$. This decomposition is derived through the following steps:

$$\Delta_7 \cong \mathbb{R}^8 \cong \mathbb{R} \oplus \mathbb{R}^7 \cong \langle \eta_0 \rangle \oplus \Lambda^1 \cong \Lambda^0 \oplus \Lambda^1,$$

corresponding to the real and purely imaginary octonions. The identification of Λ^1 inside Δ_7 is simply the embedding $\alpha \in \Lambda^1 \mapsto \alpha^\# \cdot \eta_0 \in \Delta_7$, cf. [FKMS97, p. 262]. The relation between the descriptions of G_2, as the stabiliser of a 3-form φ_0 in SO(7), and, as the stabiliser of a spinor η_0 in Spin(7), are related by [ACFH15, p. 545]:

$$\varphi_0(X, Y, Z) := -\langle X \cdot Y \cdot Z \cdot \eta_0, \eta_0 \rangle, \tag{3.5.3}$$

where in the right-hand side, the action is the Clifford multiplication. On the other hand, given the 3-form φ_0, this implies the decomposition $\Delta_7 = \Lambda^0 \oplus \Lambda^1$, so just define $\eta_0 \in \Lambda^0$ with unit-lenght. This shows the equivalence between G_2-structures via 3-forms and spinors.

Following the discussion, the spinor η_0 induces some invariant maps, which we will use later in Chapter 4. Firstly, 2-forms act in the spinor η_0 via Clifford multiplication, so we can consider the natural G_2-equivariant map:

$$\begin{array}{rccc} \mu : & \Lambda^2 & \to & \Delta_7 \\ & \beta & \mapsto & \beta \cdot \eta_0. \end{array}$$

Using the isomorphisms $\Lambda^2 \cong \Lambda^1 \oplus \mathfrak{g}_2$ and $\Delta_7 \cong \langle \eta_0 \rangle \oplus \Lambda^1$, one can easily see that $\mu|_{\mathfrak{g}_2} \equiv 0$ by invariance. Furthermore, as demonstrated in [FKMS97, p. 262], $\mu|_{\Lambda^1}$ is an embedding. This leads to a characterization of the Lie algebra $\mathfrak{g}_2$:

$$\mathfrak{g}_2 = \{ \beta \in \Lambda^2 : \beta \cdot \eta_0 = 0 \} \subset \Lambda^2. \tag{3.5.4}$$

so, the equivalence below will be useful for us when we consider a different notion of instanton in Chapter 6, the two notions of G_2-instantons will be equivalent due to the characterization above. For reference:

$$\beta \cdot \eta_0 = 0 \iff \beta \wedge \psi_0 = 0 \qquad (\iff \beta \lrcorner \psi_0 = -\beta). \tag{3.5.5}$$

Analogously, consider the action of 3-forms on the spinor η_0 via Clifford multiplication

$$\begin{array}{rccc} \nu : & \Lambda^3 & \to & \Delta_7 \\ & \gamma & \mapsto & \gamma \cdot \eta_0. \end{array}.$$

Using $\Lambda^3 \cong \langle \varphi_0 \rangle \oplus \Lambda^1 \oplus \Lambda^3_{27}$, the different pieces in this decomposition act on η_0 via the following formulae (see [FI02])

$$\varphi_0 \cdot \eta_0 = -7\eta_0, \qquad (X \lrcorner \psi_0) \cdot \eta_0 = 4X \cdot \eta_0, \qquad \gamma \cdot \eta_0 = 0, \tag{3.5.6}$$

for all $X \in \mathbb{R}^7$ and $\gamma \in \Lambda^3_{27}$. Consider now the induced 'Dirac-type' map $\not{\psi} : \Lambda^3 \to \Delta_7$, defined by

$$\begin{array}{rccc} \not{\psi} & : \Lambda^3 & \to & \Delta_7 \\ & \gamma & \mapsto & \not{\gamma} \cdot \eta_0 \end{array} \qquad \text{where } \not{\gamma} \cdot \eta_0 := \sum_j e^j \cdot (e_j \lrcorner \gamma) \cdot \eta_0 \tag{3.5.7}$$

Using the decomposition $\Delta_7 \cong \langle \eta_0 \rangle \oplus \Lambda^1$, we can describe how the irreducible components of Λ^3 behave under the G_2-invariant map $\not{\psi}$:

Lemma 3.5.2. *Under the decomposition $\Lambda^3 \cong \langle \varphi_0 \rangle \oplus \Lambda^1 \oplus \Lambda^3_{27}$, the irreducible components of Λ^3 under the map $\not{\psi} : \Lambda^3 \to \Delta_7$ in (3.5.7) acts as*

$$\not{\varphi}_0 \cdot \eta_0 = -\frac{21}{2}\eta_0, \qquad \not{i_X\psi_0} \cdot \eta_0 = 6X \cdot \eta_0, \qquad \not{\gamma} \cdot \eta_0 = 0, \tag{3.5.8}$$

for all $X \in \mathbb{R}^7$ and $\gamma \in \Lambda^3_{27}$.

Proof. The last equation $\not{\gamma} \cdot \eta_0 = 0$ in (3.5.8) holds immediately by invariance of the map $\not{\psi}$ because the representation $S^2_0(\mathbb{R}^7) \cong \Lambda^3_{27}$ is not a component in Δ_7. Now, in general for $\xi \in \Lambda^3$, we can write $\xi = \frac{1}{3!}\xi_{ijk}e^{ijk}$ an arbitrary 3-form, then we have $e_\mu \lrcorner \xi = \frac{1}{2!}\xi_{\mu jk}e^{jk}$ and consequently

$$\not{\xi} \cdot \eta_0 = e^\mu \cdot (e_\mu \lrcorner \xi) \cdot \eta_0 = \frac{1}{2!}\xi_{\mu jk}e^\mu \cdot (e^j \wedge e^k) \cdot \eta_0.$$

Using the canonical embedding $\mathfrak{spin}(7) \cong \mathfrak{so}(7) = \Lambda^2 \to \mathrm{Cl}(\mathbb{R}^7)$ given by (cf. [LM90, Prop. 6.2]) $e^j \wedge e^k \in \Lambda^2 \mapsto \frac{1}{2}e^j \cdot e^k \in \mathrm{Cl}(\mathbb{R}^7)$, in accordance with the convention $v \cdot v = -|v|^2$, we then have

$$\not{\xi} \cdot \eta_0 = \frac{1}{4}\xi_{\mu jk}\, e^\mu \cdot e^j \cdot e^k \cdot \eta_0.$$

To perform the calculations, we will use the explicit representation in (3.5.1) and the canonical spinor $\eta_0 = (1, 0, \cdots, 0) \in \Delta_7 \cong \mathbb{R}^8$. The first equation is held by a direct computation.

Finally, for the second equation in (3.5.8), it is enough to prove for the specific element $X = e_1 \in \mathbb{R}^7$, hence by invariance, it will be true for the whole irreducible representation. Firstly, performing the action of e^1 on the spinor, we have:

$$e^1 \cdot \eta_0 = (E_{18} + E_{27} - E_{36} - E_{45}) \cdot \eta_0 = E_{18} \cdot \eta_0 = (0, \cdots, 0, 1).$$

Now, we have in this case using (3.1.5) that

$$i_X \psi_0 = e^{256} + e^{234} + e^{457} + e^{367},$$

and, on the other hand, an extensive calculation gives us

$$\cancel{e_1 \lrcorner \psi_0} \cdot \eta_0 = (0, \cdots, 0, 6),$$

and the result follows. $\square$

We have discussed the spinor $\eta_0 \in \Delta_7$ induced the 3-form φ_0 and vice versa. In manifolds, a G_2-structure is equivalent to a unitary spinor field $\eta \in \Gamma(S)$, where $S = P_M \times_{\mathrm{G}_2} \Delta_7$ is the spinor bundle for $P_M \to M$ the principal Spin(7)-bundle (since a manifold with G_2-structure is spin [Gra69]) and $\Delta_7 \cong \mathbb{R}^8$ the irreducible real spinor representation. The equivalence of $\varphi \in \Omega^3(M)$ non-degenerate and $\eta \in \Gamma(S)$ unitary is in the formula as before [FKMS97, ACFH15]:

$$\varphi(X, Y, Z) := -\langle X \cdot Y \cdot Z \cdot \eta, \eta \rangle. \tag{3.5.9}$$

Conversely, any non-degenerate 3-form $\varphi \in \Omega^3(M)$ determines a non-vanishing spinor via the identification

$$S \cong \langle \varphi \rangle \oplus \Omega^3_7 \tag{3.5.10}$$

provided by (3.5.6).

4 Killing spinors and the heterotic G_2-system

This chapter explores the practical implications of the theory of Killing spinor equations and coupled instanton equations, as introduced in Chapter 2, within the framework of G_2-structures previously discussed in Chapter 3 in seven dimensions. Delving deeper, we examine specific characteristics of these structures and provide several illustrative examples.

In Section 4.1, we present the *heterotic* G_2*-system* (cf. Definition 4.1.1), a system of differential equations initially described in the physics literature and which we will explore later in Chapter 5, mainly focusing on the approximate solutions as constructed in [LSE23]. We proceed to present the main theorem of this section, Proposition 4.1.2, establishing the equivalence between the heterotic G_2-system and the Killing spinor equations discussed in Chapter 2.

Next, Section 4.2 delves into the implications of solutions to the heterotic G_2-system on curvature tensors. Our primary focus lies in establishing generalized Ricci flatness for solutions within generalized geometry and transitive Courant algebroids (when divergence is induced by τ_1) cf. Theorem 4.2.1. This theorem is a particular solution for the Problem 2 (open) elucidated in Chapter 2. With an alternative approach, we further explore this problem and its implications in Chapter 6. Additionally, this section explores some consequences of generalized scalar curvature due to the heterotic G_2-system, as outlined in Theorem 4.2.3.

In the following, in Section 4.3, we extend the applications introduced in Chapter 2 to the realm of G_2-structures by introducing the *coupled* G_2*-instanton equations*. We revisit the characterization of these equations within the context of G_2, elucidating the significance of the Ricci-Bismut form (cf. Definition 4.3.1) in measuring the non-integrability of the G_2-structure (cf. Lemma 4.3.3). Concluding the section, we establish a pivotal result: Theorem 4.3.6 demonstrates how the gravitino equation implies the existence of coupled G_2-instantons, a specific solution to Problem 1. This result is revisited in Chapter 6 with an alternative notion of instanton.

In Section 4.4, we revisit the generalized Ricci flatness resulting from the gravitino equation, cf. Theorem 4.4.3 (offering a partial solution to Problem 2). Our alternative approach in this section draws inspiration from [IS23b], serving as a prototype for the forthcoming discussions in Chapter 6, where we explore a more generalized scenario. Concluding the section, we examine the breakdown of generalized Ricci-flatness upon relaxing the instanton condition on θ, cf. Lemma 4.4.4. This technical insight will play a pivotal role in the proof of the main results outlined in Section 5.4, mainly when the instanton condition is 'approximate'. Finally, Section 4.5 presents several examples of the theory developed in this chapter.

4.1 The heterotic G_2-system

In this section, we establish a relation between the Killing spinor equations (2.1.2) in seven dimensions and the heterotic G_2-system. The Killing spinor equations are (cf. Definition 2.1.2):

$$\nabla^+\eta = 0, \qquad F_\theta \cdot \eta = 0, \qquad \left(\not{\nabla}^{1/3} - \frac{1}{2}\zeta\right) \cdot \eta = \lambda\eta$$

for the data $(g, H, \theta, \eta, \zeta)$.

We define the heterotic G_2-system on a fixed oriented and spin manifold M^7 endowed with a principal K-bundle P. We assume that $\mathfrak{k} = \mathrm{Lie}(K)$ is endowed with a non-degenerate bi-invariant symmetric bilinear form

$$\langle \cdot, \cdot \rangle : \mathfrak{k} \otimes \mathfrak{k} \longrightarrow \mathbb{R}.$$

Definition 4.1.1 (Heterotic G_2-system, [dlOLS18a]). *Let M^7 be an oriented spin manifold endowed with a principal K-bundle P. A pair (φ, θ), where φ is a G_2-structure on M and θ is a principal connection on P, satisfies the* heterotic G_2-system *if*

$$F_\theta \wedge \psi = 0, \qquad \tau_2 = 0, \qquad dH_\varphi = \langle F_\theta \wedge F_\theta \rangle. \tag{4.1.1}$$

where $H_\varphi \in \Omega^3(M)$ is the flux of the G_2-structure

$$H_\varphi := \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3.$$

Note that the condition $\tau_2 = 0$ (the structure being integrable) implies that the flux H_φ is the torsion of the characteristic connection, i.e., $\nabla^+ = \nabla^g + \frac{1}{2}g^{-1}H_\varphi$. And on the other hand, if $\nabla\varphi = 0$ for some connection ∇ with skew-symmetric torsion, then $T_\nabla = H_\varphi$ (cf. Proposition 3.4.3 and Theorem 3.4.4).

We now show that the heterotic G_2 system (4.1.1) is equivalent to the Killing spinor equations with parameter λ on M^7, cf. (2.1.5) and (2.1.6).

Proposition 4.1.2. *Let E be a transitive Courant algebroid over an oriented spin manifold M^7. Let $(\mathbf{G}, \mathrm{div}, \eta)$ be a solution of the Killing spinor equations with parameter $\lambda \in \mathbb{R}$ on E, cf. Definition (2.1.2). Assume that the spinor η is nowhere-vanishing and consider the tuple (g, H, θ, ζ) determined by $(\mathbf{G}, \mathrm{div})$ and the G_2-structure φ defined by η via (3.5.9). Then (φ, θ) satisfies the heterotic G_2 system (4.1.1) and*

$$g = g_\varphi, \qquad H = H_\varphi := \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3, \qquad \zeta = 4\tau_1, \qquad \tau_0 = \frac{12}{7}\lambda. \tag{4.1.2}$$

Conversely, any solution (φ, θ) *of the heterotic* G_2*-system* (4.1.1) *with constant* τ_0 *determines a transitive Courant algebroid as in Definition 1.1.5, endowed with a solution* $(\mathbf{G}, \mathrm{div}, \eta)$ *of the Killing spinor equations with parameter* λ*, as in* (2.1.6)*, and a nowhere-vanishing spinor. The tuple* (g, H, θ, ζ) *determined by* $(\mathbf{G}, \mathrm{div})$ *satisfies* (4.1.2) *and the spinor* η *is given by* (3.5.10)*, where we identify* S*, the spinor bundle for* V_+*, with a spinor bundle for* (T, g) *via*

$$\begin{array}{cccc} \sigma_+ \colon & (TM, g) & \longrightarrow & V_+ \\ & X & \longmapsto & X + gX \end{array}.$$

Proof. We need to prove the equivalence between solutions of the heterotic G_2 system (4.1.1) and solutions $(g, H, \theta, \zeta, \eta)$ of the coupled system defined by (2.1.5), (2.1.6), and the heterotic Bianchi identity (1.1.8). Given such a tuple (g, H, θ, ζ), consider the G_2-structure φ defined by the real spinor η via (3.5.9). Note that, as $\mathrm{G}_2 \leq \mathrm{SO}(7)$, we have $g = g_\varphi$. Then, $\nabla^+\eta = 0$ implies that $\nabla^+\varphi = 0$ and hence, applying Proposition 3.4.3 and Theorem 3.4.4,

$$\tau_2 = 0, \qquad H = H_\varphi := \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3.$$

Furthermore,(3.5.5), shows that we have the equivalence of being an instanton

$$F_\theta \wedge \psi = 0 \iff F_\theta \cdot \eta = 0.$$

Using $\nabla^{+1/3} = \nabla^g + \frac{1}{6}H = \nabla^+ - \frac{1}{3}H$, we have for (2.1.6)

$$\lambda\eta = \left(\not{\nabla}^{+1/3} - \frac{1}{2}\zeta\right) \cdot \eta = \not{\nabla}^+\eta - \frac{1}{3}\not{H} \cdot \eta - \frac{1}{2}\zeta \cdot \eta = -\frac{1}{3}\not{H} \cdot \eta - \frac{1}{2}\zeta \cdot \eta.$$

On the other hand, applying (3.5.8) in Lemma 3.5.2, it follows that

$$\begin{aligned} \not{H} \cdot \eta &= \left(\frac{1}{6}\tau_0\not{\varphi} - \cancel{\tau_1 \lrcorner \psi} - \not{\tau}_3\right) \cdot \eta = -\frac{21}{12}\tau_0 \cdot \eta - 6\tau_1 \cdot \eta \Rightarrow \\ \lambda\eta &= -\frac{1}{3}\not{H} \cdot \eta - \frac{1}{2}\zeta \cdot \eta = \frac{7}{12}\tau_0 \cdot \eta + 2\tau_1 \cdot \eta - \frac{1}{2}\zeta \cdot \eta \end{aligned}$$

and, consequently

$$\left(2\tau_1 - \frac{1}{2}\zeta\right) \cdot \eta + \left(\frac{7}{12}\tau_0 - \lambda\right) \cdot \eta = 0.$$

Using now that $(2\tau_1 - \frac{1}{2}\zeta) \cdot \eta \in \langle\eta\rangle^\perp \cong \Omega^1 \subset S$, cf. Section 3.5, it follows that:

$$\zeta = 4\tau_1, \qquad \lambda = \frac{7}{12}\tau_0.$$

Conversely, given a solution (φ, θ) of the heterotic G_2 system (4.1.1), consider the real nowhere-vanishing spinor η defined by (3.5.10). Then, by the third equation in (4.1.1), $\tau_2 = 0$, hence

$$\nabla^+\varphi = 0 \Rightarrow \nabla^+\eta = 0,$$

where $\nabla^+ = \nabla^g + \frac{1}{2}g^{-1}H$ and $H = H_\varphi$, see (4.1.2). As before, $F_\theta \wedge \psi = 0$ implies $F_\theta \cdot \eta = 0$, while the second equation in (4.1.1) implies $\lambda = \frac{7}{12}\tau_0$, by Theorem 3.4.4. Finally, setting $\zeta = 4\tau_1$, we have

$$\left(\not{\nabla}^{+1/3} - \frac{1}{2}\zeta\right) \cdot \eta = \left(\not{\nabla}^{+} - \frac{1}{3}\not{H} - \frac{1}{2}\zeta\right) \cdot \eta = \frac{7}{12}\tau_0 \cdot \eta + 2\tau_1 \cdot \eta - \frac{1}{2}\zeta \cdot \eta$$
$$= \frac{7}{12}\tau_0 \cdot \eta = \lambda \cdot \eta.$$

and we have obtained the Killing spinor equations (2.1.5) and (2.1.6). □

Remark 4.1.3. Given a solution (φ, θ) of the heterotic G_2 system, the associated transitive Courant algebroid is $E_{P,H_\varphi,\theta}$ as in Definition 1.1.5, and the corresponding generalized metric is

$$\mathbf{G}_\varphi = \begin{pmatrix} 0 & 0 & g_\varphi^{-1} \\ 0 & -\mathrm{Id} & 0 \\ g_\varphi & 0 & 0 \end{pmatrix},$$

with eigenbundles

$$V_+ = \{X + g_\varphi X : X \in T\}, \quad V_- = \{X + r - g_\varphi X : X \in T, r \in \mathrm{ad}P\}.$$

The divergence operator associated with a solution, given by

$$\mathrm{div} = \mathrm{div}^{\mathbf{G}_\varphi} - 2\langle e, \cdot\rangle,$$

is uniquely determined, provided that we impose the natural condition $e \in T^*M$. In this case, $e = 4\tau_1 \in T^*M$. ○

4.2 Curvature constraints on the heterotic G_2-system

We will derive various curvature constraints for solutions of the heterotic G_2 system. Our results follow from the characterisation of the system using generalized geometry in Proposition 4.1.2, combined with Proposition 2.1.6 and Proposition 2.4.2.

We will keep the notation from the previous section. In particular, we fix an oriented spin manifold M^7 endowed with a principal K-bundle P. Our first result interprets the heterotic G_2-system as a special class of generalized Ricci-flat metrics.

Theorem 4.2.1. *Given a solution (φ, θ) of the heterotic G_2-system (4.1.1) on (M, P), the associated Riemannian metric $g = g_\varphi$ satisfies:*

$$\begin{aligned} \mathrm{Ric}^g - \frac{1}{4}H^2 + F_\theta \circ F_\theta + \frac{1}{2}\mathcal{L}_{\zeta^\#} g &= 0, \\ d^*H - d\zeta + i_{\zeta^\#} H &= 0, \\ d^{\theta *} F_\theta - F_\theta \lrcorner H + i_{\zeta^\#} F_\theta &= 0, \end{aligned} \tag{4.2.1}$$

where $H = H_\varphi \in \Omega^3(M)$ *is the flux and* $\zeta = 4\tau_1$*. In particular, the solution* $(\mathbf{G}_\varphi, \mathrm{div}^\varphi, \eta_\varphi)$ *induced by* (φ, θ)*, cf. Proposition 4.1.2 is generalized Ricci flat:*

$$\mathrm{Ric}^+_{\mathbf{G}^\varphi, \mathrm{div}^\varphi} = 0.$$

Proof. By Proposition 4.1.2, (φ, θ) determines a solution $(\mathbf{G}_\varphi, \mathrm{div}^\varphi, \eta_\varphi)$ of the Killing spinor equations with parameter λ on the transitive Courant algebroid $E_{P,H,\theta}$. More explicitly, the generalized metric $\mathbf{G}_\varphi$ is as in Remark 4.1.3 and

$$\mathrm{div}^\varphi = \mathrm{div}^{\mathbf{G}_\varphi} - 2\langle 4\tau_1, \cdot\rangle.$$

Applying now Proposition 2.1.6, we have $\mathrm{Ric}^+_{\mathbf{G}_\varphi, \mathrm{div}^\varphi} = 0$, and the result follows from (2.1.8). □

Remark 4.2.2. By the proof of the previous result, a solution (φ, θ) of the heterotic G_2 system determines a generalized Ricci-flat metric. Alternatively, we can think of (φ, θ) as solving the equations of motion of heterotic supergravity for the metric, the three-form flux, and the gauge field, in the mathematical physics literature, see Remark 2.1.7. In our following result, the analogue of the equation of motion for the *dilaton field* is satisfied up to an overall constant on the manifold, explicitly given in terms of the parameter λ in (4.1.1). In other words, solutions of the heterotic G_2 system have constant generalized scalar curvature, proportional to the square of the torsion component τ_0. ◯

Theorem 4.2.3. *Given a solution* (φ, θ) *of heterotic* G_2*-system* (4.1.1) *on* (M, P)*, one has*

$$\mathcal{S}^+ = R_g - \frac{1}{2}|H|^2 + |F_\theta|^2 - 8d^*\tau_1 - 16|\tau_1|^2 = \frac{49}{36}\tau_0^2, \tag{4.2.2}$$

where $H = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3$ *is the flux 3-form.*

Proof. As in the proof of Theorem 4.2.1, (φ, θ) determines a solution $(\mathbf{G}_\varphi, \mathrm{div}^\varphi, \eta_\varphi)$ of the Killing spinor equations with parameter $\lambda = \frac{7}{12}\tau_0$ on the transitive Courant algebroid $E_{P,H,\theta}$. Applying now Proposition 2.4.2, we have

$$(\mathcal{S}^+ - 8d\tau_1)\eta = 4\left(\left(\not{D}^+\right)^2 - \Delta^S_- - D^S_{\tilde{e}_-}\right)\eta = 4\lambda^2\eta = \frac{49}{36}\tau_0^2\eta$$

where

$$\mathcal{S}^+ = R_g - \frac{1}{2}|H|^2 + |F_\theta|^2 - 8d^*\tau_1 - 16|\tau_1|^2. \tag{4.2.3}$$

First note that $d\psi = 4\tau_1 \wedge \psi$, consequently

$$0 = d^2\psi = d\tau_1 \wedge \psi - \tau_1 \wedge d\psi = d\tau_1 \wedge \psi - \tau_1 \wedge \tau_1 \wedge \psi = d\tau_1 \wedge \psi$$

consequently $d\tau_1 \in \Omega^2_7 \cong \Omega^1$ and $d\tau_1 \cdot \eta \in \Omega^1$ via (3.5.5) in the orthogonal decomposition $S = \langle\eta\rangle \oplus \Omega^1$. This implies $\mathcal{S}^+ = \frac{49}{36}\tau_0^2$ by the equation above. □

We conclude this section with an alternative form of the scalar equation in (4.2.2).

Corollary 4.2.4. *Given a solution* (φ, θ) *of heterotic* G_2*-system* (4.1.1) *on* (M, P)*, one has*

$$\frac{7}{6}\tau_0^2 + 12|\tau_1|^2 + 4d^*\tau_1 - |\tau_3|^2 + |F_\theta|^2 = 0. \tag{4.2.4}$$

Proof. Applying the result in Theorem 4.2.3 and the fact that

$$\begin{aligned}
|H|^2 = H \lrcorner H &= \frac{1}{6}\tau_0\varphi \lrcorner \frac{1}{6}\tau_0\varphi + (\tau_1 \lrcorner \psi) \lrcorner (\tau_1 \lrcorner \psi) + \tau_3 \lrcorner \tau_3 \\
&= \frac{7}{36}\tau_0^2 + 4|\tau_1|^2 + |\tau_3|^2,
\end{aligned}$$

where we have used $|\varphi|^2 = 7$ and $(\tau_1 \lrcorner \psi) \lrcorner \psi = -4\tau_1$ (cf. (4.4.5) or Lemma 6.3.3). On the other hand, we have (cf. [Bry05, Equation (4.2)])

$$R_g = \frac{21}{8}\tau_0^2 + 30|\tau_1|^2 - \frac{1}{2}|\tau_3|^2 + 12d^*\tau_1.$$

which gives us

$$\begin{aligned}
\frac{49}{36}\tau_0^2 &= R_g - \frac{1}{2}|H|^2 - 8d^*\tau_1 - 16|\tau_1|^2 + |F_\theta|^2 \\
&= \frac{21}{8}\tau_0^2 + 30|\tau_1|^2 - \frac{1}{2}|\tau_3|^2 + 12d^*\tau_1 - \frac{1}{2}|H|^2 - 8d^*\tau_1 - 16|\tau_1|^2 + |F_\theta|^2 \\
&= \frac{21}{8}\tau_0^2 + 14|\tau_1|^2 - \frac{1}{2}|\tau_3|^2 + 4d^*\tau_1 - \frac{1}{2}|H|^2 + |F_\theta|^2 \\
&= \frac{21}{8}\tau_0^2 + 14|\tau_1|^2 - \frac{1}{2}|\tau_3|^2 + 4d^*\tau_1 - \frac{1}{2}\left(\frac{7}{36}\tau_0^2 + 4|\tau_1|^2 + |\tau_3|^2\right) + |F_\theta|^2 \\
&= \left(\frac{21}{8} - \frac{7}{72}\right)\tau_0^2 + 12|\tau_1|^2 + 4d^*\tau_1 - |\tau_3|^2 + |F_\theta|^2 \\
&= \frac{91}{36}\tau_0^2 + 12|\tau_1|^2 + 4d^*\tau_1 - |\tau_3|^2 + |F_\theta|^2
\end{aligned}$$

and the result follows. □

4.3 Coupled instantons and the gravitino equation

We introduce the coupled G_2-instanton equations, a particular instance of the system (2.2.3) in seven dimensions. We will also establish the relation to the gravitino equation (2.1.5) by application of Theorem 2.3.2 in the present setup.

We fix an oriented spin manifold M^7 as in the previous section. Given a G_2-structure φ on M and a three-form $H \in \Omega^3$, we introduce the following quantity, which plays a similar role to the Bismut–Ricci form in the theory of coupled $\mathrm{SU}(n)$-instantons, see [GFGM23, GFJS23]. Recall that the vector cross product $\times\colon TM \otimes TM \to TM$ associated to φ defined by (cf. Section 3.1)

$$g_\varphi(X \times Y, Z) = \varphi(X, Y, Z),$$

for any $X, Y \in TM$.

Definition 4.3.1 (Bismut–Ricci form)**.** *The* Bismut–Ricci form *associated to a pair* (φ, H)*, where* φ *is a* G_2*-structure and* $H \in \Omega^3(M)$*, is the vector-valued* 2*-form*

$$\rho = \rho(\varphi, H) \in \Omega^2(M, T)$$

defined by

$$\rho(X, Y) := \frac{1}{2} \sum_j (R_{\nabla^+}(X, Y) e_j) \times e_j \tag{4.3.1}$$

in terms of the vector cross product, where $\{e_j\}$ *is a local orthonormal frame on* T*, and* ∇^+ *is the metric connection with skew-symmetric torsion*

$$\nabla^+ = \nabla^g + \frac{1}{2} g^{-1} H.$$

Remark 4.3.2. An interesting particular case of the previous definition follows when we take

$$H = H_\varphi := \frac{1}{6} \tau_0 \varphi - \tau_1 \lrcorner \psi - \tau_3.$$

In this case, we say that $\rho_\varphi = \rho(\varphi, H_\varphi)$ is the *Bismut–Ricci form* of the G_2-structure. ○

As we observe in the following result, the Bismut–Ricci form is an obstruction to the integrability of the G_2-structure.

Lemma 4.3.3. *Assume that* (φ, H) *satisfies* $\nabla^+ \varphi = 0$*. Then,* (φ, H) *has vanishing Bismut–Ricci form:*

$$\rho(\varphi, H) = 0.$$

Proof. Assuming $\nabla^+ \varphi = 0$, the endomorphism part of the curvature tensor R_{∇^+} lives in $\Omega^2_{14} \subset \Omega^2$, i.e. for any vector fields X, Y on M,

$$g(R_{\nabla^+}(X, Y)\cdot, \cdot) \in \Omega^2_{14}.$$

The proof follows from the identity

$$\begin{aligned} g(\rho(X, Y), e_l) &= \frac{1}{2} \sum_j g((R_{\nabla^+}(X, Y) e_j) \times e_j, e_l) \\ &= \frac{1}{2} \sum_{j,k} R_{\nabla^+}(X, Y)^k_j g(e_k \times e_j, e_l) = \frac{1}{2} \sum_{j,k} \varphi_{kjl} R_{\nabla^+}(X, Y)^k_j. \end{aligned}$$

Now, by Proposition 3.4.3, the assumption $\nabla^+ \varphi = 0$ implies that φ and ∇^+ unique, so we have $H = H_\varphi := \frac{1}{6} \tau_0 \varphi - \tau_1 \lrcorner \psi - \tau_3$. □

To introduce our equations of interest, we fix a principal K-bundle $P \to M$. The Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)$ is endowed with a non-degenerate bi-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$.

Definition 4.3.4. *Let $P \to M^7$ be a principal K-bundle over an oriented spin 7-manifold. For a triple (φ, H, θ), where φ is a G_2-structure on M, $H \in \Omega^3(M)$, and θ is a principal connection on P, the* coupled G_2-instanton equation *is*

$$\begin{aligned} \rho(\varphi, H) + \langle F_\theta, (F_\theta \lrcorner \varphi)^\# \rangle &= 0, \\ (\nabla^{\theta,+} F_\theta) \lrcorner \varphi &= 0, \\ [F_\theta \lrcorner \varphi, \cdot\,] - \left(F_\theta \lrcorner^1 \langle \cdot, \cdot \rangle^{-1} F_\theta\right) \lrcorner \varphi &= 0, \\ dH - \langle F_\theta \wedge F_\theta \rangle &= 0, \end{aligned} \tag{4.3.2}$$

where $\mathbb{F} \wedge \mathbb{F}^\dagger \in \Omega^2(\mathrm{End}(\mathrm{ad}P))$ is $F_\theta \lrcorner^1 \langle \cdot, \cdot \rangle^{-1} F_\theta$ via (2.3.7).

In the next result, we establish a bijection between the solutions of the coupled G_2-instanton equation (4.3.2) and the coupled instanton equations formulated in terms of spinors, in Remark 2.2.3, namely,

$$\begin{aligned} (R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F}) \cdot \eta &= 0, \\ \nabla^{\theta,+} F_\theta \cdot \eta &= 0, \\ [F_\theta \cdot \eta, \cdot] - \mathbb{F} \wedge \mathbb{F}^\dagger \cdot \eta &= 0, \\ dH - \langle F_\theta \wedge F_\theta \rangle &= 0. \end{aligned}$$

Recall that a G_2-structure on M is equivalent to a nowhere-vanishing spinor field $\eta \in \Omega^0(S)$, via (3.5.9) and (3.5.10). Note that the system (2.2.7), introduced in Remark 2.2.3, can be regarded as a system for tuples (g, H, θ, η).

Proposition 4.3.5. *Let $P \to M^7$ be a principal K-bundle over an oriented spin 7-manifold. Then, any solution of the coupled G_2-instanton equations (4.3.2),*

$$\begin{aligned} \rho(\varphi, H) + \langle F_\theta, (F_\theta \lrcorner \varphi)^\# \rangle &= 0, \\ (\nabla^{\theta,+} F_\theta) \lrcorner \varphi &= 0, \\ [F_\theta \lrcorner \varphi, \cdot\,] - \left(F_\theta \lrcorner^1 \langle \cdot, \cdot \rangle^{-1} F_\theta\right) \lrcorner \varphi &= 0, \\ dH - \langle F_\theta \wedge F_\theta \rangle &= 0, \end{aligned}$$

determines a solution $(g_\varphi, H, \theta, \eta_\varphi)$ of (2.2.7). Conversely, any solution (g, H, θ, η) of (2.2.7) determines a solution of the coupled G_2-instanton equations (4.3.2) of the form $(\varphi_\eta, H, \theta)$, where φ_η is defined by (3.5.9).

Proof. The equivalence between the second and third equations in (2.2.7) and (4.3.2) follows easily from (3.5.4). It remains, therefore, to prove the equivalence between the first equation in (4.3.2) and the first equation in (2.2.7), so long as the Bianchi identity $dH = \langle F_\theta \wedge F_\theta \rangle$ is satisfied. Arguing as in the last part of the proof of Theorem 2.3.2, the desired equivalence now follows from (3.5.4):

$$\begin{aligned}\left(R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F}\right) \cdot \eta = 0 &\iff R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} \in \Omega^2_{14} \\ &\iff \left(R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F}\right) \lrcorner \varphi = 0 \\ &\iff \frac{1}{2!1!}\left(\langle F_\theta, {F^l}_k\rangle_{ij} + (R_{\nabla^+})_{ij}{}^l{}_k\right) \varphi_{ijp} e^p \otimes e^k \otimes e_l = 0 \\ &\iff \langle (F_\theta \lrcorner \varphi)^\#, {F^l}_k e^k \otimes e_l\rangle + {\rho^l}_k = 0 \\ &\iff \langle (F_\theta \lrcorner \varphi)^\#, F_\theta\rangle + \rho = 0.\end{aligned}$$

as desired. □

As a direct consequence of the previous result and Remark 2.2.3, it follows that any solution of the coupled G_2-instanton equation (4.3.2) corresponds to a G_2-instanton on $TM \oplus \mathrm{ad}P$, given by the connection (2.2.2).

To finish this section, we prove that any solution of the gravitino equation (2.1.5) in seven dimensions provides a solution of the coupled G_2-instanton equation (4.3.2), by application of Theorem 2.3.2. Note that, in the present setup, the gravitino equation is given by

$$\nabla^+ \varphi = 0, \qquad F_\theta \wedge \psi = 0, \tag{4.3.3}$$

where the unknowns are pairs (φ, θ) as before and $\nabla^+ = \nabla^g + \frac{1}{2} g^{-1} H_\varphi$ for (see Proposition 3.4.3)

$$H_\varphi := \frac{1}{6}\tau_0 \varphi - \tau_1 \lrcorner \psi - \tau_3. \tag{4.3.4}$$

We are mainly interested in solutions of the gravitino equation and also the heterotic Bianchi identity (1.1.8)

$$dH_\varphi = \langle F_\theta \wedge F_\theta \rangle. \tag{4.3.5}$$

Our findings offer an alternative proof of, and are inspired by, certain results presented in [dlOLS18a, dlOLS18b].

Theorem 4.3.6. *Let $P \to M^7$ be a principal K-bundle over an oriented spin 7-manifold. Then any solution (φ, θ) of the gravitino equation (4.3.3) and the Bianchi identity (1.1.8) determines a solution $(\varphi, H_\varphi, \theta)$ of the coupled G_2-instanton equation (4.3.2), and the connection on $T \oplus \mathrm{ad}P$ defined in (2.2.2) by*

$$D = \begin{pmatrix} \nabla^- & \mathbb{F}^\dagger \\ -\mathbb{F} & d^\theta \end{pmatrix}$$

is a G_2-instanton with respect to $\varphi =: *\psi$, i.e.

$$F_D \wedge \psi = 0.$$

In particular, given a solution (φ, θ) of the heterotic G_2-system (4.1.1), the triple $(\varphi, H_\varphi, \theta)$ solves the coupled G_2-instanton equation (4.3.2).

Proof. The proof follows by direct application of Theorem 2.3.2, combined with Proposition 4.3.5. The last part of the statement follows from Proposition 4.1.2. □

As a straightforward consequence is related to instanton towers in Section 2.5. Theorem 4.3.6 and Proposition 2.5.1 gives us that:

Corollary 4.3.7. *Let* (M^7, φ) *be a 7-manifold with an integrable* G_2*-structure* φ *and closed torsion, endowed with a* G_2*-instanton connection* θ *on a principal* K*-bundle* $P \to M$ *with respect to* $\varphi =: *\psi$*, that is, solving the equations,*

$$\tau_2 = 0, \qquad dH_\varphi = 0, \qquad F_\theta \wedge \psi = 0,$$

cf. Proposition 3.4.3. Then there exists a sequence of G_2*-instanton bundles* $\{(V_k, \nabla^k)\}_{k\in\mathbb{N}}$ *over* M *with respect to* φ*, such that each* V_k *is a real orthogonal bundle of rank*

$$r_k = 7 + r_{k-1}(r_{k-1} - 1), \qquad r_1 = \dim K,$$

and ∇^k *is a linear orthogonal connection on* V_k*.*

4.4 Gravitino solutions and generalized Ricci-flat metrics

We will answer Problem 2 for the particular case of an oriented spin manifold M^7 in the context of G_2-structures. The approach consists of considering solutions (φ, θ) of the gravitino equation (4.3.3) and the heterotic Bianchi identity (1.1.8) and proving that they induce generalized Ricci-flat metrics for a canonical choice of divergence determined by the Lee form of the G_2-structure φ. In particular, this implies that any solution of the coupled G_2-instanton equation (4.3.2) constructed via Theorem 4.3.6 induces a generalized Ricci-flat metric, as stated in Problem 2. We start with a technical Lemma about the failure of a G_2-instanton to satisfy the Yang-Mills equations, i.e., $d^{\theta*}F_\theta = 0$, which is valid for arbitrary G_2-structures.

Lemma 4.4.1. *Let* P *be a principal* K*-bundle over 7-manifold* M^7 *with* G_2*-structure* φ*. Given a* G_2*-instanton* θ *on* P*, that is, a principal connection* θ *satisfying* $F_\theta \wedge \psi = 0$*, one has*

$$d^{\theta*}F_\theta - F_\theta \lrcorner H + 4 i_{\tau_1^\#} F_\theta = 0. \tag{4.4.1}$$

Proof. Recall that the instanton condition for θ is equivalent to the following equations:

$$F_\theta \wedge \psi = 0 \iff F_\theta \lrcorner \varphi = 0 \iff F_\theta \lrcorner \psi = -F_\theta \iff F_\theta \wedge \varphi = - * F_\theta.$$

Taking covariant derivatives in the last expression and using the usual Bianchi identity $d^\theta F_\theta = 0$, we obtain:

$$d^\theta * F_\theta = d^\theta(-F_\theta \wedge \varphi) = -F_\theta \wedge d\varphi,$$

which implies (the so-called *Yang-Mills equation with torsion*, cf. [Tor15])

$$d^{\theta *}F_\theta + F_\theta \lrcorner\, d^*\psi = 0.$$

Applying (3.3.3), we have $d^*\psi = *d\varphi = \tau_0\varphi - 3\tau_1 \lrcorner\, \psi + \tau_3$, and therefore

$$\begin{aligned} F_\theta \lrcorner\, d^*\psi &= \tau_0 F_\theta \lrcorner\, \varphi \ - 3F_\theta \lrcorner\, (\tau_1 \lrcorner\, \psi) + F_\theta \lrcorner\, \tau_3 \\ &= \Big(-\tfrac{1}{6} + \tfrac{7}{6}\Big)\tau_0 F_\theta \lrcorner\, \varphi \ + (1-4)\tau_1 \lrcorner\, (F_\theta \lrcorner\, \psi) + F_\theta \lrcorner\, \tau_3 \\ &= -F_\theta \lrcorner\, H + \tfrac{7}{6}\cancelto{0}{\tau_0 F_\theta \lrcorner\, \varphi} - 4\tau_1 \lrcorner\, (F_\theta \lrcorner\, \psi) = -F_\theta \lrcorner\, H + 4\tau_1 \lrcorner\, F_\theta. \end{aligned}$$

combining this result with the Yang-Mills equation with torsion, the result follows. □

Our following result establishes the desired relation between solutions of the seven-dimensional gravitino equation and generalized Ricci-flat metrics. Via Theorem 4.3.6, it can be regarded as a partial answer to Problem 2. For the proof, we will use a general formula for the Ricci tensor of the characteristic connection of an integrable G_2-structure, from [IS23b, Theorem 4.5].

Lemma 4.4.2 ([IS23b]). *Let* (M^7, φ) *be an integrable* G_2*-structure, then the Ricci curvature* Ric^+ *of its characteristic connection* $\nabla^+ = \nabla^g + \frac{1}{2}g^{-1}H$ *satisfies the following identity in coordinates:*

$$\mathrm{Ric}^+_{ij} - \frac{1}{12}(dH)^{ab\mu}{}_i\psi_{ab\mu j} + 4\nabla^+_i\tau_{1j} = 0. \tag{4.4.2}$$

Proof. We start with the definition of the Ricci curvature for an arbitrary affine connection ∇: $\mathrm{Ric}^\nabla{}_{ij} = R^{\nabla\mu}{}_{j\mu i}$. Additionally, the G_2-structure is integrable, so the curvature R^+ of ∇^+ has its endomorphism part living in $\mathfrak{g}_2 = \Lambda^2_{14}$, so it satisfies:

$$R^+{}_{abij}\psi^{ab}{}_{kl} = -2R^+{}_{klij}. \tag{4.4.3}$$

Now, let's delve into the computation of the Ricci coefficients for the characteristic connection ∇^+. We will perform this calculation directly using an orthonormal frame, which allows us the flexibility to raise and lower indices arbitrarily. We start as follows:

$$\begin{aligned} 2\mathrm{Ric}^+_{ij} := 2R^+{}_{\mu j\mu i} &\overset{(4.4.3)}{=} -R^+{}_{ab\mu i}\psi_{ab\mu j} = -\frac{1}{3}\left(R^+{}_{ab\mu i} + R^+{}_{ab\mu i} + R^+{}_{ab\mu i}\right)\psi_{ab\mu j} \\ &= -\frac{1}{3}\left(R^+{}_{ab\mu i}\psi_{ab\mu j} + R^+{}_{\mu abi}\psi_{\mu abj} + R^+{}_{b\mu ai}\psi_{b\mu aj}\right) \\ &= -\frac{1}{3}\left(R^+{}_{ab\mu i} + R^+{}_{\mu abi} + R^+{}_{b\mu ai}\right)\psi_{ab\mu j}, \end{aligned}$$

We have relabelled the indices in the last two equations and used the skew-symmetry property of the ψ indices.

Considering the identity in Lemma B.2.2 (Bianchi identity for metric connection with skew-symmetric torsion) applied in the characteristic connection ∇^+, the expression for the Ricci tensor obtained above becomes:

$$\begin{aligned} 2\mathrm{Ric}^+_{ij} &= -\frac{1}{3}\Big(-\frac{1}{2}(dH)_{ab\mu i} + (\nabla^+_i H)_{ab\mu}\Big)\psi_{ab\mu j} \\ &= \frac{1}{6}(dH)_{ab\mu i}\psi_{ab\mu j} - \frac{1}{3}(\nabla^+_i H)_{ab\mu}\psi_{ab\mu j} \end{aligned} \tag{4.4.4}$$

We can still simplify the last term $\nabla^+ H \lrcorner \psi$. For this, let's consider the G_2-invariant map $\gamma \in \Omega^3 \mapsto \gamma \lrcorner \psi \in \Omega^1$ which is zero in all components of Ω^3 but $\Omega^3_7 \cong \Omega^1$, consequently (by the compatibility of ∇^+ and the flux theorem which relates H and $\delta\psi$)

$$\nabla^+ H \lrcorner \psi = \nabla^+(H \lrcorner \psi) = \nabla^+(\pi_7 H \lrcorner \psi) = \nabla^+(-(\tau_1 \lrcorner \psi) \lrcorner \psi) = 4\nabla^+\tau_1$$

where we have used that $(X \lrcorner \psi) \lrcorner \psi = -4X$. To see this, use the identities in Proposition 3.1.6:

$$(X \lrcorner \psi) \lrcorner \psi = \frac{1}{1!1!3!} X^\mu {\psi_\mu}^{ijk}\psi_{ijkl}e^l = -\frac{1}{6}X^\mu \cdot 24\delta_{\mu l}\, e^l = -4X^\flat, \tag{4.4.5}$$

and the result follows. □

Theorem 4.4.3. *Let $P \to M^7$ be a principal K-bundle over a connected, oriented, spin 7-manifold, and let (φ, θ) be a solution of the gravitino equation (4.3.3) and the Bianchi identity (1.1.8):*

$$\nabla^+\varphi = 0, \qquad F_\theta \wedge \psi = 0, \qquad dH = \langle F_\theta \wedge F_\theta \rangle.$$

Then the Riemannian metric $g = g_\varphi$ on M determined by the G_2-structure satisfies:

$$\begin{aligned} \mathrm{Ric}^g - \frac{1}{4}H^2 + F_\theta \circ F_\theta + 2\mathcal{L}_{\tau_1^\#} g = 0, \\ d^* H - 4d\tau_1 + 4i_{\tau_1^\#} H = 0, \\ d^{\theta *} F_\theta - F_\theta \lrcorner H + 4i_{\tau_1^\#} F_\theta = 0, \end{aligned} \tag{4.4.6}$$

where $H = H_\varphi$. In particular,

$$\mathrm{GRic}^+_{\mathbf{G}_\varphi, \mathrm{div}^\varphi} = 0,$$

where $\mathbf{G}_\varphi$ is obtained as in Remark 4.1.3 and the divergence operator is uniquely determined by the G_2-structure via the explicit formula given by Remark 2.2.4:

$$\mathrm{div}^\varphi = \mathrm{div}^{\mathbf{G}_\varphi} - 2\langle 4\tau_1, \cdot \rangle.$$

Proof. The third equation in (4.4.6) follows from Lemma 4.4.1 and the hypothesis (4.3.3). For the first two equations in (4.4.6), we use the integrability of φ, which implies from Lemma 4.4.2 that

$$(\mathrm{Ric}_{\nabla^+})_{ij} - \frac{1}{12}(dH)_{ab\mu i}\psi_{ab\mu j} + 4\nabla^+_i \tau_{1j} = 0. \tag{4.4.7}$$

Choosing an orthonormal frame $\{\zeta_\alpha\}$ for the pairing $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{k}$, we express the curvature of θ by

$$F_\theta = \frac{1}{2!}F^\alpha{}_{ij}e^i\wedge e^j\otimes\zeta_\alpha,$$

where $\{e_j\}$ form a local orthonormal frame for the tangent bundle. Using this, we now have

$$\langle F_\theta\wedge F_\theta\rangle = \left\langle\frac{1}{2!}F^\alpha{}_{ab}\,e^{ab}\otimes\zeta_\alpha\wedge\frac{1}{2!}F^\beta{}_{kl}\,e^{kl}\otimes\zeta_\beta\right\rangle = \frac{1}{4}F^\alpha{}_{ab}F^\beta{}_{kl}e^{abkl}\langle\zeta_\alpha,\zeta_\beta\rangle$$

By the heterotic Bianchi identity (1.1.8), we have $dH = \langle F_\theta\wedge F_\theta\rangle = \frac{1}{4}F^\alpha{}_{ab}F_{\alpha kl}e^{abkl}$, which reads in local components:

$$(dH)_i = F^\alpha{}_{i\mu}F_{\alpha\nu\rho}e^{\mu\nu\rho}.$$

Contracting this expression with $\psi_j = \frac{1}{3!}\psi_{j\mu\nu\rho}e^{\mu\nu\rho}$, and using the instanton condition for θ, we conclude:

$$(dH)_i\lrcorner\,\psi_j = F^\alpha{}_i{}^\mu F_\alpha{}^{\nu\rho}\psi_{j\mu\nu\rho} = -2F^\alpha{}_i{}^\mu F_{\alpha j\mu}.$$

On the other hand,

$$(dH)_i\lrcorner\,\psi_j = \frac{1}{6}(dH)_{ab\mu i}\psi_{ab\mu j} = -2F^\alpha{}_{i\mu}F_{\alpha j\mu}\Rightarrow(dH)_{ab\mu i}\psi_{ab\mu j} = -12F^\alpha{}_{\mu i}F_{\alpha\mu j},$$

and hence

$$(\mathrm{Ric}_{\nabla^+})_{ij} = -F^\alpha{}_{\mu i}F_{\alpha\mu j} - 4\nabla^+_i\tau_{1j} = -(\langle i_{e_k}F_\theta, i_{e_k}F_\theta\rangle)_{ij} - 4\nabla^+_i\tau_{1j}. \tag{4.4.8}$$

The first and second equations in (4.4.6) now follow from the unique decomposition of Ric_{∇^+} and $\nabla^+\tau_1$ into symmetric and skew-symmetric 2-tensors given by (2.1.9), since Ric^g and $F_\theta\circ F_\theta := \langle i_{e_j}F_\theta, i_{e_j}F_\theta\rangle$ are symmetric tensors. □

We will use this approach in Chapter 6 to prove generalized Ricci flatness for more general structures, in particular for Spin(7)-structures.

To finish this section, in the following result, we investigate the failure of generalized Ricci-flatness when we remove the instanton condition on θ from the hypotheses of Theorem 4.4.3. We focus on the Yang–Mills-type equation given by the third equation in (4.4.6), which we relate to the second equation in the coupled G_2-instanton equations (4.3.2). This situation can be then compared to the case of $SU(n)$-structures (ω,Ψ) with integrable complex structure studied in [GFGM23, Proposition 4.9], see Remark 4.4.5. A similar analysis can be adopted for the first and second equations in (4.4.6), following the proof of Theorem 4.4.3 carefully. This technical result will be key for the proof of the main results in Section 5.4.

Lemma 4.4.4. *Let* (M^7, φ) *be a 7-manifold endowed with an integrable* G_2*-structure* φ. *Let* P *be a principal* K*-bundle over* M *and* θ *an arbitrary principal connection on* P. *Then the following identity holds:*

$$d^{\theta *} F_\theta + 4\tau_1 \lrcorner F_\theta - F_\theta \lrcorner H = 6\tau_1 \lrcorner \pi_7 F_\theta + \frac{1}{3}\tau_0 \pi_7 F_\theta \lrcorner \varphi - 3\pi_7 F_\theta \lrcorner \tau_3 - 3\sum_j i_{e_j} \pi_7 \nabla^{\theta,+}_{e_j} F_\theta \quad (4.4.9)$$

for a local orthonormal frame $\{e_j\}$ *on* M *and* $\pi_7 : \Omega^2 \to \Omega^2_7$ *being the projection. In particular, if* $\nabla^{\theta,+} F_\theta \in \Omega^2_{14}$, *we have*

$$d^{\theta *} F_\theta + 4\tau_1 \lrcorner F_\theta - F_\theta \lrcorner H = 6\tau_1 \lrcorner \pi_7 F_\theta + \frac{1}{3}\tau_0 \pi_7 F_\theta \lrcorner \varphi - 3\pi_7 F_\theta \lrcorner \tau_3. \quad (4.4.10)$$

or, equivalently

$$d^{\theta *} F_\theta + F_\theta \lrcorner d^* \psi = (F_\theta \lrcorner \psi + F_\theta) \lrcorner H. \quad (4.4.11)$$

Proof. Writing the expression for $\nabla^{\theta,+}_X F_\theta$ explicitly, we have

$$\begin{aligned}
\nabla^{\theta,+}_X F_\theta(V,W) &= d^\theta_X(F_\theta(V,W)) - F_\theta(\nabla^+_X V, W) - F_\theta(V, \nabla^+_X W) \\
&= d^\theta_X(F_\theta(V,W)) - F_\theta(\nabla^g_X V, W) - F_\theta(V, \nabla^g_X W) \\
&\quad - \frac{1}{2}\Big(F_\theta(H(X,V),W) + F_\theta(V, H(X,W))\Big) \\
&= \nabla^{\theta,g}_X F_\theta(V,W) - \frac{1}{2}\Big(F_\theta(H(X,V),W) - F_\theta(H(X,W),V).\Big)
\end{aligned}$$

Define $\mathcal{K} \in \Omega^1(\Lambda^2 T^* \otimes \mathrm{ad}P)$ by $\nabla^{\theta,+}_X F_\theta =: \nabla^{\theta,g}_X F_\theta - \frac{1}{2}\mathcal{K}_X$. Now, using

$$\pi_7 \nabla^{\theta,+}_X F_\theta = \frac{1}{3}\left(\nabla^{\theta,+}_X F_\theta + \nabla^{\theta,+}_X F_\theta \lrcorner \psi\right),$$

in a local orthonormal frame $\{e_j\}$, we obtain

$$\begin{aligned}
3\sum_j i_{e_j} \pi_7 \nabla^{\theta,+}_{e_j} F_\theta &= \sum_j i_{e_j} \nabla^{\theta,+}_{e_j} F_\theta + i_{e_j}(\nabla^{\theta,+}_{e_j} F_\theta \lrcorner \psi) = \sum_j i_{e_j} \nabla^{\theta,+}_{e_j} F_\theta + \nabla^{\theta,+}_{e_j} F_\theta \lrcorner i_{e_j}\psi \\
&= \sum_j i_{e_j} \nabla^{\theta,+}_{e_j} F_\theta - \nabla^{\theta,+}_{e_j} F_\theta \lrcorner *(e_j \wedge \varphi) = \sum_j i_{e_j} \nabla^{\theta,+}_{e_j} F_\theta - *(\nabla^{\theta,+}_{e_j} F_\theta \wedge e^j \wedge \varphi).
\end{aligned}$$

We compute the first summand in the last expression:

$$\begin{aligned}
\sum_j i_{e_j} \nabla^{\theta,+}_{e_j} F_\theta &= \sum_j i_{e_j} \nabla^{\theta,g}_{e_j} F_\theta - \frac{1}{2}\Big(F_\theta(H(e_j,e_j),\cdot) + F_\theta(e_j, H(e_j,\cdot))\Big) \\
&= \sum_j e_j \lrcorner \nabla^{\theta,g}_{e_j} F_\theta - \frac{1}{2} H_{jkl} F_{jl}\, e^k = -d^{\theta *} F_\theta + F_\theta \lrcorner H
\end{aligned}$$

To compute the second summand, we use the identity for the covariant exterior derivative, $d^\theta = \sum_j e^j \wedge \nabla^{\theta,g}_{e_j}$, so that the Bianchi identity $d^\theta F_\theta = 0$ gives

$$\sum_j e^j \wedge \nabla^{\theta,+}_{e_j} F_\theta = \underbrace{\sum_j e^j \wedge \nabla^{\theta,g}_{e_j} F_\theta}_{d^\theta F_\theta} - \frac{1}{2} e^j \wedge \mathcal{K}_{e_j} = -\tfrac{1}{2}\sum_j e^j \wedge \mathcal{K}_{e_j}.$$

Therefore

$$-d^{\theta *}F_\theta + F_\theta \lrcorner H + \tfrac{1}{2}\sum_j *(e^j \wedge \mathcal{K}_{e_j} \wedge \varphi) = 3\sum_j i_{e_j}\pi_7\nabla^{\theta,+}_{e_j}F_\theta.$$

Setting $\mathcal{K}_j = \mathcal{K}_{e_j}$, and computing directly (now using summation convention for efficiency)

$$\begin{aligned} e^j \wedge \mathcal{K}_j &= \tfrac{1}{2}(\mathcal{K}_j)_{\alpha\beta}\, e^{j\alpha\beta} = \tfrac{1}{2}\Big(H_{j\alpha}{}^{\gamma}F_{\gamma\beta} - H_{j\beta}{}^{\gamma}F_{\gamma\alpha}\Big)\, e^{j\alpha\beta} \\ &= \tfrac{1}{2}\Big((H_{j\alpha}{}^{\gamma}\, e^{j\alpha}) \wedge (F_{\gamma\beta}\, e^{\beta}) + (H_{j\beta}{}^{\gamma}\, e^{j\beta}) \wedge (F_{\gamma\alpha}\, e^{\alpha})\Big) \\ &= 2i_{e_\gamma}(H^\gamma \wedge F_\theta), \end{aligned}$$

we obtain

$$*(e^j \wedge \mathcal{K}_j \wedge \varphi) = 2 * (i_{e_j}(H^j \wedge F_\theta) \wedge \varphi) = 2 * (i_{e_j}(H^j \wedge F_\theta \wedge \varphi) - H^j \wedge F_\theta \wedge i_{e_j}\varphi).$$

Using now that $\tau_2 = 0$, we have (cf. Lemma B.3.2) that $d\varphi = H \lrcorner^1 \varphi = H^j \wedge \varphi_j$, we then deduce

$$\sum_j *(H^j \wedge F_\theta \wedge \varphi_j) = *(F_\theta \wedge d\varphi) = *(F_\theta \wedge **\, d**\varphi) = F_\theta \lrcorner d^*\psi,$$

and so

$$\begin{aligned} *\Big(i_{e_j}(H^j \wedge F_\theta \wedge \varphi)\Big) &= *(i_{e_j}(H^j \wedge *(F_\theta \lrcorner \psi))) = e^j \wedge *(H^j \wedge *(F_\theta \lrcorner \psi)) \\ &= e^j \wedge (H^j \lrcorner (F_\theta \lrcorner \psi)) = \tfrac{1}{2}H^{jkl}(F_\theta \lrcorner \psi)_{kl}e^j \\ &= \tfrac{1}{2}(F_\theta \lrcorner \psi)_{kl}H_{jkl}e^j = (F_\theta \lrcorner \psi) \lrcorner H. \end{aligned}$$

From this, we conclude that

$$*(e^j \wedge \mathcal{K}_j \wedge \varphi) = 2\Big(F_\theta \lrcorner d^*\psi - (F_\theta \lrcorner \psi) \lrcorner H\Big)$$

and, as desired,

$$\begin{aligned} d^{\theta *}F_\theta + 4\tau_1 \lrcorner F_\theta - *(F_\theta \wedge *H) =&\ 4\tau_1 \lrcorner F_\theta + (F_\theta \lrcorner \psi) \lrcorner H - F_\theta \lrcorner d^*\psi - 3i_{e_j}\pi_7\nabla^{\theta,+}_{e_j}F_\theta \\ =&\ 4\tau_1 \lrcorner F_\theta + \tfrac{1}{3}\tau_0\pi_7 F_\theta \lrcorner \varphi - \tau_1 \lrcorner ((F_\theta \lrcorner \psi) \lrcorner \psi) \\ &- (F_\theta \lrcorner \psi) \lrcorner \tau_3 - \tau_0 F_\theta \lrcorner \varphi + 3\tau_1 \lrcorner (F_\theta \lrcorner \psi) - F_\theta \lrcorner \tau_3 - 3_{e_j}\pi_7\nabla^{\theta,+}_{e_j}F_\theta \\ =&\ 6\tau_1 \lrcorner \pi_7 F_\theta + \tfrac{1}{3}\tau_0\pi_7 F_\theta \lrcorner \varphi - 3\pi_7 F_\theta \lrcorner \tau_3 - 3i_{e_j}\pi_7\nabla^{\theta,+}_{e_j}F_\theta. \end{aligned}$$

and the result follows. □

Remark 4.4.5. Equation (4.4.10) above shows us that the second equation in the coupled G_2-instanton equations (4.3.2), given by

$$\nabla^{\theta,+}F_\theta \wedge \psi = 0,$$

does not imply, in general, the Yang-Mills equation with torsion, given by the third equation in (4.4.6). An explicit example where this is indeed the case is not known, [dSJGFLSE24]. This situation stands out in comparison to $SU(n)$-structures (ω, Ψ) with integrable complex structure studied in [GFGM23, Proposition 4.9], for which the equation $(\nabla^{\theta,+} F_\theta) \wedge \omega^{n-1} = 0$ combined with $F_\theta^{0,2} = 0$ is equivalent to the corresponding Yang–Mills equation with torsion. More details about this example, cf. Example 6.5.4. ○

4.5 Examples

In this section, we discuss some examples of coupled G_2-instantons which can be found scattered in the literature but which have not been identified as such. The first examples arise from solutions of the heterotic G_2 system (4.1.1), by Theorem 4.3.6 and Proposition 4.1.2. Such solutions with exact torsion one-form $\tau_1 = d\phi$ have been constructed in, e.g. [GN95, FIUV11, Nol12, FIUV15, dlOG21, CGFT22, GS24], motivated by the concept's origins in heterotic string theory, which requires a globally defined dilaton field ϕ whose vacuum expectation value determines the string-coupling constant. The approximate solutions constructed in [LSE23] deserve special treatment since they do not precisely solve the first equation in (4.1.1), and we postpone their analysis to Section 5.

Since we are mainly concerned with solving the coupled instanton equation (4.3.2), we will work with Theorem 4.3.6 and consider solutions of the gravitino equation (4.3.3) and the Bianchi identity (1.1.8). Incidentally, by Theorem 4.4.3, these conditions are sufficient to imply a solution of the heterotic G_2 system, with our relaxed definition (4.1.1). Note that our equations barely impose any constraint on the torsion one-form τ_1, and therefore are more flexible than the ones usually considered in the mathematical physics literature, yet are still strong enough to prove Theorem 4.2.1 and Theorem 4.3.6.

Our first two examples are given by the product of a flat torus with a manifold carrying an $SU(n)$-structure which is integrable and has closed torsion, also known in the literature as *twisted Calabi-Yau* [GFRT20] or, more generally, *Bismut Hermitian–Einstein metrics* [GFJS23]. The seven-dimensional geometry is given by a strong integrable G_2-structure (see Proposition 3.4.3), i.e. such that

$$\tau_2 = 0 \qquad \text{and} \qquad dH_\varphi = 0.$$

Example 4.5.1. Let N^4 be a four-dimensional manifold endowed with $SU(2)$-structure (ω, Ψ), with almost complex structure J and Hermitian metric $g = \omega(\cdot, J\cdot)$. Its Lee form $\theta_\omega := -J^* d^* \omega \in \Omega^1(N)$ is defined by

$$d\omega = \theta_\omega \wedge \omega.$$

In this setup, a solution of the gravitino equation is a triple (ω, Ψ, H) such that, cf. [FI02, Theorem 10.1],

$$H = -d^c\omega + g(N_J, \cdot),$$

where N_J is the Nijenhuis tensor of J, which in particular must be skew-symmetric.

Suppose that (ω, Ψ) satisfies the *twisted Calabi-Yau equation*, introduced in [GFRST22]:

$$d\Psi = \theta_\omega \wedge \Psi, \quad d\theta_\omega = 0, \quad dd^c\omega = 0. \tag{4.5.1}$$

Then, it was proved in [GFRST22, Lemma 2.2] that $N_J = 0$ and that it determines a solution of the gravitino equation with $H = -d^c\omega$, which also solves the Bianchi identity $dH = 0$. Note that compact solutions of these equations in four dimensions are rather rigid, as they only exist on tori and K3 surfaces, with $H = 0 = \theta_\omega$, and diagonal Hopf surfaces, with $H \neq 0 \neq \theta_\omega$, cf. [GFRST22, Proposition 2.10].

To build the seven-dimensional geometry from a solution of (4.5.1) we follow closely [FMMR23]. We consider $M = N \times T^3$, where T^3 is a three-dimensional flat torus. Denote

$$\psi_+ \coloneqq \mathrm{Re}(\Psi), \qquad \psi_- \coloneqq \mathrm{Im}(\Psi).$$

Define a G_2-structure on M by

$$\varphi = dx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge \omega + dx^2 \wedge \psi_+ - dx^3 \wedge \psi_-, \tag{4.5.2}$$

where $(x_1, x_2, x_3) \in \mathbb{R}$ are coordinates in the universal cover of T^3. Then, φ is strong and integrable with

$$\tau_0 = 0, \qquad \theta_\omega = 4\tau_1, \qquad H_\varphi = d^c\omega.$$

For the proof, we follow [FMMR23, Proposition 3.5]. For instance, since ω, ψ_+ and ψ_- are Hodge self-dual on N^4,

$$*\varphi = \tfrac{1}{2}\omega^2 + dx^2 \wedge dx^3 \wedge \omega - dx^1 \wedge dx^3 \wedge \psi_+ - dx^1 \wedge dx^2 \wedge \psi_-.$$

Since $d\omega = \theta_\omega \wedge \omega$ and $d\Psi = \theta_\omega \wedge \Psi$, we obtain $d * \varphi = \theta_\omega \wedge *\varphi$. Thus, φ is an integrable G_2-structure with Lee form $\theta = \theta_\omega$, and

$$d\varphi \wedge \varphi = \theta_\omega \wedge \left(dx^1 \wedge \omega + dx^2 \wedge \psi_+ - dx^3 \wedge \psi_-\right) \wedge \varphi = 0,$$

since the self-dual forms ω, ψ_+ and ψ_- are pairwise orthogonal. The torsion of φ is

$$H_\varphi = *(\theta \wedge \varphi - d\varphi) = *(\theta_\omega \wedge dx^1 \wedge dx^2 \wedge dx^3) = - *_4 \theta_\omega = Jd\omega = d^c\omega,$$

since $\theta_\omega = J *_4 d *_4 \omega = *_4 Jd\omega$ and $J\omega = \omega$.

Applying now Theorem 2.3.2, we obtain a coupled G_2-instanton on TM given by the connection

$$\nabla^- = \nabla^{g_7} - \frac{1}{2} g_7^{-1} H_\varphi. \tag{4.5.3}$$

Incidentally, this connection is actually flat [FMMR23], and the tower of coupled G_2-instantons over this manifold given by Corollary 4.3.7 is also flat. △

The following example is given by the product of the Calabi–Eckmann 6-manifold $S^3 \times S^3$ with a circle. Similarly, as in the previous example, by application of [FMMR23, Proposition 3.5], the seven-dimensional torsion classes are inherited from the six-dimensional geometry. Consequently, this example is also strong and integrable, but unlike the previous one, it has $d\tau_1 \neq 0$, which reflects the fact that the Calabi–Eckmann complex threefold does not admit balanced hermitian metrics. Note that this example provides a solution of the heterotic G_2 system (4.1.1) according to our lax Definition 4.1.1, but it escapes from the orthodoxy for these systems of equations in the literature, precisely because τ_1 is non-closed.

Example 4.5.2. Let

$$N^6 = \{\mathbb{C}^2_\times \times \mathbb{C}^2_\times\}/\mathbb{C} \simeq S^3 \times S^3,$$

with its (non-Kähler) Calabi–Eckmann SU(3)-structure (ω, Ψ). Following [GFS20, Example 8.35], if we let $\pi_j : S^3 \to \mathbb{CP}^1$ denote the Hopf fibration on each of the two factors in N, for $j = 1, 2$, and let μ_j denote the 1-form on S^3 such that

$$d\mu_j = \pi_j^* \omega_{\mathbb{CP}^1}$$

for $j = 1, 2$, where $\omega_{\mathbb{CP}^1}$ is the Kähler form for the Fubini–Study metric on $\mathbb{CP}^1$, then we can write ω explicitly as:

$$\omega = \pi_1^* \omega_{\mathbb{CP}^1} + \pi_2^* \omega_{\mathbb{CP}^1} + \mu_1 \wedge \mu_2.$$

It is straightforward to show that if we let

$$\theta_\omega = \mu_2 - \mu_1$$

then

$$d\Psi = \theta_\omega \wedge \Psi \qquad \text{and} \qquad dd^c\omega = 0.$$

However, note that

$$d\theta_\omega = \pi_2^* \omega_{\mathbb{CP}^1} - \pi_1^* \omega_{\mathbb{CP}^1} \neq 0$$

and so the second equation (4.5.1) in the definition of twisted Calabi–Yau is not satisfied, though the rest are.

As in Example 4.5.1, we now let $\psi_\pm$ denote the real and imaginary parts of Ψ respectively. We may then define a product G_2-structure on $M^7 = N^6 \times S^1$ by

$$\varphi = \omega \wedge dt + \psi_+,$$

where dt is the standard nowhere vanishing 1-form on S^1. The Hodge dual ψ of φ is then given by

$$\psi = \frac{1}{2}\omega^2 + \psi_- \wedge dt.$$

As in [FMMR23, Proposition 3.5], one sees from these formulae that the G_2-structure φ is integrable with

$$\tau_0 = 0, \quad \tau_1 = \theta_\omega, \quad H_\varphi = d^c\omega = \pi_1^*\omega_{\mathbb{CP}^1} \wedge \mu_1 - \pi_2^*\omega_{\mathbb{CP}^1} \wedge \mu_2.$$

Hence, $dH_\varphi = dd^c\omega = 0$ and thus φ is also strong.

Even though τ_1 is not closed, one may still apply Theorem 2.3.2 and obtain a coupled G_2-instanton ∇^- on TM as in (4.5.3), which is again flat. The tower of coupled G_2-instantons we obtain from Corollary 4.3.7 are also flat. Notice that the G_2-structure presented here is fundamentally different from that obtained on $G = \mathrm{SU}(2)^2 \times S^1$ as a Lie group, in [FMMR23, Proposition 6.2]. △

Remark 4.5.3. Given the observations in Examples 4.5.1 and 4.5.2, it would be interesting to find G_2-structures which are both strong and integrable but for which the connection ∇^- in (4.5.3) is *not flat*, or even irreducible. ○

Remark 4.5.4. No irreducible compact homogeneous spaces admitting invariant G_2-structures, up to a covering, admit (invariant) strong integrable G_2-structures, cf. [FMMR23, §5]. On the other hand, the same authors find numerous examples of such structures on *reducible* spaces, which, according to their preference, have closed Lee form, cf. [FMMR23, §6]. Several of those examples can be easily adapted to provide more general solutions of the gravitino equation (4.5.1). ○

Our final example, originally found in [II05, §6], provides a solution of the heterotic G_2-system (4.1.1) with $\tau_0 \neq 0$ and non-flat instanton θ in the nearly parallel seven-dimensional sphere. In particular, the coupled G_2-instanton obtained from this solution via Theorem 4.3.6 is non-flat.

Example 4.5.5. Let $M = S^7$ be the standard 7-sphere, viewed as a sphere in the octonions. It is well-known that the embedding of S^7 in the octonions induces a natural $\mathrm{Spin}(7)$-invariant G_2-structure φ on S^7 which is *nearly parallel* in the sense that

$$d\varphi = 4\kappa\psi \tag{4.5.4}$$

where $\psi = *\varphi$ as usual and $\kappa \neq 0$ is constant. Note that the metric determined by φ has constant curvature κ^2. We clearly see that all the torsion forms vanish, except $\tau_0 = 4\kappa \neq 0$ and is constant. Hence, φ is an integrable G_2-structure but since

$$H = \frac{2}{3}\kappa\varphi, \tag{4.5.5}$$

we see that

$$dH = \frac{8}{3}\kappa^2\psi \neq 0 \tag{4.5.6}$$

by (4.5.4) and thus φ is not strong.

Take P to be the G_2-frame bundle of S^7, and let θ be the connection on P determined by ∇^+. It is observed in [II05, §6] that $\nabla^+ H_\varphi = 0$, and consequently, cf. (2.3.5),

$$g(R_{\nabla^+}(X,Y)Z,W) = g(R_{\nabla^+}(Z,W)X,Y).$$

As in the proof of Theorem 2.3.2, θ is a G_2-instanton. Furthermore, it is shown in [II05, §6] that the curvature F_θ of θ satisfies

$$\operatorname{tr} F_\theta \wedge F_\theta = -\frac{32\kappa^4}{27}\psi. \tag{4.5.7}$$

Combining (4.5.6) and (4.5.7), we see that the heterotic Bianchi identity (1.1.8) is satisfied for a suitable choice of scaling of the Killing form on the Lie algebra of G_2. Overall, we see that (φ, H, θ) defines a coupled G_2-instanton on S^7. △

Remark 4.5.6. An interesting example in six dimensions where Theorem 2.3.2 applies is the 6-sphere with the standard nearly Kähler structure inherited from the imaginary octonions. According to [II05, §6], this provides a solution of the gravitino equation and the Bianchi identity with instanton connection ∇^+ and non-closed torsion given by the Nijenhuis tensor of the SU(3)-structure, with a structure very similar to Example 4.5.5. ◯

5 Approximate solutions on contact Calabi-Yau 7-manifolds

We have seen a connection between solutions of the heterotic G_2-system, solutions of the coupled G_2-instanton equations and the vanishing of generalised Ricci curvature. In [LSE23], "approximate" solutions to the heterotic G_2 system were given in the sense that the connections involved were only "approximate" G_2-instantons: here the "approximate" pertains to dependence on the non-zero constant α' which appears in the heterotic Bianchi identity as $\alpha' \to 0$. Motivated by this and our results thus far, in this section, we propose a new definition of α'-approximate G_2-instantons and show that it not only leads to approximate solutions to the coupled G_2-instanton equations but also to generalised Ricci curvature which is approximately zero in a quantitative sense as $\alpha' \to 0$. We also demonstrate that the examples as mentioned above from [LSE23] provide α'-approximate G_2-instantons and thus lead to approximate coupled G_2-instantons and approximate generalised Ricci-flatness.

This chapter is structured as follows: In Section 5.1, we provide an introduction to the fundamental concepts of contact structures, setting the stage for the main focus of this chapter: *contact Calabi-Yau* manifolds. Additionally, we investigate how a 7-dimensional contact Calabi-Yau manifold possesses a natural family of integrable G_2-structures. Following this, we calculate the essential quantities associated with these structures as the flux and torsion forms.

In Section 5.3, a parametrised family of connections $\theta^{\varepsilon,k}_{m,\delta}$ on the tangent bundle of a contact Calabi-Yau manifold is introduced, characterised by accurate parameters ε, k, m, and δ. Moreover, as made in the analysis by [LSE23], we show how much these connections are not G_2-instantons.

In sequence, Section 5.4, we introduce the notion of approximate G_2-instantons by introducing a new imposition with relation to approximate instantons in [LSE23] (cf. Definition 5.4.1). Furthermore, we apply all the theory of coupled instanton equations and generalised Ricci flatness studied in Chapters 2 and 4 to the notion of approximate instantons. Under certain conditions, we prove that approximate instantons imply approximate coupled instantons and approximate generalised Ricci flatness, cf. Theorem 5.4.7.

Finally, in Section 5.5, we extend the findings from the preceding section to the specific setting of contact Calabi-Yau manifolds outlined in Section 5.1. Here, we establish the proof that the approximate solutions introduced in [LSE23] indeed qualify as approximate generalised Ricci flat solutions.

5.1 Contact Calabi–Yau manifolds

In [LSE23], the heterotic G_2 system was studied within the framework of contact Calabi–Yau 7-manifolds, which naturally admit a one-parameter family of G_2-structures that we will now review.

Contact structures on odd-dimensional manifolds M^{2m+1} are $U(m)$-structures in the sense of a reduction of the frame bundle of $SO(2m+1)$ to $U(m)$ when we identify $U(m) \cong U(m) \times 1$ inside $SO(2m+1)$. In terms of tensors, this is equivalent to a data $(M^{2m+1}, \eta, g, \omega)$, where $\eta \in \Omega^1(M)$ called the *contact form* defines a volume form by the expression:

$$\mathrm{vol}_M = \frac{1}{m!}\eta \wedge d\eta^m; \tag{5.1.1}$$

$\omega \in \Omega^2(M)$ being a symplectic form on the $2m$-rank bundle $\ker \eta$ satisfying

$$d\eta = \omega; \tag{5.1.2}$$

g a Riemannian metric on M given by a Riemannian metric g_η on $\ker \eta$ (compatible with ω on this subbundle) with an additional term with η as

$$g = g_\eta + \eta \otimes \eta; \tag{5.1.3}$$

With this, there exists a unique vector field, called *Reeb field* such that (cf. [BG08, Bla10]):

$$\xi \lrcorner \eta = 1; \qquad \xi \lrcorner d\eta = 0. \tag{5.1.4}$$

With this, we have an orthogonal splitting of the tangent bundle $TM = \ker \eta \oplus \langle \xi \rangle$. This data defines an almost complex structure J on $\ker \eta$ via ω and the Riemannian metric g_η. This tensor J can be defined in TM vanishing in $\langle \xi \rangle$, consequently for $X = X_\eta + \lambda\xi \in TM$, we have

$$J^2(X_\eta + \lambda\xi) = -X_\eta = -(X_\eta + \lambda\xi) + \lambda\xi = -X + \eta(X)\xi$$

so, the so-called *transverse almost complex structure* satisfies:

$$J^2 = -\mathrm{Id} + \eta \otimes \xi. \tag{5.1.5}$$

Note that in this case

$$\begin{aligned} g(JX, JX) &= g_\eta(JX, JX) + \eta(JX) \otimes \eta(JX) = g_\eta(JX_\eta, JX_\eta) \\ &= g_\eta(X_\eta, X_\eta) - \eta(X) \otimes \eta(X) + \eta(X) \otimes \eta(X) \\ &= g(X, X) - \eta \otimes \eta(X, X) \end{aligned}$$

consequently, the metric satisfies

$$g(J, J) = g - \eta \otimes \eta \tag{5.1.6}$$

Sometimes, the 'contact' setup will denoted by $(M^{2k+1}, \eta, g, \omega, \xi, J)$ with the non-independent quantities satisfying the relations (5.1.1), (5.1.2), (5.1.3), (5.1.4), (5.1.5) and (5.1.6). Sometimes, we will refer to $(M^{2k+1}, \eta, g, \omega, \xi, J)$ as *almost contact structures* if it satisfies (5.1.1), (5.1.3), (5.1.4), (5.1.5) and (5.1.6).

Example 5.1.1 (Boothby-Wang fibrations)**.** We now present an important class of examples, namely principal circle bundles over symplectic manifolds. A celebrated theorem by Boothby and Wang [BW58] states that a compact regular[1] contact manifold is always of this type (cf. [Bla10, Theorem 3.9] for a proof). Examples of this type are often simply referred to as *Boothby-Wang fibrations*.

These examples consist of a principal $\mathbb{S}^1$-bundle $\pi : M^{2m+1} \to V^{2m}$, where the base manifold (V, ω) is symplectic and $\eta \in \Omega^1(M)$ is a connection form which curvature satisfies $d\eta = \pi^*\omega$. With this and a metric g_V in V, we can define via the relations discussed above a contact structure $(M, \eta, g, \omega, \xi, J)$, [Bla10]. △

A particular example in the Boothby-Wang fibration $\pi : M \to V$ is when the basis manifold V is Kähler, so the contact manifold M is called *Sasakian manifold*. In terms of only the tensor, the contact structure will be *Sasakian* if the *transversal Nijenhius tensor* vanishes

$$N_J(X,Y) = [JX, JY] + J^2[X,Y] - J[JX,Y] - J[X,JY] + d\eta(X,Y)\xi \equiv 0. \tag{5.1.7}$$

If the base manifold of the Boothby-Wang fibration is Calabi-Yau (V, Ψ) for $\Omega = \mathrm{Re}\Omega + \mathbf{i}\,\mathrm{Im}\Omega$ is the holomorphic volume form, then Ψ is defined as M naturally vanishing on the vertical part $\langle \xi \rangle$ and the Sasakian manifold is called *contact Calabi-Yau* manifold. Due to [HV15], we have a classification of contact Calabi-Yau manifolds (cf. the definition above). We are particularly interested when the base manifold is a Calabi-Yau 3-orbifold, so the contact manifold is seven-dimensional, and the interesting thing, in this case, is that they have a natural G_2 structure.

Definition 5.1.2 (Contact Calabi-Yau manifold)**.** *Let (V, ω, Ω) be a Calabi–Yau 3-fold, i.e. a Kähler 3-fold with Kähler form ω and holomorphic volume form Ω satisfying*

$$\mathrm{vol}_V = \frac{\omega^3}{3!} = \frac{1}{4}\mathrm{Re}\Omega \wedge \mathrm{Im}\Omega,$$

where vol_V the volume form is associated with the Kähler metric g_V on V. A contact Calabi-Yau manifold is the total space of an S^1-bundle $\pi : M^7 \to V$ endowed with a connection 1-form η such that $d\eta = \omega$.

[1] A contact structure is regular if the Reeb vector field ξ is regular as a vector field, that is, every point of the manifold has a neighbourhood such that any integral curve of the vector field passing through the neighbourhood passes through only once.

5.2 Family G_2-structures φ_ε on contact Calabi–Yau 7-manifolds

As mentioned earlier, seven-dimensional contact Calabi-Yau manifolds are naturally equipped with a family of G_2-structures, which we will now define.

Definition 5.2.1. *For every $\varepsilon > 0$, we define an S^1-invariant G_2-structure φ_ε on M^7, with dual 4-form ψ_ε, by*

$$\varphi_\varepsilon = \varepsilon\eta \wedge \omega + \mathrm{Re}\Omega \quad \textit{and} \quad \psi_\varepsilon = \frac{1}{2}\omega^2 - \varepsilon\eta \wedge \mathrm{Im}\Omega. \tag{5.2.1}$$

The metric induced from this G_2-structure and its corresponding volume form on M are:

$$g_\varepsilon = \varepsilon^2 \eta \otimes \eta + g_V, \qquad \mathrm{vol}_\varepsilon = \varepsilon\eta \wedge \mathrm{vol}_V. \tag{5.2.2}$$

Note that varying ε in (5.2.2) amounts to rescaling the $\mathbb{S}^1$ fibres of $\pi : M \to V$, so that $\varepsilon \to 0$ corresponds to collapsing the fibres to zero size.

We now recall some basic observations about the family of G_2-structures φ_ε in (5.2.1) on the contact Calabi–Yau 7-manifold M from [LSE23, Lemmas 2.4 & 2.5]. We see that

$$d\varphi_\varepsilon = \varepsilon\omega^2, \qquad d\psi_\varepsilon = 0,$$

so the G_2-structures are co-closed. The torsion forms of φ_ε are [LSE23, Lemma 2.4]:

$$\tau_0 = \frac{6}{7}\varepsilon; \quad \tau_1 = 0; \quad \tau_2 = 0; \quad \tau_3 = \frac{8}{7}\varepsilon^2\eta \wedge \omega - \frac{6}{7}\varepsilon\mathrm{Re}\Omega.$$

In particular, we observe that the structures are integrable (i.e. $\tau_2 = 0$) and admit a connection with totally skew-symmetric torsion (cf. Proposition 3.4.3 and Theorem 3.4.4) given by [LSE23, Lemma 2.5]

$$H_\varepsilon = -\varepsilon^2\eta \wedge \omega + \varepsilon\mathrm{Re}\Omega, \tag{5.2.3}$$

which satisfies

$$dH_\varepsilon = -\varepsilon^2\omega^2.$$

The above facts show that one can build approximate solutions to the heterotic G_2-system on M, using the G_2-structures φ_ε. To describe these approximate solutions, it is necessary to introduce a useful (and natural) local coframe adapted to the geometry of M.

Definition 5.2.2. *Given $\varepsilon > 0$, let $(M^7, \varphi_\varepsilon)$ be as in Definition 5.1.2. We choose a local Sasakian real orthonormal coframe on M:*

$$e^0 = \varepsilon\eta, \quad e^1, \quad e^2, \quad e^3, \quad Je^1, \quad Je^2, \quad Je^3, \tag{5.2.4}$$

where J is the transverse complex structure (from the Calabi–Yau V).

In particular, we have $\{e^1, e^2, e^3, Je^1, Je^2, Je^3\}$ a basic SU(3)-frame for V. In this frame, the Kähler and holomorphic volume forms are given by:

$$\omega = e^1 \wedge Je^1 + e^2 \wedge Je^2 + e^3 \wedge Je^3$$
$$\Omega = (e^1 + iJe^1) \wedge (e^2 + iJe^2) \wedge (e^3 + iJe^3).$$

In particular, expanding the last expression we have [LSE23, Remark 2.7]:

$$\mathrm{Re}\Omega = e^1 \wedge e^2 \wedge e^3 - e^1 \wedge Je^2 \wedge Je^3 - e^2 \wedge Je^3 \wedge Je^1 - e^3 \wedge Je^1 \wedge Je^2,$$
$$\mathrm{Im}\Omega = Je^1 \wedge e^2 \wedge e^3 + Je^2 \wedge e^3 \wedge e^1 + Je^3 \wedge e^1 \wedge e^2 - Je^1 \wedge Je^2 \wedge Je^3.$$

By [LSE23, Proposition 3.2], we know that if we write $e = (e^1\ e^2\ e^3)^{\mathrm{T}}$ and $Je = (Je^1\ Je^2\ Je^3)^{\mathrm{T}}$ then the following structure equations hold:

$$d\begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} = -\begin{pmatrix} 0 & \frac{\varepsilon}{2}Je^{\mathrm{T}} & -\frac{\varepsilon}{2}e^{\mathrm{T}} \\ -\frac{\varepsilon}{2}Je & a & b - \frac{\varepsilon}{2}e_0 I \\ \frac{\varepsilon}{2}e & -b + \frac{\varepsilon}{2}e_0 I & a \end{pmatrix} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix}, \tag{5.2.5}$$

where a is a skew-symmetric 3×3 matrix of 1-forms, b is a symmetric traceless 3×3 matrix of 1-forms, and I is the 3×3 identity matrix. Therefore, if we define

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & 0 & -e_0 I \\ e & e_0 I & 0 \end{pmatrix} \tag{5.2.6}$$

we see that $\mathbf{A} + \frac{\varepsilon}{2}\mathbf{B}$ is the local matrix representing the Levi-Civita connection of g_ε with respect to the local orthonormal coframe introduced in Definition 5.2.2. In particular, $\mathbf{A}$ is the matrix representation of the Levi-Civita connection of g_V [LSE23, Lemma 3.1]. If we then let

$$\mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & -[e] & [Je] \\ e & [Je] & [e] \end{pmatrix} - e_0\mathbf{I}, \tag{5.2.7}$$

where

$$\left[\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}\right] = \begin{pmatrix} 0 & e_3 & -e_2 \\ -e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{pmatrix}, \tag{5.2.8}$$

These quantities will be useful for us to define squashing connections on the tangent bundle to find for (approximate) G_2-instantons.

5.3 The approximate G_2-instantons $\theta^{\delta,k}_{\varepsilon,m}$ in TM

Using the quantities defined in the last section, we can 'squash' the Levi-Civita connection ∇^{g_ε} for g_ε in TM. We can define a family of connections on TM as follows [LSE23, Proposition 3.21].

Definition 5.3.1. *Let $(M^7, \varphi_\varepsilon)$ be as in Definition 5.1.2 for some $\varepsilon > 0$. Recall the local coframe on M in Definition 5.2.2 and the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{I}$ defined concerning this coframe in (5.2.6)–(5.2.7). For $k \in \mathbb{R} \setminus \{0\}$ and $\delta, m \in \mathbb{R}$ we define a connection $\theta^{\delta,k}_{\varepsilon,m}$ on TM by the formula*

$$\theta^{\delta,k}_{\varepsilon,m} = \mathbf{A} + \frac{k\varepsilon}{2}\mathbf{B} + \frac{k\varepsilon\delta}{2}\mathbf{C} + \frac{km\varepsilon}{2}e_0\mathbf{I}. \tag{5.3.1}$$

Note that this local expression determines a globally defined connection on TM, that taking $\delta = m = 0$ and $k = 1$ in (5.3.1) yields the Levi-Civita connection ∇^{g_ε} of the metric g_ε on M, and taking $\delta = k = 1$ and $m = 0$ in (5.3.1) yields the Bismut connection ∇^+ associated with g_ε and torsion H_ε.

Remark 5.3.2. We can interpret the various parameters in Definition 5.3.1 as follows. First, the parameter k can be viewed as a "squashing" parameter, allowing us to rescale the connection along the fibres of $\pi : M \to V$ independently of the parameter ε. The matrix $\mathbf{C}$ is equivalent (up to a factor of ε) to the torsion H_ε in (5.2.3) by [LSE23, Proposition 3.10], so the parameter δ varies the torsion of the connection along a canonical line, which contains the Bismut, Hull and Levi-Civita connections when $k = 1$ and $m = 0$. Finally, the parameter m can be viewed as an additional "twist" parameter acting in the transverse directions for the fibration of M over V. ◯

In [LSE23, Corollary 3.27], it was described how $\theta^{\delta,k}_{\varepsilon,m}$ in Definition 5.3.1 fails to be a G_2-instanton.

Proposition 5.3.3. *Using the notation of Definition 5.3.1, (5.2.7) and (5.2.8), the curvature $R^{\delta,k}_{\varepsilon,m}$ of the connection $\theta^{\delta,k}_{\varepsilon,m}$ on TM satisfies:*

$$R^{\delta,k}_{\varepsilon,m} \wedge \psi_\varepsilon = \frac{k\varepsilon^2(6(1-\delta+m) + k(1-\delta)(1+3\delta))}{4} \cdot \frac{\omega^3}{3!}\mathbf{I} + \frac{k^2\varepsilon^2}{4}\eta \wedge \frac{\omega^2}{2!} \wedge \mathbf{M}^\delta_m, \tag{5.3.2}$$

where

$$\mathbf{M}^\delta_m = \begin{pmatrix} 0 & (1+m-5\delta)(1+\delta)e^T & (1+m-5\delta)(1+\delta)Je^T \\ (5\delta-1-m)(1+\delta)e & (\delta^2-2(2+m)\delta-1)[Je] & (\delta^2-2(2+m)\delta-1)[e] \\ (5\delta-1-m)(1+\delta)Je & (\delta^2-2(2+m)\delta-1)[e] & -(\delta^2-2(2+m)\delta-1)[Je] \end{pmatrix}.$$

In particular, $\theta^{\delta,k}_{\varepsilon,m}$ is never a G_2-instanton.

The main result on the heterotic G_2 system in this contact Calabi–Yau setting is the following [LSE23, cf. Theorem 1].

Theorem 5.3.4. *Let $\pi : M^7 \to V$ as in Definition 5.1.2 be a contact Calabi–Yau 7-manifold. Let A be the pullback of the Levi-Civita connection of the Calabi–Yau metric on V, defined on $E = \pi^*TV$.*

For all $\alpha' > 0$ there exist $\varepsilon = \varepsilon(\alpha') > 0$, $k = k(\alpha') > 0$, with $k(\alpha') \to \infty$ and $\varepsilon(\alpha') \to 0$ as $\alpha' \to 0$, and $\delta, m \in \mathbb{R}$ so that if M is endowed with the G_2-structure φ_ε as in (5.2.1), the connection $\theta^{\delta,k}_{\varepsilon,m}$ in Definition 5.3.1 on TM and the connection A on E, then we have a solution to the heterotic G_2 system, except that $\theta^{\delta,k}_{\varepsilon,m}$ is never a G_2-instanton but instead satisfies

$$|R^{\delta,k}_{\varepsilon,m} \wedge \psi_\varepsilon|_{g_\varepsilon} = \mathcal{O}(\alpha')^2 \quad \text{as } \alpha' \to 0. \tag{5.3.3}$$

Concretely, three separate regimes are presented in [LSE23, §4.4] of choices of the parameters $\varepsilon, k, \delta, m$ so that the conclusion of Theorem 5.3.4 holds for any positive α' sufficiently close to 0.

Case 1. $\delta \in \mathbb{R} \setminus \{0, -1\}$, $m = \delta - 1$, $k^2 = (\alpha')^{-3}$, $\varepsilon^2 = \dfrac{8}{\delta^2(1+\delta)^2}(\alpha')^5$.

Case 2. $\delta = 0$, $m < -1$, $k = (\alpha')^{-3}$, $\varepsilon^2 = -\dfrac{8}{(1+m)(1+3(\alpha')^3)}(\alpha')^8$.

Case 3. $\delta = -1$, $m > -2$, $k = (\alpha')^{-3}$, $\varepsilon^2 = \dfrac{8}{(2+m)(4-3(\alpha')^3)}(\alpha')^8$.

We shall return to the examples in Theorem 5.3.4 at the end of this section to understand in what sense the condition (5.3.3) gives "approximate" G_2-instantons and thus approximate solutions to the heterotic G_2 system.

5.4 Approximate coupled G_2-instantons and generalized Ricci curvature

We return to the general setting of 7-manifolds with integrable G_2-structures. Given the relationship between solutions of the heterotic G_2 system, coupled G_2-instantons and the vanishing of the generalised Ricci curvature as seen in Section 4.3, and based on the results in Theorem 5.3.4, we are motivated to define a suitable notion of approximate G_2-instantons, and then to show that this leads to an appropriate sense of both approximate coupled G_2-instantons and approximate generalised Ricci-flatness.

Given this goal, we propose the following definition of approximate G_2-instantons in our context.

Definition 5.4.1. *Suppose that for a sequence of non-zero real numbers $\alpha' \to 0$ we have the following data.*

Let (M^7, φ) be a 7-manifold endowed with an integrable G_2-structure with induced metric g, dual 4-form ψ and torsion 3-form H. Let $P \to M$ be a principal K-bundle over M, where the Lie algebra $\mathfrak{k}$ is endowed with an α'-independent, non-degenerate, bilinear, symmetric pairing $\langle \cdot, \cdot \rangle : \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}$. Let $\theta \in \Omega^1(P, \mathfrak{k})$ define a connection on P

with curvature F_θ and recall the induced connection $\nabla^{\theta,+}$ (cf. (2.2.4)). Suppose finally that the heterotic Bianchi identity is satisfied:

$$dH = \alpha' \langle F_\theta \wedge F_\theta \rangle. \tag{5.4.1}$$

We say that the connections θ are α'-approximate G_2-instantons *if*

$$\Big| F_\theta \wedge \psi \Big|_g = \mathcal{O}(\alpha')^2 \qquad \text{and} \qquad \Big| \nabla^{\theta,+} F_\theta \wedge \psi \Big|_g = \mathcal{O}(\alpha')^2, \tag{5.4.2}$$

as $\alpha' \to 0$.

Remark 5.4.2. Note that if φ is integrable, then ∇^+ preserves φ and hence ψ, and so

$$\nabla^{\theta,+}(F_\theta \wedge \psi) = \nabla^{\theta,+} F_\theta \wedge \psi. \tag{5.4.3}$$

In particular, if θ is a G_2-instanton then $\nabla^{\theta,+} F_\theta \wedge \psi = 0$. Hence G_2-instantons give trivial examples of α'-approximate G_2-instantons. ◯

In general, the first condition in (5.4.2), which is the one considered in [LSE23] (see Theorem 5.3.4), does not imply the second. Definition 5.4.1, therefore, gives a more robust notion of approximate G_2-instanton, which appears to be more natural, at least in our context.

Remark 5.4.3. As it is well-known, G_2-instantons can bubble, meaning that their curvature can blow up pointwise in a family. To avoid this, it is natural to impose that their curvature stays bounded (pointwise), so we can ask the same of our approximate G_2-instantons. In this setting, we can achieve our main results concerning approximate solutions. ◯

In the following sense, we now show that α'-approximate G_2-instantons yield approximate coupled G_2-instantons.

Theorem 5.4.4. *Suppose that we have an α'-approximate G_2-instantons θ on a principal K-bundle over (M^7, φ) with integrable G_2-structure φ and torsion H satisfying (5.4.1) as in Definition 5.4.1. Recall $\rho(\varphi, H)$ given in Definition 4.3.1 and $\mathbb{F} \wedge \mathbb{F}^\dagger$ given in Lemma 2.2.1.*

If the curvature F_θ of θ is bounded as $\alpha' \to 0$, then (φ, H, θ) give approximate solutions to the coupled G_2-instanton equation (4.3.2) in Definition 4.3.4 in the following sense as $\alpha' \to 0$:

$$\begin{aligned} |\rho(\varphi, H) + \langle F_\theta, (F_\theta \lrcorner \varphi)^\# \rangle|_g &= \mathcal{O}(\alpha')^2, \\ |(\nabla^{\theta,+} F_\theta) \lrcorner \varphi|_g &= \mathcal{O}(\alpha')^2, \\ |[F_\theta \lrcorner \varphi,] - \mathbb{F} \wedge \mathbb{F}^\dagger \lrcorner \varphi|_g &= \mathcal{O}(\alpha')^2, \\ dH - \alpha' \langle F_\theta \wedge F_\theta \rangle &= 0. \end{aligned} \tag{5.4.4}$$

Proof. Since φ is integrable, $\rho(\varphi, H) = 0$ by Lemma 4.3.3. The first equation in (5.4.4) is then an immediate consequence of the boundedness of F_θ and the first condition in (5.4.2) of α'-approximate G_2-instantons. The second equation in (5.4.4) is precisely the second condition in (5.4.2). The fourth equation in (5.4.4) is satisfied by assumption. We are, therefore, only left with the third equation in (5.4.4).

The first term in the third equation is of order $\mathcal{O}(\alpha')^2$ by the first condition in (5.4.2). In the proof of Theorem 2.3.2, we saw locally we can write $\mathbb{F} \wedge \mathbb{F}^\dagger$ as:

$$-\sum_j \Big(e_j \lrcorner \langle \cdot, \cdot \rangle^{-1}(\zeta_l \lrcorner F_\theta)\Big) \wedge \Big(e_j \lrcorner (\zeta_k \lrcorner F_\theta)\Big) \otimes \zeta^k \otimes \zeta_l = -F^l \lrcorner^1 F_k,$$

where $\{e_j\}$ form a local orthonormal frame on M^7 and $\{\zeta_j\}$ give an orthonormal basis for the Lie algebra of K. By Lemma 6.4.3 and the calculations in the proof, we deduce that $\pi_7(F_\theta) = 0$ forces $\pi_7(\mathbb{F} \wedge \mathbb{F}^\dagger) = 0$ and, moreover, there is a universal constant $C > 0$ so that

$$|\pi_7(\mathbb{F} \wedge \mathbb{F}^\dagger)|_g \leq C|F_\theta|_g |\pi_7(F_\theta)|_g.$$

The third equation in (5.4.4) now follows from the boundedness of F_θ and the first condition in (5.4.2). □

Now, as we have seen, taking G_2-instantons θ in Definition 5.4.1 leads to generalised Ricci-flatness because in this case, the following two terms, which are the components of the generalised Ricci curvature as in Theorem 1.5.5, must vanish:

$$d^{\theta *} F_\theta + 4 i_{\tau_1^\#} F_\theta - F_\theta \lrcorner H, \qquad \mathrm{Ric}^+ + \alpha' F_\theta \circ F_\theta + 4\nabla^+ \tau_1. \tag{5.4.5}$$

We now examine these terms for α'-approximate G_2-instantons.

Proposition 5.4.5. *Let θ be α'-approximate G_2-instantons over (M^7, φ) as in Definition 5.4.1. Then the curvature F_θ satisfies*

$$\Big| d^{\theta *} F_\theta + 4\tau_1 \lrcorner F_\theta - F_\theta \lrcorner H \Big|_g = \mathcal{O}(\alpha')^2 \quad \text{as } \alpha' \to 0.$$

Proof. Recall that, by Lemma 4.4.4, we have

$$\begin{aligned} d^{\theta *} F_\theta + 4\tau_1 \lrcorner F_\theta - F_\theta \lrcorner H = 6\tau_1 \lrcorner \pi_7 F_\theta + \frac{1}{3}\tau_0 \pi_7 F_\theta \lrcorner \varphi - 3\pi_7 F_\theta \lrcorner \tau_3 \\ + 3\sum_{j=1}^{7} e_j \lrcorner \pi_7 \nabla^{\theta,+}_{e_j} F_\theta. \end{aligned} \tag{5.4.6}$$

The result follows from the α'-approximate G_2-instanton condition (5.4.2). □

We now turn to the second term in (5.4.5).

Proposition 5.4.6. *Let (M^7, φ) be a 7-manifold with an integrable* G_2*-structure and θ be a connection on a principal K-bundle over M as in Definition 5.4.1 so that the heterotic Bianchi identity (5.4.1) is satisfied. Then*

$$\mathrm{Ric}^+ + \alpha' \, F_\theta \circ F_\theta + 4\nabla^+ \tau_1 = 3\alpha' \, F_\theta \circ \pi_7 F_\theta \tag{5.4.7}$$

In particular, if θ are α'-approximate G_2*-instantons as in Definition 5.4.1 with bounded curvature F_θ as $\alpha' \to 0$, then*

$$\left| \mathrm{Ric}^+ + \alpha' F_\theta \circ F_\theta + 4\nabla^+ \tau_1 \right|_g = \mathcal{O}(\alpha')^3 \quad \text{as } \alpha' \to 0. \tag{5.4.8}$$

Proof. Using Lemma (4.4.2), we have

$$\mathrm{Ric}^+_{ij} - \frac{1}{12}(dH)_{ab\mu i}\psi_{ab\mu j} + 4\nabla^+_i(\tau_1)_j = 0, \tag{5.4.9}$$

so it suffices to study the second term in (5.4.9) to obtain (5.4.7). Note that

$$(dH)_i \lrcorner \psi_j = \frac{1}{6}(dH)_{ab\mu i}\psi_{ab\mu j}.$$

This observation, together with the heterotic Bianchi identity (5.4.1), then implies that

$$\frac{1}{12}(dH)_{ab\mu i}\psi_{ab\mu j} = \frac{1}{2}(dH)_i \lrcorner \psi_j = \frac{\alpha'}{2}(F_\theta)^{\beta}{}_i{}^{\mu}(F_\theta)_{\beta}{}^{\nu\rho}\psi_{j\mu\nu\rho}.$$

Using the decomposition

$$F_\theta \lrcorner \psi = 2\pi_7 F_\theta - \pi_{14} F_\theta = 3\pi_7 F_\theta - F_\theta. \tag{5.4.10}$$

we deduce that

$$\frac{1}{12}(dH)_{ab\mu i}\psi_{ab\mu j} = \alpha' \left(-(F_\theta)^{\beta}{}_{\mu i}(F_\theta)_{\beta\mu j} + 3(F_\theta)^{\beta}{}_i{}^{\mu}\pi_7 F_{\theta\beta j\mu} \right). \tag{5.4.11}$$

Inserting (5.4.11) in (5.4.9) gives (5.4.7). The final result then follows from the condition (5.4.2) in Definition 5.4.1 of α'-approximate G_2-instantons, together with the assumption that F_θ is bounded as $\alpha' \to 0$. □

Combining Propositions 5.4.5 and 5.4.6, we immediately obtain the following result about approximate generalised Ricci flatness.

Theorem 5.4.7. *Let θ be an α'-approximate* G_2*-instanton on a principal K-bundle over (M^7, φ), endowed with an integrable* G_2*-structure with torsion form H satisfying the heterotic Bianchi identity (5.4.1), as in Definition 5.4.1. Suppose further that the curvature F_θ is bounded as $\alpha' \to 0$.*

Let $E = TM \oplus \mathrm{ad}P \oplus T^*M$ have the transitive Courant algebroid structure defined by the pair (H, θ) and generalized metric $\mathbf{G}_\varphi$ as in Example 1.2.4. If the divergence is given by

$$\mathrm{div} = \mathrm{div}^{V_+} - 2\langle 4\tau_1, \cdot \rangle, \tag{5.4.12}$$

then we obtain approximate generalized Ricci flatness *in the sense of*

$$\left|\mathrm{GRic}\left(\mathbf{G}_\varphi, \mathrm{div}^\varphi\right)\right|_{g_\varphi} = \mathcal{O}(\alpha')^2, \tag{5.4.13}$$

as $\alpha' \to 0$.

5.5 Approximate solutions on contact Calabi–Yau 7-manifolds

In this subsection, we revisit the setting of contact Calabi–Yau 7-manifolds $\pi : M^7 \to V$ endowed with the G_2-structures φ_ε as in Definition 5.1.2. Recall that φ_ε are integrable G_2-structures with torsion H_ε. Recall also the connections $\theta^{\delta,k}_{\varepsilon,m}$ on TM in Definition 5.3.1 and that we can define the bundle $E = \pi^*TV$ and endow it with the pullback A of the Levi-Civita connection from the Calabi–Yau 3-orbifold V. This leads us to the following definition.

Definition 5.5.1. *Let $K = \mathrm{G}_2 \times \mathrm{SU}(3)$ and identify G_2 and $\mathrm{SU}(3)$ with their standard matrix representations (acting on $\mathbb{R}^7$ and $\mathbb{C}^3$ respectively). We can define a principal K-bundle P over M whose natural associated vector bundle is $TM \oplus E$. We can then define a connection θ on P using the pair of connections $\theta^{\delta,k}_{\varepsilon,m}$ and A.*

We also endow the Lie algebra $\mathfrak{k}$ of K with the pairing $\langle\cdot,\cdot\rangle : \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}$ with respect to the splitting $\mathfrak{k} = \mathfrak{g}_2 \oplus \mathfrak{su}(3)$:

$$\langle (X_1, Y_1), (X_2, Y_2)\rangle = -\mathrm{tr}(X_1X_2) + \mathrm{tr}(Y_1Y_2). \tag{5.5.1}$$

Note that $\langle\cdot,\cdot\rangle$ is non-degenerate, bilinear and symmetric.

Theorem 5.3.4 then states that if we are given *any* sequence $\alpha' \to 0$, then we can choose positive parameters $\varepsilon = \varepsilon(\alpha')$, $k = k(\alpha')$ and real parameters δ, m *independent of* α' so that $H = H_\epsilon$ and θ given in Definition 5.5.1 satisfy the heterotic Bianchi identity (5.4.1) with $\langle\cdot,\cdot\rangle$ as in (5.5.1) (which we notice is α'-independent). Moreover, since A is a G_2-instanton by [LSE23, Lemma 3.1], (5.3.3) in Theorem 5.3.4 also gives that the curvature F_θ of θ satisfies the first condition in (5.4.2).

Theorems 5.4.4 and 5.4.7 imply that if θ also satisfies the second condition in (5.4.2) and has bounded curvature as $\alpha' \to 0$, then the "approximate" solutions to the heterotic G_2 system given by Theorem 5.3.4 give rise to approximate coupled G_2-instantons and an approximate generalised Ricci-flat connection on the associated Courant algebroid. This is what we now show.

Theorem 5.5.2. *Let M^7 be a contact Calabi–Yau 7-manifold as in Definition 5.1.2. Suppose we are given any sequence of positive numbers $\alpha' \to 0$. Let $\varepsilon = \varepsilon(\alpha') > 0$, $k = k(\alpha') > 0$, $\delta, m \in \mathbb{R}$ be the associated parameters given by Theorem 5.3.4 and let M be* *endowed the integrable* G_2*-structure* φ_ε *given in* (5.2.1) *with torsion* $H = H_\varepsilon$*. Let* $P, \theta, \langle\cdot,\cdot\rangle$ *be the principal* K*-bundle, connection and pairing on* $\mathfrak{k}$ *given in Definition 5.5.1.*

Then the heterotic Bianchi identity (5.4.1) *is satisfied, and* θ *are* α'*-approximate* G_2*-instantons in the sense of Definition 5.4.1 with bounded curvature as* $\alpha' \to 0$*. Hence,* $(\varphi_\varepsilon, H_\varepsilon, \theta)$ *are* α'*-approximate coupled* G_2*-instantons in the sense of* (5.4.4) *and the Courant algebroid* $Q = TM \oplus \mathrm{ad}P \oplus T^*M$ *with structure* (H, θ) *and divergence as in* (5.4.12) *has a torsion-free* V_+*-compatible generalized connection with generalized Ricci curvature with norm of order* $\mathcal{O}(\alpha')^2$ *as* $\alpha' \to 0$*.*

Proof. As explained before the statement, we need only show that the curvature F_θ is bounded and that the second condition in (5.4.2) holds. By (5.4.3), we see that this second condition is equivalent to

$$\left|\nabla^{\theta,+}(F_\theta \wedge \psi_\varepsilon)\right|_{g_\varepsilon} = \mathcal{O}(\alpha')^2 \quad \text{as } \alpha' \to 0. \tag{5.5.2}$$

We already remarked that the connection A on E is a G_2-instanton and is pulled back from V. Hence, its curvature F_A, and the norm of F_A, are α'-independent since the metric g_ε on M is α'-independent when restricted to basic forms by (5.2.2). Moreover, $F_A \wedge \psi_\varepsilon = 0$ and so (5.5.2) is trivially satisfied for $F_\theta = F_A$.

Given this discussion and the definition of θ, it now suffices to show that the curvature $R^{\delta,k}_{\varepsilon,m}$ of $\theta^{\delta,k}_{\varepsilon,m}$ has bounded norm as $\alpha' \to 0$ and satisfies

$$\left|\nabla^{\theta^{\delta,k}_{\varepsilon,m},+}(R^{\delta,k}_{\varepsilon,m} \wedge \psi_\varepsilon)\right|_{g_\varepsilon} = \mathcal{O}(\alpha')^2 \quad \text{as } \alpha' \to 0. \tag{5.5.3}$$

In [LSE23, Proposition 3.17], the curvature $R^{\delta,k}_{\varepsilon,m}$ was written in terms of the local orthonormal coframe given in Definition 5.2.2 as:

$$R^{\delta,k}_{\varepsilon,m} = F_A + \frac{1}{2}k\varepsilon^2(1-\delta+m)\omega\mathbf{I} + \frac{k^2\varepsilon^2}{4}\mathbf{Q}^\delta_m,$$

where F_A is the curvature of the connection A as above, $\mathbf{I}$ is given in (5.2.7) and $\mathbf{Q}^\delta_m$ depends only on δ and m (and the local coframe), so is independent of α'. Since all the terms except F_A involve at least a factor of $k\varepsilon$, which tends to zero as $\alpha' \to 0$, we deduce that

$$\left|R^{\delta,k}_{\varepsilon,m} - F_A\right|_{g_\varepsilon} \to 0 \quad \text{as } \alpha' \to 0.$$

Since we already established that F_A has bounded norm as $\alpha' \to 0$, the same must be true for $R^{\delta,k}_{\varepsilon,m}$.

We already saw the expression for $R^{\delta,k}_{\varepsilon,m} \wedge \psi_\varepsilon$ in (5.3.2). Note that the matrix $\mathbf{M}^\delta_m$ is again independent of α'. Recall that $\theta^{\delta,k}_{\varepsilon,m}$ is given in (5.3.1) and note that taking $\delta = 1$, $m = 0$ and $k = 1$ in this expression leads to the Bismut connection ∇^+, and instead taking $\delta = m = 0$ and $k = 1$ yields the Levi-Civita connection. Altogether, we see that taking derivatives using $\nabla^{\theta^{\delta,k}_{\varepsilon,m},+}$ cannot decrease the powers of k and ε that already appear in (5.3.2). Therefore, the norm of $\nabla^{\theta^{\delta,k}_{\varepsilon,m},+}(R^{\delta,k}_{\varepsilon,m} \wedge \psi_\varepsilon)$ must have at least the same order as $\alpha' \to 0$ as the norm of $R^{\delta,k}_{\varepsilon,m} \wedge \psi_\varepsilon$. Since we are already given that this latter quantity is of order $\mathcal{O}(\alpha')^2$ as $\alpha' \to 0$ by (5.3.3), we deduce that (5.5.3) holds as desired. $\square$

Remark 5.5.3. Theorem 5.5.2 shows that it is justified to say that the results in [LSE23], summarised in Theorem 5.3.4, indeed lead to "approximate" solutions to the heterotic G_2 system. $\bigcirc$

6 Generalized Ricci flatness and coupled equations for geometrical structures

This chapter revisits Problem 1 and Problem 2, initially explored in Chapter 2, employing an alternative notion of instanton. We reformulate these problems using this new approach (cf. Problem 3 and Problem 4) and provide the solutions for them. Additionally, we examine the equivalence between G_2 and $\mathrm{SU}(m)$ structures under this alternative framework, as presented firstly in Chapter 2 and Chapter 4. Using the theory developed in this chapter, we will extend our discussion exploring these proposed problems within the context of Spin(7)-structures in Chapter 7.

This chapter is structured as follows: In Section 6.1, we introduce an alternative notion of instanton (cf. Definitions 6.1.1, 6.1.3, and 6.1.4), which is based on the existence of a specific 4-form that defines the manifold's geometric structure and characterizes the Lie algebra (within the space of 2-forms) as an eigenvalue of an invariant map associated with this 4-form. We conclude the section by exploring key examples, including G_2, Spin(7), $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, and contact-instantons, and demonstrate the equivalence of this alternative notion with the one considered in Chapter 2.

In Section 6.2, we delve into the fundamental rationale behind the consideration of instantons as outlined in Section 6.1: the elementary derivation (as a direct derivative of the instanton condition) of the Yang-Mills equations, pivotal in understanding the generalized Ricci curvature (as demonstrated in Theorem 1.5.5). We initially present the first version of these equations in Lemma 6.2.1, followed by their refined formulation derived from the former under additional conditions. This sophisticated version appears in Theorem 6.2.3 for computing the generalized Ricci curvature.

In Section 6.3, we investigate generalized Ricci curvature within our alternative notion of instantons. Here, we explore geometrical structures which admit compatible connections with totally skew-symmetric torsion (as reviewed in Appendix B), utilizing them to derive a formula closely resembling the final term for the generalized Ricci curvature in Theorem 1.5.5 (cf. Lemma 6.3.1). Leveraging the heterotic Bianchi identity, we rewrite this formula in the way it appears in the expression for the generalized Ricci curvature (Corollary 6.3.2). We apply several technical lemmas to identify the conditions necessary for a structure to exhibit generalized Ricci flatness, as encapsulated in Theorem 6.3.6. This not only resolves Problem 3 (a reformulation of Problem 2) but also recovers the results observed for the G_2 case in Chapter 4.

In Section 6.4, we delve into the coupled instanton equations using the bundle $TM \oplus \mathrm{ad}P$ and the connection D, as introduced in Chapter 2, albeit with the alternative approach to instantons. Key results include Proposition 6.4.2, which provides a characterization of coupled instanton equations under this approach, and Theorem 6.4.4, demonstrating that solutions to the gravitino equations in this framework imply solutions to the coupled instanton equations. This resolves the proposed Problem 4, a reformulation of Problem 1.

Finally, in Section 6.5, we explore the concept of semi-instantons (a weak version of instanton) and their implications for the behaviour of the Yang-Mills equation, as discussed in Section 6.1. This inquiry was sparked by a finding in [GFGM23], revealing that the Yang-Mills equation can be derived using a weak notion of instanton in the $\mathrm{SU}(m)$ case. Here, we establish more precise conditions for this phenomenon to occur in a broader context, and recover this result of [GFGM23] with the main theorem in the section, Theorem 6.5.2. Concluding the section, we pose Problem 5, suggesting that a specific hypothesis of Theorem 6.5.2 should be satisfied in a more general setting, and propose a further investigation into $\mathrm{Sp}(k)$-instantons, a case where the problem can be tested.

6.1 G-instantons and the Yang-Mills equation

In Chapter 2, we have considered a spin manifold endowed with a non-vanishing spinor η which induces on the manifold a G-structure for G the stabilizer of η in $\mathrm{Spin}(n)$. In this context, we say that a connection θ (in a vector or principal bundle over M) is a *G-instanton* if its curvature satisfies:

$$F_\theta \cdot \eta = 0 \tag{6.1.1}$$

In particular, we have proved that the Killing spinor equations,

$$\nabla^+ \eta = 0, \qquad F_\theta \cdot \eta = 0, \qquad \left(\not{\nabla}^{1/3} - \tfrac{1}{2}\zeta\right) \cdot \eta = \lambda \eta, \qquad dH = \langle F_\theta \wedge F_\theta \rangle$$

imply on generalized Ricci flatness on the transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ (cf. Proposition 2.1.6 for details). In particular, generalized Ricci flatness is equivalent to the equations (cf. Theorem 1.5.5 and Proposition 2.1.6),

$$\begin{aligned} \mathrm{Ric}^+ + F_\theta \circ F_\theta + \nabla^+ \zeta &= 0, \\ d^{\theta *} F_\theta + i_{\zeta^\#} F_\theta - F_\theta \lrcorner H &= 0. \end{aligned}$$

We will propose a non-spinorial description for instantons where the first equation is a consequence of the heterotic Bianchi identity and the existence of the connection $\nabla^+ = \nabla^g + \frac{1}{2} g^{-1} H$ compatible with the structure. The second equation is a consequence of an alternative notion of instantons (based mainly in [Car98, Don96], see also [FSE19]).

This alternative notion holds significance as it is equivalent to spinorial descriptions in cases such as G_2, $\mathrm{Spin}(7)$, and $\mathrm{SU}(m)$ instantons. Moreover, its applicability extends to structures lacking spinorial descriptions, such as contact structures (as discussed in [PSE20]) and $\mathrm{Sp}(k)$-instantons (though yet unexplored in this context, [FSE23] provides a natural way for defining it. It is interesting in the light of Problem 5, elaborated in Section 6.5).

A general notion of instanton, which we will use here, is related to the Lie algebra (inside the space of 2-forms canonically identified with $\mathfrak{so}(n)$) of the geometrical structure in question. It was firstly considered by [Car98], Cf. also [FSE19, LM22]. Our context will be a manifold M^n endowed with a G-structure (for $G \subset \mathrm{SO}(n)$ being a closed Lie subgroup) such that the group G is the stabilizer of some 4-form $\psi_0 \in \Lambda^4(\mathbb{R}^n)^*$

$$G = \{g \in \mathrm{SO}(n) : g^*\psi_0 = \psi_0\}$$

so, the G-structure is equivalent to a 4-form $\psi \in \Omega^4(M)$ punctually modelled by ψ_0.

Definition 6.1.1 (Instanton 4-form)**.** *Let M^n be a manifold with a $N(H)$-structure*[1] *$\psi \in \Omega^4(M)$ (for $H < \mathrm{SO}(n)$ being a closed subgroup), then ψ is called an H-*instanton form *if it defines the Lie algebra $\mathfrak{h} \subset \mathfrak{so}(n) = \Lambda^2$ as:*

$$\mathfrak{h} = \left\{\beta \in \Omega^2 : \beta \lrcorner \psi = -\beta\right\} \leq \mathfrak{so}(n) = \Omega^2 \tag{6.1.2}$$

Naturally, the instanton form defines a subspace of vector-valued 2-forms.

We will call the invariant operator map defined below within the space of 2-forms as the *instanton map*:

$$\begin{array}{rccc} \Xi: & \Omega^2(M) & \to & \Omega^2(M) \\ & \beta & \mapsto & \beta \lrcorner \psi \end{array} \tag{6.1.3}$$

so instanton form ψ makes $\mathfrak{h}$ (contained in) an eigenspace of the instanton map with eigenvalue -1 (the value -1 is just for convention, but, up to rescaling for ψ, it could be every non-zero real value).

Remark 6.1.2 (Simple Lie Groups)*.* For our purposes, (6.1.2) in the definition of instanton form is more than necessary. We can impose

$$\mathfrak{h} \subset \{\beta \in \Omega^2 : \beta \lrcorner \psi = -\beta\}.$$

Because of that, if the group H is simple and ψ non-degenerate, then $\mathfrak{h}$ is irreducible and necessarily $\mathfrak{h}$ is contained in a unique eigenspace of instanton map. Hence, the relation above $\mathfrak{h} \subset \{\beta \in \Omega^2 : \beta \lrcorner \psi = -\beta\}$ holds up to a rescaling of ψ. So, for our purposes, it is

[1] Let G be a group and $H \leq G$ a subgroup, then the *normalizer* of H is $N(H) = \{g \in G : gHg^{-1} = H\}$.

enough for H to be simple and ψ non-degenerate (in the sense of the instanton map being an isomorphism between 2-forms). All of our cases of interest occur for H being simple. ○

Now, we can define the notion of instanton as a connection in which curvature 2-form is in the Lie algebra as above. Precisely:

Definition 6.1.3 (H-instanton). *Let ψ be an H-instanton form on M (in particular, it defines an $N(H)$-structure on the manifold), then a connection ∇ on a vector bundle $E \to M$ is said to be an H-*instanton *if* [2]

$$R_\nabla \in \mathfrak{h} \tag{6.1.4}$$

*where $R_\nabla \in \Omega^2(M, \mathrm{End}(E))$ is the curvature 2-form. More general, if $\theta \in \Omega^1(P, \mathfrak{g})$ is a connection form on a principal G-bundle $P \to M$, it is called an H-*instanton *if*

$$F_\theta \in \mathfrak{h} \tag{6.1.5}$$

where $F_\theta \in \Omega^2(P, \mathfrak{g})^G \cong \Omega^2(M, \mathrm{ad}P)$. So the instanton condition is characterized by the 2-form part of the curvature living in the Lie algebra inside the space of 2-forms.

We also define the weak notion of semi-instanton (which will be explored in Section 6.5) as follows:

Definition 6.1.4 (H-semi-instanton). *Let ψ be an H-instanton form on M (in particular, it defines an $N(H)$-structure on the manifold), then a connection ∇ on a vector bundle $E \to M$ is said to be an H-*semi-instanton *if*

$$R_\nabla \in \mathfrak{n}(\mathfrak{h}) \geq \mathfrak{h} \tag{6.1.6}$$

where $R_\nabla \in \Omega^2(M, \mathrm{End}(E))$ is the curvature 2-form. Analogously, for general connection forms on principal bundles.

Now, we provide some essential examples which we will through the text:

Example 6.1.5 (G_2-instantons). The group G_2 is normal in $\mathrm{SO}(7)$, i.e., $N(\mathrm{G}_2) = \mathrm{G}_2$, so the notion of G_2-instantons makes sense for G_2-structures. The group G_2 is the stabilizer of the 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ as in (3.1.4). However, in the seven dimensions, the Hodge dual of a 3-form is a 4-form, so we have $\psi_0 = *\varphi_0 \in \Lambda^4(\mathbb{R}^7)^*$ as in (3.1.5). It happens that G_2 is also the stabilizer of the form ψ_0, so on a manifold (M^7, φ) with G_2-structure, the 4-form $\psi \in \Omega^4(M)$ is an instanton form by because (3.3.2).

[2] We are using the simplified notation $\mathfrak{h}$ for $\mathfrak{h} \otimes \mathrm{End}(E)$.

Furthermore, due to (3.3.2), the Lie algebra (which is simple in this case) can also be characterized by

$$\mathfrak{g}_2 = \left\{\beta \in \Omega^2 : \beta \wedge \psi = 0\right\} = \left\{\beta \in \Omega^2 : \beta \lrcorner \varphi = 0\right\}.$$

We have noticed that G_2 is also the stabilizer of the spinor $\eta_0 = (1, 0, \cdots, 0) \in \Delta_7 \cong \mathbb{R}^8$. However, as we have pointed in (3.5.5), the Lie algebra $\mathfrak{g}_2$ is characterized as

$$\beta \in \mathfrak{g}_2 \iff \beta \wedge \psi = 0 \iff \beta \lrcorner \psi = -\beta \iff \beta \cdot \eta = 0$$

so the two notions of instantons in G_2-structures are equivalent seeing G_2 in $\mathrm{SO}(7)$ and in $\mathrm{Spin}(7)$. △

Example 6.1.6 (Spin(7)-instantons)**.** The group Spin(7) is a normal subgroup of SO(8), i.e., $N(\mathrm{Spin}(7)) = \mathrm{Spin}(7)$, so the notion of Spin(7)-instantons makes sense for Spin(7)-structures. As we will see later in Chapter 7, the group Spin(7) is the stabilizer of the self-dual 4-form $\Omega_0 \in \Lambda^4(\mathbb{R}^8)^*$ as in (7.1.2). In this way, a Spin(7)-structure (M^8, Ω) defines an Spin(7)-instanton form on M simply by the Ω itself because the characterization of $\mathfrak{spin}(7)$ inside Ω^2 as in (7.1.5), which is irreducible. △

For G_2 and Spin(7) instantons, the concept of semi-instantons does not expand beyond the notion of instantons themselves. The Lie algebra is irreducible in these cases, so $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$. However, there are scenarios where this isn't the case.

Example 6.1.7 ($\mathrm{SU}(m)$-instantons)**.** Fix $m \geq 3$, so the group $\mathrm{U}(m)$ is the stabilizer of the standard Hermitian 2-form $\omega_0 \in \Lambda^2(\mathbb{R}^{2m})^*$ in $\mathbb{R}^{2m}$ in $\mathrm{SO}(2m)$:

$$\omega_0 = e^{12} + e^{34} + \cdots + e^{2m-1} \wedge e^{2m}$$

and consider the subgroup $\mathrm{SU}(m) \subset \mathrm{U}(m)$, then we have $N(\mathrm{SU}(m)) = \mathrm{U}(m)$, so the notion of $\mathrm{SU}(m)$-instantons makes sense in $\mathrm{U}(m)$-structures (i.e., almost Hermitian Riemannian manifolds (M^{2m}, g, ω)).

The $\mathrm{SU}(m)$-instanton form here $\psi = \frac{1}{2}\omega \wedge \omega$ because of the characterization of 2-forms as in cf. Chapter 7. Note that here, the notion of instantons is equivalent in the spinorial and via ω because

$$\beta \in \mathfrak{su}(m) \iff \beta \lrcorner \psi = -\beta \iff \beta \cdot \eta = 0$$

for η the spinorial description (to be explored in Chapter 7); however, the spinorial description only makes sense for $\mathrm{SU}(m)$-structures since via instanton-form we just need almost-Hermitian structures. △

Example 6.1.8 (SU(2)-instantons)**.** In dimension 4, it happens that $N(\mathrm{SU}(2)) = \mathrm{SO}(4)$, so the notion of SU(2)-instantons makes sense for all oriented Riemannian four-dimensional manifolds (it is not necessary to be almost Hermitian). Consider $\psi = \mathrm{vol}_M$ in this case, and we have the instanton-form. The study of instantons in four dimensions first motivated defining instantons as we have done due to the work by Donaldson [DK90] and generalized in [DT98, DS11]. △

Example 6.1.9 (Sp(k)-instantons)**.** The example of Sp(k)-instantons is, in some sense, similar to SU(m)-instantons. Firstly $\mathrm{Sp}(k) \subset \mathrm{SO}(4k)$ and its normalizer $N(\mathrm{Sp}(k)) = \mathrm{Sp}(k)\mathrm{Sp}(1)$ (cf. [McI91, pg. 11]). So Sp(k)-instantons make sense in Sp(k)Sp(1)-structures. It happens Sp(k)Sp(1) is the stabilizer of some $\Omega_0 \in \Omega^4(\mathbb{R}^{4k})^*$ (cf. [FSE23]) and since $\mathfrak{sp}(k)$ is irreducible, up to rescaling the Sp(k)Sp(1)-structure $\Omega \in \Omega^4(M^{4k})$ we obtain an instanton form. △

Example 6.1.10 (Contact-instantons)**.** Identifying $\mathrm{U}(m) \cong \mathrm{U}(m) \times 1$ inside $\mathrm{SO}(2m+1)$, so contact manifolds are U(m)-structures in $(2m+1)$-dimensional manifolds, known as contact structures. In this way, we have the transversal Hermitian form, and we can normally define the instanton form as

$$\psi = \frac{1}{2}\omega \wedge \omega \Rightarrow *\psi = \frac{1}{(n-2)!}\eta \wedge \omega^{n-2}$$

for $\eta \in \Omega^1(M)$ the contact form. These instantons were considered in [PSE20], particularly in seven dimensions and the relation with G_2-structures. △

To finish the section, we will introduce some concepts that emerge from the existence of an instanton form.

Definition 6.1.11. *Let $\psi \in \Omega^4(M)$ be an instanton form on M^n and let*

$$\Omega^3(M) = \Omega^3_{k_1}(M) \oplus \cdots \oplus \Omega^3_{k_p}(M),$$

the orthogonal decomposition of 3-forms into irreducible components due to the instanton form and denote $\pi_{k_j} : \Omega^3(M) \to \Omega^3_{k_j}(M)$ the respective projection, define

$$\tau^{k_j} := \pi_{k_j}(d^*\psi) \in \Omega^3_{k_j}(M). \tag{6.1.7}$$

Then τ^{k_j} is called the associated torsion form *in the irreducible component $\Omega^3_{k_j}(M)$.*

Example 6.1.12 (G_2-structures)**.** For G_2-structures, we have $d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3$ and consequently

$$d^*\psi = *d\varphi = \tau_0\varphi - 3\tau_1 \lrcorner \psi + \tau_3$$

so, the torsion forms in the context of instanton form relates to the torsion forms for G_2-structures as follows (considering the decomposition $\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$)

$$\tau^1 = \tau_0\varphi, \qquad \tau^7 = -3\tau_1 \lrcorner \psi, \qquad \tau^{27} = \tau_3.$$

Even though we use the name ‘torsion forms’ in two contexts, they are related. △

Another important notion is the operator $\mathbf{H}$ as we have considered in Appendix B and in Theorem 3.4.4. See Appendix A.2 for the notations $\lrcorner^k$ about partial contractions.

Definition 6.1.13. *Given an instanton form $\psi \in \Omega^4(M)$, we define the* associated flux operator $\mathbf{H} : \Omega^3(M) \to \Omega^3(M)$ *by the (equivalent) formulas*[3]

$$\mathbf{H}(\gamma) \coloneqq \frac{1}{4}\gamma^{ij}{}_a\psi_{ijbc}e^{abc} = \gamma \lrcorner^2 \psi = (-1)^{n+1} * (\gamma \lrcorner^1 *\psi). \tag{6.1.8}$$

If $\mathbf{H}$ is an isomorphism, then the instanton form is called determinant.

Example 6.1.14. In G_2-structures the instanton form is always determinant. In fact, by Theorem 3.4.4, we have

$$a_1 = 6, \qquad a_7 = 3, \qquad a_{27} = -1.$$

so $\mathbf{H}$ is an isomorphism. △

Note that this map is invariant, consequently, it decomposes the space of 3-forms into eigenspaces and since eigenspaces are representations, then every irreducible component $\Omega^3_{k_j}$ is contained in a unique eigenspace. In particular, if ψ is determinant, every eigenvalue is non-zero. With this and the torsion forms, we introduce the following natural 3-form for a determinant instanton form.

Definition 6.1.15. *Given a determinant instanton form $\psi \in \Omega^4(M)$ and let*

$$\Omega^3(M) = \Omega^3_{k_1}(M) \oplus \cdots \oplus \Omega^3_{k_p}(M),$$

the orthogonal decomposition of 3-forms into irreducible components due to the instanton form, we define the associated flux 3-form $H \in \Omega^3(M)$ *by the expression*

$$H = \sum_{j=1}^{p} \frac{1}{a_{k_j}} \tau^{k_j}, \tag{6.1.9}$$

where $\tau^{k_j} \in \Omega^3_{k_j}(M)$ is the associated torsion form in $\Omega^3_{k_j}(M)$ and $a_{k_j} \in \mathbb{R}$ the eigenvalue of the flux operator $\mathbf{H} : \Omega^3(M) \to \Omega^3(M)$ in the component $\Omega^3_{k_j}(M)$.

The necessity of this mapping, as outlined in Appendix B, arises from its function in defining a method for determining the torsion of a compatible connection with skew-symmetric torsion when such a connection exists. This procedure was used in Theorem 3.4.4. We will further explore this concept in contexts with a compatible connection with skew-symmetric torsion. It's important to note that for a determinant instanton form if the connection exists, it will be unique, as established in Theorem B.3.6 in Appendix B.

[3] cf. Appendix A.

6.2 Yang-Mills equation with torsion

A fundamental consequence of our chosen definition for the instanton form lies in its pivotal role in deriving a Yang-Mills equation with torsion, akin to the one presented in Theorem 1.5.5. Let $\psi \in \Omega^4(M)$ denote the instanton form on a manifold, and θ represents an H-instanton. In this context, the curvature 2-form of θ adheres to the following relation:

$$F_\theta \lrcorner \psi = -F_\theta \iff F_\theta \wedge *\psi = - * F_\theta.$$

Taking the exterior covariant derivative of this expression and using the Bianchi identity $d^\theta F_\theta = 0$, we then have

$$\begin{aligned} d^{\theta *} F_\theta &= (-1)^{n+1} * d^\theta * F_\theta = (-1)^n * d^\theta (F_\theta \wedge *\psi) = (-1)^n * \left(\cancelto{0}{d^\theta F_\theta} \wedge *\psi + F_\theta \wedge d * \psi\right) \\ &= - * (F_\theta \wedge * * d * \psi) = (-1)^n * (F_\theta \wedge *d^* \psi) = -F_\theta \lrcorner d^* \psi \end{aligned}$$

then we conclude the so-called torsion Yang-Mills equation:

Proposition 6.2.1. *Let $\psi \in \Omega^4$ an H-instanton form on a manifold M, if θ is an H-instanton on some bundle over M, then it satisfies the so-called* Yang-Mills equation with torsion*:*

$$d^{\theta *} F_\theta + F_\theta \lrcorner d^* \psi = 0 \tag{6.2.1}$$

where F_θ is the curvature 2-form of the connection θ.

If $d^* \psi = 0$, the above equation reduces to the classical Yang-Mills equation: $d^{\theta *} F_\theta = 0$. Because of that, (6.2.1) is called '*with torsion*'.

We aim to modify the torsion Yang-Mills above to obtain the first part of Theorem 1.5.5 for appropriate ζ and H. Since we are in the context of the existence of a $N(H)$-structure on the manifold, we can impose this structure to accept a unique compatible connection with totally skew-symmetric torsion $T \in \Omega^3(M)$ and with this define $H := T$ (as we have made for G_2-structures in Chapter 3, where this imposition was $\tau_2 = 0$). So, how about ζ? Since we have the instanton form $\psi \in \Omega^4(M)$ non-degenerate, we have an embedding

$$X \in \Omega^1 \mapsto X \lrcorner \psi \in \Omega^3$$

so $d^* \psi$ has a component of type $X \lrcorner \psi$ and this will be useful for us to define ζ.

To clarify some subsequent statements, let's classify the components of $\Omega^3(M)$. We are considering the impositions above. Firstly, we have the only component of Ω^3 isomorphic to Ω^1 given by

$$\Omega^3_n := \left\{ X \lrcorner \psi : X \in \Omega^1 \right\} \cong \Omega^1, \tag{6.2.2}$$

the respective torsion form τ^n will be denoted by $\tau_1 \lrcorner \psi$. The other components will be classified into two classes with relation to their eigenvalue in $\mathbf{H}$-operator: if it has the eigenvalue given by -1 or not

$$\mathcal{O}^{-1} = \bigoplus_{\substack{k_j \neq n,\\ a_{k_j}=-1}} \Omega^3_{k_j} \qquad \text{and} \qquad \mathcal{O}^{0} = \bigoplus_{\substack{k_j \neq n,\\ a_{k_j}\neq-1}} \Omega^3_{k_j} \tag{6.2.3}$$

this is because if $a_{k_j} \neq -1$, it will not make a difference in the theory, as we will see later. Note that this classification $\Omega^3 = \Omega^3_n \oplus \mathcal{O}^{-1} \oplus \mathcal{O}^0$ doesn't depend on the existence of a compatible connection with totally skew-symmetric torsion.

Example 6.2.2. In G_2-structures, we have decomposition $\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$ and in this case $\mathcal{O}^{-1} = \Omega^3_{27}$ and $\mathcal{O}^0 = \Omega^3_1$. The representation Ω^3_7 is precisely the vectorial representation, cf. Theorem 3.4.4. △

So, we can finally obtain the desired Yang-Mills equation as it appears in Theorem 1.5.5.

Theorem 6.2.3. *Let $\psi \in \Omega^4(M)$ be an H-instanton form on M^n, suppose Ω^1 is irreducible and $\Omega^1 \hookrightarrow \Omega^3$ only once. Decompose Ω^3 into irreducible components $\Omega^3_{k_j}$, along with $\mathcal{O}^0$ and $\mathcal{O}^{-1}$ as in (6.2.3):*

$$\Omega^3 = \Omega^3_{k_1} \oplus \cdots \oplus \Omega^3_{k_p} = \Omega^3_n \oplus \mathcal{O}^0 \oplus \mathcal{O}^{-1}.$$

If θ is an H-instanton *on some bundle over M, and its curvature 2-form F_θ satisfies*

$$F_\theta \lrcorner \tau^{k_j} = 0, \quad \forall k_j : \Omega^3_{k_j} \subset \mathcal{O}^0,$$

then F_θ satisfies the following Yang-Mills equation with torsion*:*

$$d^{\theta *} F_\theta + i_{\zeta^\#} F_\theta - F_\theta \lrcorner H = 0, \tag{6.2.4}$$

where $\zeta \in \Omega^1(M)$ is the so-called Lee form *for the instanton form $\psi \in \Omega^4(M)$, defined by*

$$\zeta := -\left(1 + \frac{1}{a_n}\right)\tau_1, \tag{6.2.5}$$

where a_n is the eigenvalue of $\mathbf{H}$ on the component Ω^3_n and $\tau_1 \in \Omega^1$ is defined by $\tau_1 \lrcorner \psi := \tau^n$, the torsion form for the vectorial component Ω^3_n.

Proof. Since $H = \sum_j \frac{1}{a_{k_j}} \tau^{k_j}$ and $d^\psi = a_{k_j}\tau^{k_j}$, we have

$$d^*\psi = -H + \sum_{j=1}^{p} \left(1 + \frac{1}{a_{k_j}}\right)\tau^{k_j}$$

and the torsion Yang-Mills equation established in Proposition 6.2.1 can be rewritten as

$$0 = d^{\theta*}F_\theta + F_\theta \lrcorner d^*\psi = d^{\theta*}F_\theta - F_\theta \lrcorner H + \sum_{j=1}^{p}\left(1+\tfrac{1}{a_{k_j}}\right)F_\theta \lrcorner \tau^{k_j}$$

considering the classification of 3-forms established in (6.2.3) and (6.2.2), we have that $1+\frac{1}{a_{k_j}} = 0$ if $\Omega^3_{k_j} \subset \mathcal{O}^{-1}$ because in this case, $a_{k_j} = -1$ and by hypothesis $F_\theta \lrcorner \tau^{k_j}$ if $\Omega^3_{k_j} \subset \mathcal{O}^0$, so the only component which doesn't cancel in the above summand is the vectorial torsion form $\tau^n = \tau_1 \lrcorner \psi$, i.e., $k_j = n$ (which appears once) and we then have

$$\begin{aligned}-d^{\theta*}F_\theta + F_\theta \lrcorner H &= \left(1+\tfrac{1}{a_n}\right)F_\theta \lrcorner (\tau_1 \lrcorner \psi) = \left(1+\tfrac{1}{a_n}\right)\tau_1 \lrcorner (F_\theta \lrcorner \psi)\\ &= -\left(1+\tfrac{1}{a_n}\right)\tau_1 \lrcorner F_\theta,\end{aligned}$$

where in the last line, we have used the instanton condition for θ, i.e., $F_\theta \lrcorner \psi = -F_\theta$, so the result follows by the definition of the Lee form ζ in (6.2.5). □

Example 6.2.4. In the context of G_2-structures, Theorem 6.2.3 above is the Lemma 4.4.1. It's worth highlighting that in the G_2 case, the condition $F_\theta \lrcorner \tau^{k_j}$ in $\mathcal{O}^0$ becomes superfluous. This is because $\mathcal{O}^0$ reduces to Ω^3_1, and the respective torsion form is $\tau_0\varphi$. Consequently, $F_\theta \lrcorner (\tau_0\varphi) = 0$ is automatically satisfied due to the instanton condition since, in this case, $F_\theta \in \mathfrak{g}_2 \iff F_\theta \lrcorner \varphi = 0$. △

6.3 Generalized Ricci flatness For G-Structures

This section will address the following problem, which represents a reformulation of Problem 2 within this new framework of instantons.

Problem 3. *Let E be a transitive Courant algebroid over an oriented spin manifold M^n endowed with an instanton form $\psi \in \Omega^4(M)$. Find the precise conditions, in terms of the G-structure determined by ψ, which imply that*

$$\mathrm{Ric}^+_{\mathbf{G},\mathrm{div}_0} = 0,$$

for a canonical choice of divergence operator div_0 uniquely determined by the pair $(\mathbf{G}, \psi)$, where $\mathbf{G}$ is induced by the Riemannian metric due to the $G \subset \mathrm{SO}(n)$-structure.

Let's establish and prove several results and technical lemmas in sequence to tackle this.

Lemma 6.3.1. *Let $\psi \in \Omega^4(M)$ be a determinant H-instanton form on M^n, suppose Ω^1 is irreducible and $\Omega^1 \hookrightarrow \Omega^3$ only once. Suppose that the $N(H)$-structure ψ admits a compatible connection with totally skew-symmetric torsion $H \in \Omega^3$ given by $\nabla^+ = \nabla^g + \frac{1}{2}g^{-1}H$ and suppose that the endomorphism part of ∇^+ is in $\mathfrak{h}$, i.e.,*

$$R_{\nabla^+}(X,Y) \in \mathfrak{h} \le \mathfrak{n}(\mathfrak{h}) \le \mathfrak{so}(n) = \Omega^2(M), \qquad \forall X, Y \in \Gamma(TM) \tag{6.3.1}$$

then its Ricci curvature satisfies

$$\mathrm{Ric}^+_{ij} - \frac{1}{12}(dH)_{ab\mu i}\psi_{ab\mu j} + \frac{\alpha}{a_n}(\nabla^+_i \tau_1)_j = 0 \tag{6.3.2}$$

where $\alpha \in \mathbb{R}$ is such that $(X \lrcorner \psi) \lrcorner \psi = \alpha X$ for all $X \in \Gamma(TM)$ and a_n the eigenvalue of the vectorial representation Ω^3_n in the $\mathbf{H} : \Omega^3 \to \Omega^3$ operator.

Proof. The proof of this follows the approach in [IS23a, IS23b, IP23] (where is proved for G_2 and Spin(7) cases) and it was already proved in Lemma 4.4.2. It is essentially the same proof until (4.4.4):

$$2\mathrm{Ric}^+_{ij} = \frac{1}{6}(dH)_{ab\mu i}\psi_{ab\mu j} - \frac{1}{3}(\nabla^+_i H)_{ab\mu}\psi_{ab\mu j} \tag{6.3.3}$$

We still follow the same approach in Lemma 4.4.2 but being careful about the constants. For this, let's consider the invariant map $\gamma \in \Omega^3 \mapsto \gamma \lrcorner \psi \in \Omega^1$ which is zero in all components of Ω^3 but $\Omega^3_n \cong \Omega^1$, consequently (by the compatibility of ∇^+ and the flux theorem B.3.6 which relates H and $\delta\psi$)

$$\nabla^+ H \lrcorner \psi = \nabla^+(H \lrcorner \psi) = \nabla^+(\pi_n H \lrcorner \psi) = \frac{1}{a_n}\nabla^+((\tau_1 \lrcorner \psi) \lrcorner \psi) = \frac{\alpha}{a_n}\nabla^+\tau_1$$

and the result follows. □

The equation above is extremely similar to the second equation in Theorem 1.5.5, $\mathrm{Ric}^+ + F_\theta \circ F_\theta + \nabla^+\zeta = 0$ and we will see in the result below that in fact, we can obtain this equation using the Lemma 6.3.1 above because of the heterotic Bianchi identity. For this, let's introduce the notation $F_\theta \circ F_\theta$ (as in [GF14]) as the symmetrization of F_θ defined by (where was used an orthonormal basis $\{e_\mu\}$ of TM when expressed in coordinates):

$$F_\theta \circ F_\theta := \sum_{\mu=1}^{n} \langle i_{e_\mu} F_\theta \rangle = \sum_{\mu=1}^{n} F^{\alpha}{}_{\mu i} F_{\alpha\mu j} e^i \otimes e^j.$$

Corollary 6.3.2. *Under the assumptions of the Lemma 6.3.1 and the heterotic Bianchi identity $dH = \langle F_\theta \wedge F_\theta \rangle$, then the equation (6.3.2) in the Lemma 6.3.1 becomes*

$$\mathrm{Ric}^+ + F_\theta \circ F_\theta + \frac{\alpha}{a_n}\nabla^+\tau_1 = 0. \tag{6.3.4}$$

Proof. The same proof as in Theorem 4.4.3. □

We can finally prove the main theorem of the chapter below. It establishes the condition for generalized Ricci flatness under the notion of instanton and the existence of a compatible connection with skew-symmetric torsion, as we have been discussing above. We will need some results.

Lemma 6.3.3. *Let $\psi \in \Omega^4(M)$ be an instanton form and suppose Ω^1 is irreducible and consider the following invariant*

$$\begin{array}{rccc} \Xi : & \Omega^1 & \to & \Omega^1 \\ & X & \mapsto & (X \lrcorner \psi) \lrcorner \psi \end{array}$$

then there is a constant $\alpha \in \mathbb{R}$ such that $\Xi = \alpha \, \mathrm{Id}$ with α given by

$$\alpha = -\frac{4|\psi|^2}{n}. \tag{6.3.5}$$

Proof. The existence of α is due to invariance and the fact that Ω^1 is irreducible. By definition of norm, we have $\psi^{ijkl}\psi_{ijkl} = 4!|\psi|^2$. Now using $(X \lrcorner \psi) \lrcorner \psi = \alpha X$ for all $X \in \Omega^1$ in coordinates, we have

$$\alpha X_j e^j = \frac{1}{3!} X^\mu \psi_\mu{}^{ijk} \psi_{ijkl} e^l \Rightarrow \alpha 3! \delta_j{}^i = \psi^\mu{}_{ijk} \psi_{ijkl}$$

where we have taken $X = e^i \Rightarrow X^\mu = \delta^{\mu i}$. Taking $i = j$ and summing over it, we have

$$\psi^{ijkl}\psi_{ijkl} = -n\alpha 3! = 4!|\psi|^2$$

therefore $\alpha = -\frac{4|\psi|^2}{n}$. □

Continuing, we introduce now a *partial norm* of an arbitrary 4-form $\xi \in \Omega^4(M)$ by the expression

$$[\![\xi]\!]^2 \coloneqq \frac{1}{16} \xi^{ij}{}_{ab} \xi^{ab}{}_{\mu\nu} \xi^{\mu\nu}{}_{ij}.$$

Lemma 6.3.4. *Let $\psi \in \Omega^4(M)$ be an instanton form and suppose Ω^1 is irreducible and $\Omega^1 \hookrightarrow \Omega^3$ only once, then we have*

$$a_n = \frac{[\![\psi]\!]^2}{|\psi|^2}, \tag{6.3.6}$$

where a_n is the eigenvalue of $\mathbf{H}$ in the component $\Omega^3_n \cong \Omega^1$.

Proof. Consider the invariant maps

$$A : X \in \Omega^1 \mapsto X \lrcorner \psi \in \Omega^3, \qquad \text{and} \qquad B : X \in \Omega^1 \mapsto (X \lrcorner \psi) \lrcorner^2 \psi \in \Omega^3$$

so since Ω^1 is irreducible and $\Omega^1 \hookrightarrow \Omega^3$ only once, by invariance, there is a constant $k \in \mathbb{R}$ such that $A = kB$. Here, these maps are defined by

$$\begin{aligned} A(X) \lrcorner \psi &= (X \lrcorner \psi) \lrcorner \psi = \alpha X \\ B(X) \lrcorner \psi &= ((X \lrcorner \psi) \lrcorner^2 \psi) \lrcorner \psi = a_n (X \lrcorner \psi) \lrcorner \psi = a_n \alpha X \end{aligned}$$

consequently $\alpha X = A(X) \lrcorner \psi = kB(X) \lrcorner \psi = ka_n\alpha X$, thus $k = \frac{1}{a_n}$ and we conclude that $B = a_n A$. On the other hand, we can compute directly

$$\begin{aligned} B(X) \lrcorner \psi &= \frac{1}{3!}((X \lrcorner \psi) \lrcorner^2 \psi)^{ijk}\psi_{ijkl}e^l = \frac{3!}{3!2!2!}X^\mu \psi_{\mu\alpha\beta i}\psi_{\alpha\beta jk}\psi_{ijkl}e^l \\ &= -\frac{1}{4}X^\mu \psi_{i\mu\alpha\beta}\psi_{\alpha\beta jk}\psi_{jkil}e^l \end{aligned}$$

using $B(X) \lrcorner \psi = a_n A(X) \lrcorner \psi = a_n \alpha X$ and $\alpha = -\frac{4|\psi|^2}{n}$ as we have calculated above, we then obtain taking $X = e_\nu$, then $X^\mu = \delta^\mu_\nu$ and $X_l = \delta_{l\nu}$ and we have

$$\begin{aligned} -\frac{1}{4}X^\mu \psi_{\mu i\alpha\beta}\psi_{\alpha\beta jk}\psi_{jkli}e^l = -a_n\frac{4|\psi|^2}{n}X_l e^l \Rightarrow \frac{1}{4}\psi_{\mu\nu\alpha\beta}\psi_{\alpha\beta jk}\psi_{jkil}e^l = a_n\frac{4|\psi|^2}{n}e^\nu \Rightarrow \\ a_n\frac{4|\psi|^2}{n} = \frac{1}{4}\psi_{\mu\nu\alpha\beta}\psi_{\alpha\beta jk}\psi_{jkil}\delta_{l\nu} \end{aligned}$$

now, taking the sum over ν, we have $\frac{1}{4}\psi_{\mu\nu\alpha\beta}\psi_{\alpha\beta jk}\psi_{jki\nu} = a_n 4|\psi|^2$, which gives the desired expression. $\square$

The results for some important structures, just for comparison:

G_2	$[\![\psi]\!]^2 = 21$	$\lvert\psi\rvert^2 = 7$	$a_n = 3$	$\alpha = -4$
Spin(7)	$[\![\psi]\!]^2 = 84$	$\lvert\psi\rvert^2 = 14$	$a_n = 6$	$\alpha = -7$
U(3)	$[\![\psi]\!]^2 = 3$	$\lvert\psi\rvert^2 = 3$	$a_n = 1$	$\alpha = -2$
U(4)	$[\![\psi]\!]^2 = 12$	$\lvert\psi\rvert^2 = 6$	$a_n = 2$	$\alpha = -3$
U(5)	$[\![\psi]\!]^2 = 30$	$\lvert\psi\rvert^2 = 10$	$a_n = 3$	$\alpha = -4$

Example 6.3.5. For G_2 we use Proposition 3.1.6, and we have

$$[\![\psi]\!]^2 := \frac{1}{16}\psi^{ij}{}_{ab}\psi^{ab}{}_{\mu\nu}\psi^{\mu\nu}{}_{ij} = \frac{1}{16}\left(4\delta^i_\mu\delta^j_\nu - 4\delta^j_\mu\delta^i_\nu + 2\psi^{ij}{}_{\mu\nu}\right)\psi^{\mu\nu}{}_{ij} = \frac{1}{8}\psi^{ij}{}_{\mu\nu}\psi^{\mu\nu}{}_{ij} = 21$$

as in the table. $\triangle$

We will see the other structures in detail later. With these two Lemmas, we can finally prove the main theorem:

Theorem 6.3.6. *Under the hypothesis of Theorem 6.2.3 and Corollary 6.3.2, consider the transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ defined with a pair (θ, H), so if the instanton form $\psi \in \Omega^4(M)$ satisfies the following condition on its norm*

$$[\![\psi]\!]^2 = \frac{4}{n}|\psi|^4 - |\psi|^2 \tag{6.3.7}$$

then $\mathrm{GRic}^+(\mathbf{G}, \mathrm{div}) = 0$ where $\mathbf{G}$ is the standard generalized metric in Example 1.2.4 and the divergence is given by

$$\mathrm{div} = \mathrm{div}^{\mathbf{G}} - 2\langle\zeta, \cdot\rangle$$

for $\zeta \in \Omega^1$ the Lee form as in (6.2.5).

Proof. The Theorem 6.2.3 and Corollary 6.3.2 and the expression for the Lee form (6.2.5), we have the equations

$$d^{\theta *}F_\theta - \left(1 + \tfrac{1}{a_n}\right) i_{\tau_1^\#} F_\theta - F_\theta \lrcorner H = 0 \tag{6.3.8}$$
$$\mathrm{Ric}^+ + F_\theta \circ F_\theta + \frac{\alpha}{a_n}\nabla^+ \tau_1 = 0 \tag{6.3.9}$$

so by Theorem 1.5.5, we will have generalized Ricci flatness if

$$-\left(1 + \frac{1}{a_n}\right) = \frac{\alpha}{a_n}$$

which can be rewritten as $\alpha + 1 = -a_n$. But using Lemma 6.3.3 and Lemma 6.3.4, we have this equality equivalent to:

$$\alpha + 1 = -\frac{4|\psi|^2}{n} + 1 = -a_n = -\frac{[\![\psi]\!]^2}{|\psi|^2} \iff [\![\psi]\!]^2 = \frac{4}{n}|\psi|^4 - |\psi|^2$$

which is satisfied by the hypothesis. □

6.4 Coupled instanton equations

In this section, we will be in the same hypothesis as in Section 2.2, we will fix a transitive Courant algebroid $E = TM \oplus \mathrm{ad}P \oplus T^*M$ (i.e., a pair (θ, H) satisfying the heterotic Bianchi identity 1.1.8) and we define the connection D on $TM \oplus \mathrm{ad}P$ in the same way (2.2.2):

$$D = \begin{pmatrix} \nabla^- & \mathbb{F}^\dagger \\ -\mathbb{F} & d^\theta \end{pmatrix}$$

but here we are considering H being the flux, defined when we have a determinant instanton form $\psi \in \Omega^4(M)$, and H is well-defined by

$$H = \sum_{j=1}^{p} \frac{1}{a_{k_j}} \tau^{k_j}$$

So, the coupled H-instanton holds if D is an instanton.

Definition 6.4.1. *Let E be a transitive Courant algebroid with generalized metric $\mathbf{G}$ over an oriented manifold M^n endowed with a determinant instanton form $\psi \in \Omega^4(M)$. We say that ψ is a solution of the coupled instanton equations if*

$$F_D \lrcorner \psi = -F_D, \tag{6.4.1}$$

where F_D is the curvature of D (2.2.2).

Via Lemma 2.2.1 which computes explicitly the curvature F_D, the coupled instanton equations are equivalent to the system:

$$\begin{aligned} R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} &\in \mathfrak{h}, \\ \nabla^{\theta,+} F_\theta &\in \mathfrak{h}, \\ [F_\theta, \cdot] - \mathbb{F} \wedge \mathbb{F}^\dagger &\in \mathfrak{h}, \\ dH - \langle F_\theta \wedge F_\theta \rangle &= 0. \end{aligned} \tag{6.4.2}$$

We can also rewrite these equations more directly, similar to what we have done in Definition 4.3.4 and Proposition 4.3.5 for the case of G_2-structures.

Proposition 6.4.2. *Let M^n be a manifold endowed with a determinant instanton form $\psi \in \Omega^4(M)$ and θ a connection in some principal K-bundle and H the flux induced by $\mathbf{H}$. Then, the coupled instanton equations are equivalent to*

$$R_{\nabla^+} + \langle F_\theta, F_\theta \rangle \in \Omega^2 \otimes \mathfrak{h} \tag{6.4.3}$$

$$\nabla^{\theta,+} F_\theta \in \mathfrak{h}, \tag{6.4.4}$$

$$[F_\theta, \cdot] - F_\theta \lrcorner^1 \langle \cdot, \cdot \rangle^{-1} F_\theta \in \mathfrak{h}, \tag{6.4.5}$$

$$dH - \langle F_\theta \wedge F_\theta \rangle = 0. \tag{6.4.6}$$

the first equation is saying that the endomorphism part R_{∇^+} which contributes to the Lie algebra $\mathfrak{h}$.

Proof. For the first equation, we will use (2.3.6) derived in Theorem 2.3.2, which gives us the formula (where i, j is in $\mathfrak{h}$ and k, l in $\Omega^2(M)$ up to raise an index)

$$R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} = \frac{1}{2} \left(\langle F_\theta, F^l{}_k \rangle_{ij} + (R_{\nabla^+})_{ij}{}^l{}_k \right) e^{ij} \otimes e^k \otimes e_l$$

This shows exactly what we are looking for: the endomorphism part of R_{∇^+} (indices I, j) in the Lie algebra.

We have already derived the second equation since it comes directly from how F_D is written. The third is deduced in the end of the proof of Theorem 2.3.2 in (2.3.7). □

Now, we will delve into the solution of Problem 1 in the context of this new notion of instanton. Let's reformulate it:

Problem 4 (Reformulated Problem 1)**.** *Let E be a transitive Courant algebroid with generalized metric $\mathbf{G}$ over a manifold endowed with a determinant instanton form $\psi \in \Omega^4(M)$ and suppose it satisfies the* gravitino equation*:*

$$\nabla^+ \psi = 0, \qquad F_\theta \lrcorner \psi = -F_\theta, \qquad dH - \langle F_\theta \wedge F_\theta \rangle = 0.$$

So, what conditions make the data $(\mathbf{G}, \psi)$ a solution for the coupled instanton equation (6.4.1), i.e., $F_D \lrcorner \psi = -F_D$?

To solve this, let's prove a quick lemma:

Lemma 6.4.3. *Let M be a manifold with an instanton form $\psi \in \Omega^4(M)$. Let Ω^2_p be an irreducible component of $\Omega^2(M)$ and suppose $\Omega^2_p \lrcorner^1 \psi = 0$ inside Ω^4, then $\Omega^2_p \lrcorner^1 \Omega^2_p \subset \Omega^2_p$, i.e., if $\alpha = \frac{1}{2!}\alpha_{ij}e^{ij}$ and $\beta = \frac{1}{2!}\beta_{ij}e^{ij}$ are in Ω^2_p, then*

$$\gamma \coloneqq \alpha \lrcorner^1 \beta = \alpha_{ij}\beta_{ik}e^{jk} \in \Omega^2_p.$$

In particular $\mathfrak{h} \lrcorner^1 \mathfrak{h} \subset \mathfrak{h}$ and $\mathfrak{n}(\mathfrak{h}) \lrcorner^1 \mathfrak{n}(\mathfrak{h}) \subset \mathfrak{n}(\mathfrak{h})$.

Proof. Let b the eigenvalue of Ω^2_p under the instanton map (6.1.3): $\beta \mapsto \beta \lrcorner \psi$. Then, we need to prove that $\gamma \lrcorner \psi = b\gamma$. By direct computation, we have $\gamma = \frac{1}{2}(\alpha_{ki}\beta_{kj} - \alpha_{kj}\beta_{ki})e^{ij}$ and:

$$\gamma \lrcorner \psi = \frac{1}{2!2!}(\alpha_{ki}\beta_{kj} - \alpha_{kj}\beta_{ki})\psi_{ijab}e^{ab} = \frac{1}{2}\alpha_{ik}\beta_{kj}\psi_{jabi}e^{ab}$$

now, let's study the permutation

$$\alpha_{i[k}\psi_{jab]i} = \frac{1!3!}{4!}\Big(\alpha_{ik}\psi_{jabi} - \alpha_{ib}\psi_{kjai} + \alpha_{ia}\psi_{bkji} - \alpha_{ij}\psi_{abki}\Big)$$

coming back in the expression of $\gamma \lrcorner \psi$, using that $\alpha_{i[k}\psi_{jab]i}$ are the coefficients of the form $\alpha \lrcorner^1 \psi \in \Omega^4$, then they must be zero by hypothesis, i.e., $\alpha_{i[k}\psi_{jab]i} = 0$ for all k, j, a, b. Consequently, using that $\alpha, \beta \in \Omega^2_p$, consequently $\alpha_{ij}\psi_{ijab} = 2b\alpha_{ab}, \beta_{ij}\psi_{ijab} = 2b\beta_{ab}$, we have

$$\begin{aligned}
\gamma \lrcorner \psi &= \frac{1}{2}\beta_{kj}\Big(\cancelto{0}{4\alpha_{i[k}\psi_{jab]i}} + \alpha_{ib}\psi_{kjai} - \alpha_{ia}\psi_{bkji} + \alpha_{ij}\psi_{abki}\Big)e^{ab} \\
&= b_k\alpha_{ib}\beta_{ai}e^{ab} - b_k\alpha_{ia}\beta_{bi}e^{ab} - \frac{1}{2}\alpha_{ji}\beta_{jk}\psi_{ikab}e^{ab} \\
&= b(-\beta \lrcorner^1 \alpha + \alpha \lrcorner^1 \beta) - \gamma \lrcorner \psi = 2b\gamma - \gamma \lrcorner \psi
\end{aligned}$$

which gives us $\gamma \lrcorner \psi = b\gamma \Rightarrow \gamma \in \Omega^2_p$ as stated. $\square$

For instance, in the case of G_2-structures, since the adjoint representation is not a component inside Ω^4, the condition $\mathfrak{g}_2 \lrcorner^1 \psi = 0$ holds by invariance (this has already been proved in [dlOLS18a, dlOLS16] in the case G_2, and their result were modifying for us to prove the Lemma above). The same holds for Spin(7)-structures, since $\mathfrak{spin}(7)$ is not contained in Ω^4, as we will describe in the section about Spin(7)-structures.

So now, we present the conditions that make the Problem 4 true. In summary, we will need to create an imposition about the endomorphism part of R_{∇^+} (which naturally lives in $\mathfrak{n}(\mathfrak{h})$, when ∇^+ is compatible with ψ) and roughly speaking, the Lie algebra $\mathfrak{h}$ cannot be contained within 4-forms.

Theorem 6.4.4. *Let E be a transitive Courant algebroid with generalized metric $\mathbf{G}$ over a manifold endowed with a determinant instanton form $\psi \in \Omega^4(M)$ and suppose it satisfies the* gravitino equation*:*

$$\nabla^+\psi = 0, \qquad F_\theta \lrcorner\, \psi = -F_\theta, \qquad dH - \langle F_\theta \wedge F_\theta \rangle = 0.$$

If the endomorphism part of R_{∇^+} lives in $\mathfrak{h}$ and $\mathfrak{h} \lrcorner^1 \psi = 0$, then the data $(\mathbf{G}, \psi)$ satisfies the coupled instanton equation (6.4.1), i.e., $F_D \lrcorner\, \psi = -F_D$.

Proof. Note that the equation $\nabla^+\psi = 0$ is that ∇^+ is compatible and since $\mathbf{H}$ is an isomorphism, ∇^+ is the only compatible connection with skew-symmetric torsion $H \in \Omega^3(M)$ induced by $\mathbf{H}$ and the torsion forms via d^ψ. The second equation, which is the instanton condition for θ is equivalent to $F_\theta \in \mathfrak{h}$ and since ∇^+ preserves the irreducible components of Ω^2 (because it is compatible), then we have

$$\nabla^{\theta,+} F_\theta \in \mathfrak{h}$$

We must prove that the diagonal elements of F_D are contained in $\mathfrak{h}$. The fact that $[F_\theta,] \in \mathfrak{h}$ is also immediate, so we just have to prove

$$R_{\nabla^+} + \langle F_\theta, F_\theta \rangle \in \mathfrak{h}, \qquad F_\theta \lrcorner^1 \langle \cdot, \cdot \rangle^{-1} F_\theta \in \mathfrak{h}. \tag{6.4.7}$$

In the first one, we have to prove the endomorphism part to be in $\mathfrak{h}$. But it is immediate because we impose the endomorphism part of R_{∇^+} being in $\mathfrak{h}$ and θ is an instanton, so $F_\theta \in \mathfrak{h}$. For the last entry, $F_\theta \lrcorner^1 \langle \cdot, \cdot \rangle^{-1} F_\theta \in \mathfrak{h}$ follows immediately by the Lemma 6.4.3 assuming the instanton condition. So the coupled instanton equations follow by the heterotic Bianchi identity (which is satisfied since we are in the context of transitive Courant algebroids) and Proposition 6.4.2. $\square$

6.5 Semi-instantons and the $\nabla^{\theta,+} F_\theta \in \mathfrak{h}$ condition

Suppose that we are in the case of solutions for the gravitino equation; then we have the equations:

$$\nabla^+\psi = 0, \qquad F_\theta \in \mathfrak{h}, \qquad dH = \langle F_\theta \wedge F_\theta \rangle.$$

The first equation guarantees ∇^+ is compatible, and with the instanton condition in the second equation, we have $\nabla^{\theta,+} F_\theta \in \mathfrak{h}$. However, in the definition of coupled instanton equations

$$F_D \in \mathfrak{h}, \qquad dH = \langle F_\theta \wedge F_\theta \rangle,$$

the condition $\nabla^{\theta,+} F_\theta \in \mathfrak{h}$ is satisfied even when $F_\theta \in \mathfrak{h}$ doesn't. The condition $F_\theta \in \mathfrak{h}$ implies the Yang-Mills equation with torsion (6.2.3). The role is, under some conditions, $\nabla^{\theta,+} F_\theta \in \mathfrak{h}$ also can imply this. This discussion is based on this result for $\mathrm{SU}(m)$-instanton case in [GFGM23].

Proposition 6.5.1. *Let M be a manifold endowed with a determinant instanton form $\psi \in \Omega^4(M)$ and θ a connection in some vector bundle over M satisfying such that*

$$\nabla^+\psi = 0, \qquad \nabla^{\theta,+}F_\theta \in \mathfrak{h}.$$

Then θ satisfies the so-called semi-instanton torsion Yang-Mills equation:

$$d^{\theta*}F_\theta + F_\theta \lrcorner\, d^*\psi = (F_\theta + F_\theta \lrcorner\, \psi) \lrcorner\, H. \tag{6.5.1}$$

where $H \in \Omega^3$ is the flux induced by $\mathbf{H}$.

Proof. The proof is equal to Lemma 4.4.4 as pointed in (4.4.11). □

Note that when the instanton condition holds, $F_\theta \lrcorner\, \psi = -F_\theta$, we obtain exactly the torsion Yang-Mills $d^{\theta*}F_\theta + F_\theta \lrcorner\, d^*\psi = 0$ as in Proposition 6.2.1. Note that in the first torsion Yang-Mills equation, we didn't impose $\nabla^+$to be compatible, but for this version, we had to impose this because now we use the definition of ∇^+ due to hypothesis $\nabla^+F_\theta \in \mathfrak{h}$.

Interestingly, this last equation involving the flux also holds for semi-instantons under certain hypotheses (as in the case of our examples). We have the following result:

Proposition 6.5.2. *Let $\psi \in \Omega^4(M)$ be an determinant H-instanton form on M^n, suppose Ω^1 is irreducible and $\Omega^1 \hookrightarrow \Omega^3$ only once. Let θ be a connection in some vector bundle over M satisfying such that*

$$\nabla^+\psi = 0, \qquad \nabla^{\theta,+}F_\theta \in \mathfrak{h}$$

Also, suppose that the connection the curvature of the connection θ lives at most in two irreducible components of Ω^2, say $F_\theta \in \mathfrak{h} \oplus \mathfrak{b}$, and suppose that $b \in \mathbb{R}$ is the eigenvalue of $\mathfrak{b}$ under the instanton map (6.1.3). If

$$F_\theta \lrcorner\, \tau_{k_j} = 0,\ \forall \Omega^3_{k_j} \subset \mathcal{O}^0, \qquad \text{and} \qquad \pi_{\mathfrak{b}}F_\theta \lrcorner\, \tau_{k_j} = 0,\ \forall \Omega^3_{k_j} \subset \mathcal{O}^{-1}$$

where $\tau_{k_j} = \pi_{k_j}(d^*\psi)$ are the torsion forms and $\mathcal{O}^0, \mathcal{O}^{-1}$ as in (6.2.3). If $b = 1 + a_n$ (where a_n is the eigenvalue of Ω^3_n under the $\mathbf{H}$*), then θ satisfies the following torsion Yang-Mills equation:*

$$d^{\theta*}F_\theta + i_{\zeta^\#}F_\theta - F_\theta \lrcorner\, H = 0$$

where $H \in \Omega^3$ is the flux induced by $\mathbf{H}$ *and $\zeta = -(1 + \frac{1}{a_n})\tau_1$ is the Lee form, for $\tau_1 \lrcorner\, \psi = \tau_n = \pi_n(d^*\psi)$.*

Proof. As we have written before:

$$H + d^*\psi = \sum_j \left(1 + \frac{1}{a_{k_j}}\right)\tau_{k_j}$$

by the Proposition 6.5.1 (using the existence of compatible connection ∇^+ compatible and $\nabla^{\theta,+}_X F_\theta \in \mathfrak{h}$), we have the weak torsion Yang-Mills equation given by

$$d^{\theta *} F_\theta + F_\theta \lrcorner d^* \psi = (F_\theta + F_\theta \lrcorner \psi) \lrcorner H$$

since θ is not an instanton, we cannot conclude that the term $F_\theta \lrcorner d^*\psi = i_{\zeta\#} F_\theta - F_\theta \lrcorner H$. Thus, we need to work on this expression: using $a_{k_j} = -1$ in $\mathcal{O}^{-1}$ and $F_\theta \lrcorner \mathcal{O}^0 = 0$, we then have

$$\begin{aligned} F_\theta \lrcorner d^*\psi &= F_\theta \lrcorner \left(-H + \left(1 + \frac{1}{a_n}\right) \tau_1 \lrcorner \psi + \sum_{k_j \neq n} \left(1 + \frac{1}{a_{k_j}}\right) \tau_{k_j} \right) \\ &= -F_\theta \lrcorner H - \zeta \lrcorner (F_\theta \lrcorner \psi) \end{aligned}$$

Supposing b is the eigenvalue of $\mathfrak{b}$ under the instanton and $\mathfrak{h}$ has eigenvalue -1, we then obtain

$$F_\theta \lrcorner d^*\psi = -F_\theta \lrcorner H - \zeta \lrcorner (-\pi_{\mathfrak{h}} F_\theta + b \pi_{\mathfrak{b}} F_\theta)$$

on the other hand, the term $F_\theta + F_\theta \lrcorner \psi$ has only the component $\mathfrak{b}$ because the $\mathfrak{h}$ cancels out since it is the -1 eigenvalue or the instanton map. Using $\pi_{\mathfrak{b}} F_\theta \lrcorner \tau_{k_j} = 0$ on $\mathcal{O}^{-1}$ and $F_\theta \lrcorner \tau_{k_j} = 0$ on $\mathcal{O}^0$ we have $\pi_{\mathfrak{b}} F_\theta \lrcorner \tau_{k_j} = 0$ for all $k_j \neq n$, consequently:

$$\begin{aligned} (F_\theta + F_\theta \lrcorner \psi) \lrcorner H = (1+b)\pi_{\mathfrak{b}} F_\theta \lrcorner H &= (1+b)\pi_{\mathfrak{b}} F_\theta \lrcorner \left(\tfrac{1}{a_n} \tau_1 \lrcorner \psi + \sum_{k_j \neq n} \tfrac{1}{a_{k_j}} \tau_{k_j} \right) \\ &= \frac{b(b+1)}{a_n} \tau_1 \lrcorner \pi_{\mathfrak{b}} F_\theta, \end{aligned}$$

finally:

$$\begin{aligned} 0 &= d^{\theta *} F_\theta - F_\theta \lrcorner H - (1 + \tfrac{1}{a_n}) \tau_1 \lrcorner \pi_{\mathfrak{h}} F_\theta + \left(b(1 + \tfrac{1}{a_{k_j}}) - (b^2 + b) a_n^{-1} \right) \tau_1 \lrcorner \pi_{\mathfrak{b}} F_\theta \\ &= d^{\theta *} F_\theta - F_\theta \lrcorner H + i_{\zeta\#} \pi_{\mathfrak{h}} F_\theta + \left(b - b^2 a_n^{-1} \right) \tau_1 \lrcorner \pi_{\mathfrak{b}} F_\theta \end{aligned}$$

and the equation would follow if $b - b^2 a_n^{-1} = -(1 + a_n^{-1})$. Solving this equation with b as a variable, we find two possible values:

$$b = -1; \quad b = 1 + a_n$$

but we also know $b \neq -1$ because $\mathfrak{b} \neq \mathfrak{h}$. So the desired equation holds if $b = 1 + a_n$ as the hypothesis guarantees. □

Example 6.5.3 (G$_2$-structures)**.** The hypothesis of $b = 1 + a_n$ in the proposition above cannot be removed; indeed, it doesn't always hold. For example, consider G_2-structures, then $F_\theta \in \Omega^2 = \mathfrak{g}_2 \oplus \Omega^2_7$, in this case $b = 2$ and $a_n = 3$ and consequently $b = 2 \neq 4 = 1 + a_n$, so the hypothesis is not guaranteed. In fact, in Lemma 4.4.4, the expression $d^{\theta *} F_\theta + i_{\zeta\#} F_\theta - F_\theta \lrcorner H$ is calculated, and it is, in fact, not zero.

The same holds for Spin(7)-structures where $b = 3 \neq 7 = 1 + 6 = 1 + a_n$ and in this case, $b - b^2 a_n^{-1} + (1 + a_n^{-1}) = 8/3 \neq 0$. △

The following example is crucial because it motivated this analysis and is still the motivation of open problems about it, which we will discuss below.

Example 6.5.4 (Revisiting [GFGM23] of $\mathrm{U}(m)$-structures)**.** Consider $\mathrm{U}(m)$-structures on M^{2m}, where $m \geq 3$, yielding $\mathrm{SU}(m)$-instanton via instanton form $\psi = \frac{1}{2}\omega^2$. The eigenvalue a_{2m} of $\mathbf{H}$ in Ω^3_{2m} is:

$$a_{2m} = m - 2.$$

Now, let $\mathfrak{b} = \Omega^2_1 = \{f\omega : f \in \mathcal{C}^\infty(M)\}$. Direct computation yields the eigenvalue b as follows:

$$\omega \lrcorner \psi = \frac{1}{2!}\omega \lrcorner \omega^2 = \frac{1}{(m-2)!} * (\omega \wedge \omega^{m-2}) = \frac{m-1}{(m-1)!} * \omega^{m-1} = (m-1)\omega$$

which implies $b = m - 1$. Hence, we have $b = m - 1 = m - 2 + 1 = 1 + a_{2m}$, aligning with the hypothesis of the Proposition 6.5.2.

We proceed to verify the other hypothesis of the proposition. Firstly, $F_\theta = F_\theta^{1,1}$ is equivalent to $F_\theta \in \mathfrak{su}(m) \oplus \langle\omega\rangle = \mathfrak{h} \oplus \mathfrak{b}$. Additionally, considering $\nabla^+\psi = 0$, which is equivalent to the component $\mathcal{W}_2$ of the intrinsic torsion being zero (in this case, $H = -d^c\omega$ represents the torsion of the compatible connection with skew-torsion). In this scenario, the space of 3-forms decomposes as:

$$\Omega^3 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

If J is integrable, then the torsion form in $\mathcal{W}_1 = \mathcal{O}^0$ is zero. Furthermore, the class $\mathcal{W}_3$ is perpendicular to $\mathcal{W}_4 = \Omega^3_6$, then for $\gamma \in \mathcal{W}_3$, satisfies (for $X = \partial_\mu$)

$$0 = \langle X \wedge \omega, \gamma\rangle = \frac{1}{1!2!3!}\delta_{\mu a}\omega_{bc}\gamma_{ijk}\langle e^{abc}, e^{ijk}\rangle = \frac{1}{2}\delta_{\mu a}\omega_{bc}\gamma_{abc} = \frac{1}{2}\omega_{bc}\gamma_{bc\mu}$$

for all μ, implying $\omega \lrcorner \gamma = 0$ for $\gamma \in \mathcal{W}_3 = \mathcal{O}^{-1}$. Now since $\mathfrak{b} = \langle\omega\rangle$ then $\pi_{\mathfrak{b}}F_\theta = f\omega$ and consequently $\pi_{\mathfrak{b}}F_\theta \lrcorner \mathcal{O}^{-1} = 0$ as in the hypothesis.

Thus, we've placed the result from [GFGM23] under the hypothesis of Proposition 6.5.2. In the paper, $\nabla^{\theta,+}F_\theta \in \mathfrak{su}(n)$ and the almost complex structure is considered integrable. Note that Proposition 6.5.2 offers the same result for a slightly more general hypothesis: imposing only $F_\theta \lrcorner \tau = 0$ for $\tau = \pi_{\mathcal{W}_1}(d^*\psi)$ not $\tau = 0$ as in [GFGM23] (because J is integrable). In Chapter 7, the calculations cited here are explored. △

The difference between the case of G_2-structures and $\mathrm{U}(m)$-structures is that for the first one, $N(\mathrm{G}_2) = \mathrm{G}_2$ and consequently, nothing interesting holds for F_θ. However, in the second one, we are imposing $F_\theta \in \mathfrak{n}(\mathfrak{h}) = \mathfrak{u}(m)$, i.e., θ being a semi-instanton and a natural problem emerges:

Problem 5. *Let $\psi \in \Omega^4(M)$ be an H-instanton form on M^n, suppose Ω^1 is irreducible and $\Omega^1 \hookrightarrow \Omega^3$ only once and that the associated* $\mathbf{H}$ *operator being an isomorphism. Let θ be a semi-instanton, i.e., $F_\theta \in \mathfrak{n}(\mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ (in some bundle) satisfying $\mathfrak{h}^\perp$ irreducible and*

$$\nabla^+\psi = 0, \qquad \nabla^{\theta,+}F_\theta \in \mathfrak{h}.$$

If $b \in \mathbb{R}$ denotes the eigenvalue of $\mathfrak{b}$ under the instanton map (6.1.3) *and suppose the conditions (about $\tau_{k_j} = \pi_{k_j}(d^*\psi)$ the torsion forms and $\mathcal{O}^0, \mathcal{O}^{-1}$ as in* (6.2.3)*)*

$$F_\theta \lrcorner \tau_{k_j} = 0, \forall \Omega^3_{k_j} \subset \mathcal{O}^0, \qquad \text{and} \qquad \pi_{\mathfrak{b}} F_\theta \lrcorner \tau_{k_j} = 0, \forall \Omega^3_{k_j} \subset \mathcal{O}^{-1}$$

are satisfied, then $b = 1 + a_n$ (where a_n is the eigenvalue of Ω^3_n under the $\mathbf{H}$*).*

If the Problem 5 is true, then the semi-instanton θ satisfies the torsion Yang-Mills equation as follows (by the Proposition 6.5.2):

$$d^{\theta *}F_\theta + i_{\zeta^\#}F_\theta - F_\theta \lrcorner H = 0,$$

where $H \in \Omega^3$ represents the flux induced by the vector field $\mathbf{H}$, and $\zeta = -(1 + \frac{1}{a_n})\tau_1$ denotes the Lee form, with $\tau_1 \lrcorner \psi = \tau_n = \pi_n(d^*\psi)$.

This problem is of interest only in scenarios where $N(H) \neq H$, for example, the case discussed above where $N(\mathrm{SU}(m)) = \mathrm{U}(m)$. This condition does not hold for G_2 and Spin(7) structures, but it does for $\mathrm{Sp}(k)\mathrm{Sp}(1)$-structures, satisfying the relation [McI91]:

$$N(\mathrm{Sp}(k)) = \mathrm{Sp}(k)\mathrm{Sp}(1) \neq \mathrm{Sp}(k)$$

within $\mathrm{SO}(4k)$. However, a thorough investigation of such examples is lacking. If proven true, it raises questions regarding the general solution to Problem 5.

7 Spin(7) and almost Hermitian structures

In Chapter 3, we thoroughly explored the geometry of G_2-structures in seven dimensions. Shifting our focus to this chapter, we will now delve into the realm of Spin(7)-structures in eight dimensions and $U(m)$-structures (commonly known as almost Hermitian structures) in $2m$ dimensions, providing insights into similar findings. To conclude our investigation, we will utilize the frameworks established in Chapter 2 (as we have made in Chapter 4 for G_2-structure), regarding conditions for Ricci flatness and the coupled instanton equations. The main references for this section were [Iva04, Kar05, Kar08b, MM18, IP23] for the part of Spin(7) and [Sal89, AFS05, MU19] for the almost Hermitian structures.

The chapter is organized as follows: In Section 7.1, we present fundamental information about the group Spin(7), including its alternative definition as the stabilizer of a 4-form in $\mathbb{R}^8$, the properties of its fundamental 4-form Ω_0, and how Spin(7) decomposes the space of forms $\Omega^k(\mathbb{R}^8)^*$ into irreducible components.

In Section 7.2, we introduce Spin(7)-structures on a manifold, characterized by a 4-form $\Omega \in \Omega^4(M^8)$, locally modelled in $(\mathbb{R}^8, \Omega_0)$. We further discuss the torsion forms associated with such a structure, culminating in Proposition 7.2.3, which addresses the existence of a compatible connection with skew-symmetric torsion for all Spin(7)-structures. Additionally, we apply Appendix B in computing the torsion of such a connection.

In Section 7.3, we conclude our examination of Spin(7)-structures by exploring conditions for generalized Ricci flatness within transitive Courant algebroids over a manifold equipped with a Spin(7)-structure satisfying the *gravitino equation*, as detailed in Theorem 7.3. Finally, we present Theorem 7.3.2, which delineates the coupled Spin(7)-instanton equations due to the *gravitino equation* and the *Bianchi identity*.

In continuation, we delve into the realm of almost Hermitian structures ($U(m)$-structures) in even dimensions in Section 7.4. Similar to Section 7.1, this section follows a parallel approach, where we introduce the group $U(m)$, its fundamental form ω_0 (referred to as the Hermitian form), and describe the decomposition of the space of forms into irreducible components.

In Section 7.5, we delve into the intrinsic torsion of almost Hermitian structures, exploring the conditions necessary for the existence of a compatible connection with skew-symmetric torsion and providing an explicit computation for it using the techniques in Appendix B. This section draws inspiration from [AFS05], which approaches Hermitian structures through the lens of torsion forms rather than classical tensors, cf. Theorem 7.5.3.

To conclude the chapter, Section 7.6 leverages the theory of almost Hermitian structures, as explored in the preceding sections, to establish conditions of generalized Ricci flatness for transitive Courant algebroids over almost Hermitian manifolds satisfying the gravitino equation, as demonstrated in Theorem 7.6.1. Furthermore, in Theorem 7.6.2, we establish that such manifolds imply coupled $\mathrm{SU}(m)$-instanton equations.

7.1 The group $\mathrm{Spin}(7)$ and the decomposition of forms

The group $\mathrm{Spin}(7) \leq \mathrm{SO}(8)$ is a simple, simply-connected, compact, 21-dimensional Lie group [Joy00] defined as

$$\mathrm{Spin}(7) := \left\{ g \in \mathrm{GL}(8) : g^*\Omega_0 = \Omega_0 \right\} \tag{7.1.1}$$

where $\Omega_0 \in \Omega^3(\mathbb{R}^8)$ (called the standard $\mathrm{Spin}(7)$-structure in $\mathbb{R}^8$) is defined by (in the canonical basis and metric of $\mathbb{R}^8$)[1]:

$$\begin{aligned} \Omega_0 = e^{0127} &+ e^{0347} + e^{0567} + e^{0135} - e^{0146} - e^{0236} - e^{0245} \\ &+ e^{3456} + e^{1256} + e^{1234} - e^{2467} + e^{2357} + e^{1457} + e^{1367}. \end{aligned} \tag{7.1.2}$$

This 4-form is self dual, i.e., $*\Omega_0 = \Omega_0$ (with respect to the usual metric) and satisfies $|\Omega_0|^2 = 14$ and normally called the *standard* or *fundamental* $\mathrm{Spin}(7)$-*structure on* $\mathbb{R}^8$.

The $\mathrm{Spin}(7)$-structure Ω_0 is related to the Riemannian metric g_0 and orientation in a non-linear fashion, which we now describe. Let $v \in \mathbb{R}^8$ a non-zero vector, we define [Kar08b, FLMSE23]:

$$\begin{aligned} B_{ij}(v) &= ((i_{e_i} i_v \Omega_0) \wedge (i_{e_j} i_v \Omega_0) \wedge (i_v \Omega_0))(e_1, \cdots, e_7) \\ A(v) &= (i_v \Omega_0 \wedge \Omega_0)(e_1, \cdots, e_7), \end{aligned}$$

then, the metric is given by

$$(g_0(v,v))^2 = -\frac{7^3}{6^{7/3}} \frac{(\det B_{ij})^{1/3}}{A(v)^3}. \tag{7.1.3}$$

As for the G_2-structures which the motivation was the cross product in $\mathbb{R}^7$, the $\mathrm{Spin}(7)$-structure can be motived by the triple cross product in $\mathbb{R}^8$ [SW17], but is not our objective to treat this here.

Remark 7.1.1. Although we are considering (7.1.1) as the definitive definition of the group $\mathrm{Spin}(7)$, it's essential to point out that the broader concept of $\mathrm{Spin}(n)$ is as the universal covering of $\mathrm{SO}(n)$. So, (7.1.1) is just a particularity for $n = 7$. ○

[1] Note that $\Omega_0 = e^0 \wedge \varphi_0 + \psi_0$, where φ_0 and ψ_0 has the expressions as in G_2-structures (3.1.4) and (3.1.5).

Using the standard way to write the forms, we will denote the coordinates of Ω_0 (using the standard basis of $\mathbb{R}^8$) as

$$\Omega_0 = \frac{1}{4!}\Omega_{ijkl}e^{ijkl}.$$

These coefficients satisfy some relations under contractions:

Proposition 7.1.2. *The fundamental* Spin(7)*-structure on* $\mathbb{R}^8$ *(in the standard basis as in* (7.1.2)*) satisfies the following relations between the coefficients:*

$$\begin{aligned}
\Omega^{\mu\nu\rho\eta}\Omega_{\mu\nu\rho\eta} &= 336\\
\Omega^{\mu\nu\rho i}\Omega_{\mu\nu\rho a} &= 42\delta^i{}_a\\
\Omega^{\mu\nu ij}\Omega_{\mu\nu ab} &= 6\delta^i{}_a\delta^j{}_b - 6\delta^i{}_b\delta^j{}_a + 4\Omega^{ij}{}_{ab}\\
\Omega^{\mu ijk}\Omega_{\mu abc} &= \delta^i{}_a\delta^j{}_b\delta^k{}_c + \delta^i{}_b\delta^j{}_c\delta^k{}_a + \delta^i{}_c\delta^j{}_a\delta^k{}_b - \delta^i{}_a\delta^j{}_c\delta^k{}_b - \delta^i{}_b\delta^j{}_a\delta^k{}_c\\
&\quad - \delta^i{}_c\delta^j{}_b\delta^k{}_a + \delta^i{}_a\Omega^{jk}{}_{bc} + \delta^j{}_a\Omega^{ki}{}_{bc} + \delta^k{}_a\Omega^{ij}{}_{bc} + \delta^i{}_b\Omega^{jk}{}_{ca}\\
&\quad + \delta^j{}_b\Omega^{ki}{}_{ca} + \delta^k{}_c\Omega^{ij}{}_{ca} + \delta^i{}_c\Omega^{jk}{}_{ab} + \delta^j{}_c\Omega^{ki}{}_{ab} + \delta^k{}_c\Omega^{ij}{}_{ab}
\end{aligned}$$

Proof. [Kar08b, Kar05]. □

As in the case of the standard G_2-structure, the form Ω_0 implies on the decomposition of the space of forms $\Lambda^k(\mathbb{R}^8)^*$ into Spin(7)-irreducible representations. Using the Hodge star operator $* : \Lambda^k \cong \Lambda^{n-k}$, we need to describe the decomposition for $k = 0, 1, 2, 3, 4$. The space of 0-forms and 1-forms are irreducible via the 1-dimensional and 8-dimensional (natural) representations (determinant and matrix multiplication, considering $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$).

$$\Lambda^0 = \Lambda^0_1 \cong \mathbb{R}, \qquad \Lambda^1 = \Lambda^1_8 \cong \mathbb{R}^8.$$

The space of 2-forms contains the seven-dimensional representation $\mathbb{R}^7$ (irreducible over SO(7) and irreducible over Spin(7) via universal covering $\Xi : \mathrm{Spin}(7) \to \mathrm{SO}(7)$) and the adjoint representation $\mathfrak{spin}(7)$ (irreducible because Spin(7) is simple). We then have:

Proposition 7.1.3 (Decomposition of 2-forms)**.** *The space* $\Lambda^2(\mathbb{R}^8)^*$ *of 2-forms decomposes into* Spin(7)*-irreducible representations as*

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21} \cong \mathbb{R}^7 \oplus \mathfrak{spin}(7) \tag{7.1.4}$$

where

$$\Lambda^2_7 = \{\beta \in \Lambda^2 : \beta \lrcorner \Omega_0 = 3\beta\}, \qquad \Lambda^2_{21} = \{\beta \in \Lambda^2 : \beta \lrcorner \Omega_0 = -\beta\}.$$

Furthermore, we have the projection formulas for these spaces

$$\pi_7(\beta) = \frac{1}{4}(\beta + \beta \lrcorner \Omega_0) \ \text{ and } \ \pi_{21}(\beta) = \frac{1}{4}(3\beta - \beta \lrcorner \Omega_0). \tag{7.1.5}$$

Proof. Cf. [Kar08b, Puh09]. □

The space of 3-forms has $\binom{8}{3} = 56$ dimensions and contains the vectorial representation $\mathbb{R}^8$ inside Λ^3 via the embedding invariant map

$$X \in \Lambda^1 \mapsto X \lrcorner \Omega_0 \in \Lambda^3$$

and still left 48 dimensions to describe. It happens that Spin(7) have a 48-dimensional irreducible representation given by the kernel of the Clifford multiplication $\Delta_7 \otimes \mathbb{R}^7 \to \Delta_7$. It is well known that this space is an irreducible Spin(7)-representation [Fri02, §2.4]. Consequently, we have:

Proposition 7.1.4 (Decomposition of 3-forms)**.** *The space $\Lambda^3(\mathbb{R}^8)^*$ of 3-forms decomposes into* Spin(7)*-irreducible representations as*

$$\Lambda^3 = \Lambda^3_8 \oplus \Lambda^3_{48} \tag{7.1.6}$$

where

$$\Lambda^3_8 = \{X \lrcorner \Omega_0 : X \in \mathbb{R}^8\} \tag{7.1.7}$$
$$\Lambda^3_{48} = \{\gamma \in \Lambda^3 : \gamma \wedge \Omega_0 = 0\}. \tag{7.1.8}$$

Proof. We have to prove the characterization of $\Lambda^3_{48} = \left(\Lambda^3_8\right)^\perp$. But this is immediate via this perpendicularity, let $\gamma \in \Lambda^3_{48}$, then

$$\Lambda^3_8 \perp \Lambda^3_{48} \;\Rightarrow \Omega_0 \perp \gamma \Rightarrow \langle \Omega_0, \gamma \rangle = 0 \Rightarrow \gamma \lrcorner \Omega_0 = 0$$

so $\gamma \in \Lambda^3_{48} \iff \gamma \wedge \Omega_0 = 0$ and we have proved the characterization of Λ^3_{48}. □

We also have a decomposition of 4-forms (to characterize all the spaces of forms). We will not use this decomposition in the text, but it is:

Proposition 7.1.5 (Decomposition of 4-forms)**.** *The space $\Lambda^4(\mathbb{R}^8)^*$ of 4-forms decomposes into* Spin(7)*-irreducible representations as*

$$\Lambda^4(M) = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35} \tag{7.1.9}$$

where

$$\begin{aligned}
\Lambda^4_1 &= \{f\Omega_0 : f \in \mathbb{R}\}\\
\Lambda^4_7 &= \{\beta \lrcorner^1 \Omega_0 : \beta \in \Lambda^2_7\},\\
\Lambda^4_{27} &= \{\sigma \in \Lambda^4 : \sigma \wedge \Omega_0 = 0, \sigma \wedge \Lambda^4_7 = 0; *\sigma = \sigma\}\\
\Lambda^4_{35} &= \{\sigma \in \Lambda^4 : *\sigma = -\sigma\}
\end{aligned}$$

Proof. [Kar05, Section 4.2] and [Kar08b, Section 2.1]. □

7.2 Spin(7)-structures on manifolds, their torsion forms and intrinsic torsion

As G_2, the group Spin(7) is the isotropy of a form, then a Spin(7)-structure on a manifold M^8 is equivalent to a 4-form $\Omega \in \Omega^4(M)$ which can put punctually in the standard form (7.1.2).

Remark 7.2.1. In seven dimensions [Joy00], every non-degenerate 3-form can be put in the standard form (3.1.4) for G_2-structures. However, for eight dimensions this is not true, that is, a non-degenerate 4-form doesn't necessarily define a Spin(7)-structure. ◯

The Spin(7)-structures have only two torsion forms because $d\Omega \in \Omega^5$, which has only two irreducible components. Let (M^8, Ω) be a manifold with Spin(7)-structure, then there are unique differential forms $\tau_1 \in \Omega^1$ and $\tau_3 \in \Omega^3_{48}$ (called the *torsion forms*) satisfying

$$d\Omega = \tau_1 \wedge \Omega + *\tau_3. \tag{7.2.1}$$

Now, we will describe the intrinsic torsion for a Spin(7)-structure as we have made for G_2-structures based on the discussion in Appendix B.

Proposition 7.2.2. *The intrinsic torsion of a* Spin(7)*-structure* (M^8, Ω) *lives is the space* $\Omega^3(M)$ *and decomposes into irreducible components as*

$$\Gamma \in \Omega^1 \oplus \Omega^3_{48},$$

and with the appropriate identifications, the components of the intrinsic torsion are the torsion forms:

$$\Gamma = \tau_1 + \tau_3 \in \Omega^1 \oplus \Omega^3_{48}. \tag{7.2.2}$$

In particular, $\Gamma \equiv 0$ *if, and only if* $d\Omega = 0$.

Proof. Analogous to Proposition 3.3.4. □

Let's discuss the *characteristic connection* for Spin(7)-structures. Recall that an affine connection ∇ is said to be compatible with the Spin(7)-structure if

$$\nabla\Omega = 0.$$

In the context of compatible connections with totally skew-symmetric torsion, every Spin(7)-structure admits such a connection. The existence of this connection discussed in [Fri02], while the derivation of its torsion formula was first established by [Iva04] and an alternative proof of this fact utilizing spinors can be found in Lucia Martín Merchán's thesis (cf. [MM18]). The proof we present here follows the flux theorem B.3.6 in Appendix B.

Proposition 7.2.3. *Let (M^8, Ω) be a* Spin(7)*-structure, then it admits a compatible connection with totally skew-symmetric torsion $\nabla^+ = \nabla^g + \frac{1}{2}g^{-1}T$. Furthermore, this connection is unique and its torsion is given by*

$$T = -\frac{1}{6}\tau_1 \lrcorner \Omega - \tau_3. \tag{7.2.3}$$

this 3-form is called flux *and denoted by $T = H$.*

Proof. We have seen in the last result the intrinsic torsion $\Gamma = \tau_1 + \tau_3$ lives in $\Omega^1 \oplus \Omega^3_{48} \cong \Omega^3$, and this space embeds (in fact, is equal) to Ω^3. Hence, the structure admits a unique compatible connection with skew-symmetric torsion $T \in \Omega^3$ by the results in Appendix B.

To compute the formula for the torsion, we have to find the eigenvalues for the flux operator $\mathbf{H}(\gamma) = \gamma \lrcorner^2 \Omega$ in each of the irreducible components of the space of 3-forms. For the vectorial component $\Omega^3_8 = \{X \lrcorner \Omega : X \in TM\}$, we compute directly using the identities in Proposition 7.1.2:

$$\begin{aligned}\mathbf{H}(X \lrcorner \Omega) = (X \lrcorner \Omega) \lrcorner^2 \Omega &= \frac{1}{2!(3-2)!(4-2)!} X^i \Omega_i{}^{\mu\nu}{}_j \Omega_{\mu\nu ab} e^{jab} \\ &= \frac{1}{4} X^i \left(6\delta_{ia}\delta_{jb} - 6\delta_{ib}\delta_{ja} + 4\Omega_{ijab}\right) e^{jab} = X^i \Omega_{ijab} e^{jab} = 6X \lrcorner \Omega\end{aligned}$$

hence, the eigenvalue is $a_{k_8} = 6$. For Ω^3_{48}, we have no general form for an element in it, so we have to find some specific element in this space and apply $\mathbf{H}$ on it. Using the expression (7.1.2), is immediate to see that

$$\Omega_0 \wedge e^{023} = e^{1457023}, \qquad \Omega_0 \wedge e^{057} = e^{1234057} = \Omega_0 \wedge e^{023}$$

so this defines $\gamma = e^{023} - e^{057} \in \Omega^3_{48}$. Proceeding analogous (by direct computation as we have done in Theorem 3.4.4), we obtain $a_{k_{48}} = -1$. Now, the expression for co-differential of Ω is $\delta\Omega = - * d\Omega = -\tau_1 \lrcorner \Omega + \tau_3$, consequently we have by the flux theorem B.3.6 that $T = -\frac{1}{6}\tau_1 \lrcorner \Omega - \tau_3$. □

7.3 Generalized Ricci flatness and coupled equations for Spin(7)-structures

As we have seen, a Spin(7)-structure (M^8, Ω) has naturally an instanton form given by $\Omega \in \Omega^4(M)$. Furthermore, the instanton form is determinant since (cf. Theorem 7.2.3)

$$a_8 = 6, \qquad a_{48} = -1.$$

where $\Omega^3 = \Omega^3_8 \oplus \Omega^3_{48}$ such that $\Omega^3_{48} = \mathcal{O}^{-1}$ and $\mathcal{O}^0 = \emptyset$ (see (6.2.3)). By Theorem 7.2.3, we have the (particular) gravitino equation is always satisfied:

$$\nabla^+\Omega = 0.$$

The torsion forms are

$$\tau^8 = -\tau_1 \lrcorner \Omega, \qquad \tau^{48} = \tau_3,$$

the norm of the instanton form is $|\Omega|^2 = 14$, $\alpha = -\frac{4|\Omega|^2}{8} = -7$ (cf. Lemma 6.3.3). Consequently, the Lee form is given by

$$\zeta := -\left(1 + \frac{1}{a_8}\right) \cdot \frac{1}{\alpha}\tau^8 \lrcorner \Omega = \left(1 + \frac{1}{6}\right)\tau_1 = \frac{7}{6}\tau_1.$$

The partial norm using Proposition 7.1.2, we have

$$[\![\Omega]\!]^2 = \frac{1}{16}\Omega^{ij}{}_{ab}\Omega^{ab}{}_{\mu\nu}\Omega^{\mu\nu}{}_{ij} = \frac{1}{16}\Omega^{ij}{}_{ab}\left(6\delta_i^a\delta_j^b - 6\delta_i^b\delta_j^a + 4\Omega_{ij}{}^{ab}\right) = \frac{4 \cdot 336}{16} = 84,$$

Consequently, we have

$$\frac{4}{8}|\Omega|^4 - |\Omega|^2 = 98 - 14 = 84 = [\![\Omega]\!]^2$$

i.e., we are in the hypothesis of Theorem 6.3.6; consequently, we have the result about generalized Ricci flatness:

Theorem 7.3.1. *Let $P \to M^8$ be a principal K-bundle for (M^8, Ω) a* Spin(7)*-structure and suppose a connection θ satisfies the instanton condition and Bianchi identity:*

$$F_\theta \lrcorner \Omega = -F_\theta, \qquad dH = \langle F_\theta \wedge F_\theta \rangle.$$

Then the Riemannian metric $g = g_\Omega$ on M determined by the Spin(7)*-structure satisfies:*

$$\begin{aligned} \mathrm{Ric}^g - \tfrac{1}{4}H^2 + F_\theta \circ F_\theta + \tfrac{7}{12}\mathcal{L}_{\tau_1^\#} g &= 0, \\ d^*H - \tfrac{7}{6}d\tau_1 + \tfrac{7}{6}i_{\zeta^\#}H &= 0, \\ d^{\theta *}F_\theta - F_\theta \lrcorner H + \tfrac{7}{6}i_{\tau_1^\#}F_\theta &= 0. \end{aligned} \tag{7.3.1}$$

where $H = H_\Omega$ (cf. Theorem 7.2.3). In particular,

$$\mathrm{GRic}^+_{\mathbf{G}_\Omega, \mathrm{div}^\Omega} = 0,$$

where $\mathbf{G}_\Omega$ is obtained analogously as in Remark 4.1.3 and the divergence operator is uniquely determined by the Spin(7)*-structure via the explicit formula given by Remark 2.2.4:*

$$\mathrm{div}^\Omega = \mathrm{div}^{\mathbf{G}_\Omega} - 2\langle \tfrac{7}{6}\tau_1, \cdot \rangle.$$

The first two equations in (7.3.1) are the symmetric and skew-symmetric parts of the equation

$$\mathrm{Ric}^+ + F_\theta \circ F_\theta + \tfrac{7}{6}\nabla^+\tau_1 = 0.$$

Now, another thing to see for Spin(7) is the coupled Spin(7)-instanton equations. Initially, note that $\Omega^2_{21} \lrcorner^1 \Omega = 0$ due to $\mathfrak{spin}(7) = \Omega^2_{21}$ not being a component of $\Omega^4 =$

$\Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27} \oplus \Omega^4_{35}$. Thus, the conditions outlined in Lemma 6.4.3 are satisfied. Moreover, as $N(\mathrm{Spin}(7)) = \mathrm{Spin}(7)$ within SO(8) (thus, the endomorphism part of R_{∇^+} resides in $\mathfrak{spin}(7)$), Theorem 6.4.4 applies directly to this specific scenario: instanton condition for θ and heterotic Bianchi identity implies coupled instanton condition (here the gravitino equation is always satisfied, i.e., $\nabla^+\Omega = 0$).

Theorem 7.3.2. *Let E be a transitive Courant algebroid with generalized metric $\mathbf{G}$ over a* Spin(7)*-structure (M^8, Ω) and suppose:*

$$F_\theta \lrcorner \Omega = -F_\theta, \qquad dH - \langle F_\theta \wedge F_\theta \rangle = 0.$$

Then, the coupled Spin(7)*-instanton equation is satisfied, i.e., $F_D \lrcorner \Omega = -F_D$.*

Remark 7.3.3. Note that in the case of Spin(7)-structures, the first entry of the curvature F_D is equivalent to

$$\langle F_\theta, F_\theta \rangle \in \Omega^2_{21} \otimes \Omega^2$$

because R_{∇^+} always have endomorphism part in Ω^2_{21} due to Theorem 7.2.3. ○

7.4 The group $\mathrm{U}(m)$ and the decomposition of forms

Now, we will explore the notion of SU(m)-instantons, which makes sense in the context of U(m)-structures (almost Hermitian structures) in $(2m)$-dimensional manifolds.

The group $\mathrm{U}(m) \leq \mathrm{SO}(2m)$ can be defined as the one which stabilizes the standard hermitian form in $\mathbb{R}^{2m}$

$$\mathrm{U}(m) := \left\{ g \in \mathrm{SO}(2m) : g^*\omega_0 = \omega_0 \right\} \tag{7.4.1}$$

where $\omega_0 \in \Omega^2(\mathbb{R}^{2m})$ is the standard hermitian structure in $\mathbb{R}^{2m}$ defined by (using $\{e^1, e^2, \cdots, e^{2m-1}, e^{2m}\}$ the canonical basis and metric of $\mathbb{R}^{2m}$):

$$\omega_0 = e^{12} + e^{34} + \cdots + e^{2m-1} \wedge e^{2m} = \sum_{k=1}^{m} e^{2k-1} \wedge e^{2k} \tag{7.4.2}$$

The powers of ω_0 satisfy some interesting properties

$$\frac{1}{k!} * \omega_0^k = \frac{1}{(m-k)!}\omega_0^{m-k}; \qquad \mathrm{vol}_{\mathbb{R}^{2m}} = \frac{1}{m!}\omega_0^m; \qquad \left|\frac{\omega_0^k}{k!}\right|^2 = \binom{m}{k} = \frac{m!}{k!(m-k)!}. \tag{7.4.3}$$

The group U(m) is not obtained as the stabilizer of ω_0 in GL$(2m)$, so a metric is initially considered in such structures and not induced by it. In the classical approach in almost Hermitian manifolds, the first object considered is the almost complex structure J_0. For us here, it will be defined in terms of the objects we already have: ω_0 and g_0:

$$g_0(X, Y) = \omega_0(X, J_0Y).$$

In coordinates, we have

$$g_{ij} = \omega_{ik} J^k{}_j.$$

where the coefficients are given by

$$\omega_0 = \frac{1}{2!}\omega_{ij} e^{ij}; \qquad J = J^k{}_j e^j \otimes e_k.$$

We can check the essential property of the almost complex structure $J_0^2 = -\operatorname{Id}$ using the contraction identities for ω

Proposition 7.4.1. *The fundamental* $\mathrm{U}(m)$*-structure* ω_0 *on* $\mathbb{R}^{2m}$ *satisfies the following relations between the coefficients*

$$\begin{aligned} \omega^{\mu\nu}\omega_{\mu\nu} &= 2|\omega|^2 = 2m \\ \omega^{\mu i}\omega_{\mu a} &= \delta^i{}_a. \end{aligned}$$

Furthermore, if we define $\psi_0 = \frac{1}{2}\omega_0 \wedge \omega_0$*, given in coordinates by* $\psi_0 = \frac{1}{4!}\psi_{ijkl}e^{ijkl}$ *will satisfy* $|\psi_0|^2 = \frac{1}{2}m(m-1)$ *and*

$$\begin{aligned} \psi_{ijkl} &= \omega_{ij}\omega_{kl} - \omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk}, \\ \psi^{ijkl}\psi_{ijkl} &= 12m(m-1), \\ \psi^{ijkl}\psi_{ijk\mu} &= 6\delta_{l\mu}(m-1), \\ \psi^{ijkl}\psi_{ijab} &= 2\omega^{kl}\omega_{ab}(m-2) + 2\delta_a{}^k\delta_b{}^l - 2\delta_a{}^l\delta_b{}^k. \end{aligned}$$

Proof. The formulas for ω are immediate given the norm $|\omega|^2 = m$ and the fact that ω is essentially the almost complex structure up to lower an index. The expression $\psi_{ijkl} = \omega_{ij}\omega_{kl} - \omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk}$ is by definition of ψ and for the others for ψ are just elementary manipulations of this one. ☐

The decomposition of the space of 2-forms (and of k-forms in general is based on the realification of the decomposition of complex forms, and we will follow the convention and notation in [Sal89]) is decomposed using the realification of complex forms:

$$\Lambda^2 = [\![\Lambda^{2,0}]\!] \oplus [\Lambda_0^{1,1}] \oplus \langle\omega_0\rangle$$

since $\dim_{\mathbb{R}}[\![\Lambda^{2,0}]\!] = \dim_{\mathbb{C}}\left(\Lambda^{2,0} \oplus \Lambda^{0,2}\right) = 2\binom{m}{2}\binom{m}{0} = m(m-1)$ and the sum space $[\Lambda_0^{1,1}] \oplus \langle\omega\rangle \cong \mathfrak{u}(m) = \mathfrak{su}(m) \oplus \mathbf{1}$ (note that the adjoint representation $\mathfrak{u}(m)$ is not irreducible since $\mathrm{U}(m)$ is not simple, however, $\mathfrak{su}(m)$ is irreducible because $\mathrm{SU}(m)$ is simple), so we have:

Proposition 7.4.2 (Decomposition of 2-forms)**.** *The space* $\Lambda^2(\mathbb{R}^{2m})^*$ *of 2-forms decomposes into* $\mathrm{U}(m)$*-irreducible representations as*

$$\begin{aligned} \Lambda^2 &= \Lambda^2_{m(m-1)} \oplus \Lambda^2_{m^2-1} \oplus \Lambda^2_1 \\ &\cong [\![\Lambda^{2,0}]\!] \oplus [\Lambda_0^{1,1}] \oplus \langle\omega_0\rangle \cong [\![\Lambda^{2,0}]\!] \oplus \mathfrak{su}(m) \oplus \langle\omega\rangle \end{aligned} \tag{7.4.4}$$

where (considering $\psi = \frac{1}{2!}\omega_0^2$)

$$\Lambda^2_1 = \{\beta \in \Lambda^2 : \beta \lrcorner \psi = m-1\} = \{f\omega : f \in \mathbb{R}\}$$
$$\Lambda^2_{m^2-m} = \{\beta \in \Lambda^2 : \beta \lrcorner \psi = 3\} = \{\beta \in \Lambda^2 : J\beta = -\beta\}$$
$$\Lambda^2_{m^2-1} = \{\beta \in \Lambda^2 : \beta \lrcorner \psi = -1\} = \{\beta \in \Lambda^2 : J\beta = \beta, \beta \lrcorner \omega = 0\}$$

the condition $\beta \lrcorner \omega = 0$ also holds in $\Omega^2_{m^2-m}$. Furthermore, we have the projection formulas for these spaces

$$\pi^2_1(\beta) = \frac{1}{m}\langle \beta, \omega\rangle \omega \tag{7.4.5}$$
$$\pi^2_i(\beta) = \frac{1}{\lambda_i^2 - \lambda_i(\lambda_j + \lambda_k) + \lambda_j\lambda_k}\Big((\beta \lrcorner \xi) \lrcorner \xi - (\lambda_j + \lambda_j)\beta \lrcorner \xi + \lambda_j\lambda_k\beta\Big) \tag{7.4.6}$$

where λ_i are the eigenvalues of $\beta \mapsto \beta \lrcorner \psi$ and $i \neq j \neq k \neq i$.

Proof. Cf. [Sal89, FSE19, MU19]. □

We can embed Λ^1 (irreducible) into Λ^3 via the map $X \in \Lambda^1 \mapsto X \wedge \omega_0 \in \Lambda^3$, we also have two other irreducible components inside Λ^3 (Cf. [GH80, Thrm 2.1] and [Fri02, FI02, AFS05]). We have:

Proposition 7.4.3 (Decomposition of 3-forms)**.** *The space $\Lambda^3(\mathbb{R}^{2m})^*$ of 3-forms decomposes into $\mathrm{U}(m)$-irreducible representations as*

$$\Lambda^3(M) = \Lambda^3_{2m} \oplus \Lambda^3_{\frac{1}{3}m(m-1)(m-2)} \oplus \Lambda^3_{m(m+1)(m-2)} \cong \Lambda^1 \oplus [\![\Lambda^{3,0}]\!] \oplus [\![\Lambda^{2,1}_0]\!] \tag{7.4.7}$$

where the identification $\Lambda^3_{2m} \cong \Lambda^1$ is $\Lambda^3_{2m} = \{X \wedge \omega : X \in \Lambda^1\}$.

Proof. [Sal89, GH80, AFS05]. □

7.5 Torsion of almost Hermitian structures

For manifolds, a $\mathrm{U}(m)$-structure is equivalent to (M^{2m}, g, ω) where ω is punctually ω_0 and g a Riemannian metric, then there are unique differential forms

$$\tau_1 \in \Omega^1, \qquad \tau_{3,1} \in \Omega^3_{\frac{1}{3}m(m-1)(m-2)}, \qquad \tau_{3,3} \in \Omega^3_{m(m+1)(m-2)}$$

(called the torsion forms) satisfying

$$d^*\psi = \tau_1 \wedge \omega + \tau_{3,1} + \tau_{3,3} \tag{7.5.1}$$

Remark 7.5.1 (Relation between τ_1 and the Lee form). Consider $d\omega^{m-1} \in \Omega^{2m-1} \cong \Omega^1$, then there is a unique 1-form such that $d\omega^{m-1} = \theta_\omega \wedge \omega^{m-1}$ (the so-called *Lee form* in the context of almost Hermitian structures).

On the other hand, note that the map $\gamma \in \Omega^3 \mapsto \omega \lrcorner \gamma \in \Omega^1$ is invariant, so the only component which is not cancelled is Ω^3_{2m}, consequently

$$\omega \lrcorner d^*\psi = (\text{constant}) \cdot \tau_1$$

We have

$$\omega \lrcorner d^*\psi = \omega \lrcorner (\tau_1 \lrcorner \psi) = \tau_1 \lrcorner (\omega \lrcorner \psi) = \tau_1 \lrcorner * \frac{\omega^{m-1}}{(m-2)!} = (m-1)\tau_1 \lrcorner \omega = (m-1)\tau_1$$

since $X \lrcorner \omega = X \in \Omega^1$ (by invariance), but on the other hand, we have

$$\begin{aligned} \omega \lrcorner d^*\psi &= - * (\omega \wedge d * \psi) = -\frac{1}{(m-2)!} * \left(\omega \wedge d\omega^{m-2}\right) = -\frac{m-2}{(m-2)!} * \left(d\omega \wedge \omega^{m-2}\right) \\ &= -\frac{m-2}{(m-1)!} * d\omega^{m-1} = -\frac{m-2}{(m-1)!} * (\theta_\omega \wedge \omega^{m-1}) = -(m-2)\theta_\omega \lrcorner \omega \\ &= -(m-2)\theta_\omega \end{aligned}$$

We conclude that τ_1 and the Lee form θ_ω are essentially the same thing, up to a constant. They are related by the expression $\tau_1 = -\frac{m-2}{m-1}\theta_\omega$. ○

Now, let's describe the intrinsic torsion of almost Hermitian structures. Since $\mathfrak{so}(2m) = \mathfrak{u}(m) \oplus \mathfrak{m}^{m(m-1)}$, consequently, the space where the intrinsic torsion Γ lives is

$$\begin{aligned} \mathbb{R}^{2m} \otimes \mathfrak{m}^{m(m-1)} &\cong \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4 \\ &\cong \Omega^3_{\frac{1}{3}m(m-1)(m-2)} \oplus \mathcal{W}_2 \oplus \Omega^3_{m(m+1)(m-2)} \oplus \Omega^3_1 \\ &\cong \Omega^3 \oplus \mathcal{W}_2, \end{aligned}$$

where the spaces $\mathcal{W}_j$ were defined in [GH80]. Since $\mathrm{U}(m)$ is the isotropy of ω, so, as we have one before in Proposition 3.3.4 and Proposition 7.2.2 we have that the intrinsic torsion Γ is identified as

$$\Gamma = \tau_1 \wedge \omega + \tau_{3,1} + \tau_{3,3} + \pi_{\mathcal{W}_2}(\Gamma), \tag{7.5.2}$$

i.e., $\mathcal{W}_2$ and $d\omega$ define Γ completely. Some properties of these $\mathcal{W}_j$ classes are important [GH80]:

- $\Gamma \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, the so-called $\mathcal{G}_1$-manifolds is equivalent to the Nijenhius tensor $N \in \Omega^2 \otimes \Omega^1$, being completely skew-symmetric, i.e., $N \in \Omega^3$.
- $\Gamma \in \mathcal{W}_3 \oplus \mathcal{W}_4$ if, and only if $N \equiv 0$, which is equivalent to the almost complex structure being integrable, i.e., the manifold is complex (hermitian). Therefore, a $\mathcal{G}_1$-hermitian structure is a complex manifold if, and only if $\tau_{3,1} = 0$ [GH80, AFS05].

With this, we can investigate compatible connections with totally skew-symmetric torsion.

Theorem 7.5.2 ([FI02]). *Let (M^{2m}, ω, g) be an almost-Hermitian manifold. Then, there exists a compatible connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor $N(X, Y, Z) := g(N(X, Y), Z)$ is a 3-form. In this case, the connection is unique and is determined by*

$$T = H = -Jd\omega + N \tag{7.5.3}$$

In particular, if the manifold is hermitian, then $H = -Jd\omega = -d^c\omega$. The existence of such a connection is equivalent to $\pi_{\mathcal{W}_2}(\Gamma) = 0$.

The result above is interesting because it gives us the torsion in terms of ω and N. Still, for our purposes, it is interesting because it provides the torsion in terms of the torsion forms, so we have the theorem below, which is proved in [AFS05] for $m = 3$, but here we will prove the general version and using our method.

Theorem 7.5.3. *Let (M^{2m}, g, ω) be an almost-Hermitian manifold which admits a compatible connection $\nabla^+ = \nabla^g + \frac{1}{2}T$ which totally skew-symmetric torsion $T \in \Omega^3(M)$, then its torsion is given by*

$$T = \frac{1}{m-2}\tau_1 \lrcorner \psi + \frac{1}{3}\tau_{3,1} - \tau_{3,3}.$$

Proof. Following the flux theorem B.3.6, we have to compute the eigenvalues of the flux operator $\mathbf{H}(\gamma) = \gamma \lrcorner^2 \psi$. In the vector component in Ω^3_{2m}, which elements have the form $X \wedge \omega$, we have using the contraction identities in Proposition 7.4.1

$$\begin{aligned}
\mathbf{H}(X \wedge \omega) &= (X \wedge \omega) \lrcorner^2 \psi = \tfrac{1}{2!2!}(X_j\omega_{ka} + X_a\omega_{jk} + X_k\omega_{aj})\psi_{jkbc}e^{abc} \\
&= \tfrac{1}{4}(X_j\omega_{ka} + X_a\omega_{jk} + X_k\omega_{aj})(\omega_{jk}\omega_{bc} - \omega_{jb}\omega_{kc} + \omega_{kb}\omega_{jc})e^{abc} \\
&= \tfrac{1}{4}\Big(- X_j\delta_{aj}\omega_{bc} - \cancel{X_j\delta_{ac}\omega_{jb}} + \cancel{X_j\delta_{ac}\omega_{jc}} + 2mX_a\omega_{bc} \\
&\qquad - X_a\delta_{kb}\omega_{kc} - X_a\delta_{bj}\omega_{jc} - X_k\delta_{ka}\omega_{bc} + \cancel{X_k\delta_{ab}\omega_{kc}} + \cancel{X_k\delta_{ab}\omega_{jc}}\Big)e^{abc} \\
&= (m-2)X \wedge \omega
\end{aligned}$$

and we have obtained the desired eigenvalue: $a_{2m} = m - 2$. Now, let's consider the irreducible component $\Omega^3_{\frac{1}{3}m(m-1)(m-2)} \cong [\![\Omega^{3,0}]\!]$. For all $m \geq 3$, this space complexified $[\![\Omega^{3,0}]\!]^2 \otimes \mathbb{C} = \Omega^{3,0}$ contains the element dz^{123} by definition, and it is given by

$$\begin{aligned}
dz^{123} &= (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6) \\
&= e^{135} + ie^{136} + ie^{145} - e^{146} + ie^{235} - e^{236} - e^{245} - ie^{246}
\end{aligned}$$

Considering this complex form as a real form, we obtain

$$\gamma = e^{135} + e^{136} + e^{145} - e^{146} + e^{235} - e^{236} - e^{245} - e^{246} \in [\![\Omega^{3,0}]\!]$$

performing the computations for this specific form, we obtain

$$a_{\frac{1}{3}m(m-1)(m-2)} = 3.$$

Now, for the component $\Omega^3_{m(m+1)(m-2)} \cong [\![\Omega_0^{2,1}]\!]$ and this space always contains the 3-form $\gamma = \alpha - \beta = e^{123} - e^{356}$ (Cf. [AFS05, p.4]), we obtain

$$a_{m(m+1)(m-2)} = -1.$$

Using $\delta\psi = \tau_1 \lrcorner \psi + \tau_{3,1} + \tau_{3,3}$ and the Flux Theorem B.3.6, the result follows. □

Note that, in particular, the instanton form is determinant (because we are considering $m \geq 3$) and

$$\mathcal{O}^0 = \Omega^3_{\frac{1}{3}m(m-1)(m-2)} = \mathcal{W}_1, \qquad \mathcal{O}^{-1} = \Omega^3_{m(m+1)(m-2)} = \mathcal{W}_3.$$

7.6 Generalized Ricci flatness and coupled $\mathrm{SU}(m)$-instantons

As we have seen, a $\mathrm{U}(m)$-structure (M^{2m}, ω, g) has naturally an instanton form given by $\psi = \frac{1}{2}\omega^2 \in \Omega^4(M)$. Furthermore, the instanton form is determinant since (cf. Theorem 7.5.3)

$$a_{2m} = m - 2, \qquad a_{m(m+1)(m-2)} = -1, \qquad a_{m(m-1)(m-2)/3} = 3.$$

where the space of 3-forms decomposes as

$$\Omega^3 = \Omega^3_{2m} \oplus \Omega^3_{m(m+1)(m-2)} \oplus \Omega^3_{m(m-1)(m-2)/3}$$

such that $\Omega^3_{m(m+1)(m+2)} = \mathcal{O}^{-1}$ and $\mathcal{O}^0 = \Omega^3_{m(m-1)(m-2)/3}$ (see (6.2.3)). By Theorem 7.5.3, we have the (particular) gravitino equation satisfied if the component $\mathcal{W}_2$ of the intrinsic torsion vanishes:

$$\pi_{\mathcal{W}_2}(\Gamma) \equiv 0.$$

The torsion forms are

$$\tau^{2m} = \tau_1 \lrcorner \psi, \qquad \tau^{\frac{1}{3}m(m-1)(m-2)} = \tau_{3,1}, \qquad \tau^{m(m+1)(m-2)} = \tau_{3,3}.$$

the norm of the instanton form is $|\psi|^2 = \frac{1}{2}m(m-1)$, $\alpha = -\frac{4|\psi|^2}{2m} = 1 - m$ (cf. Lemma 6.3.3). Consequently, the *Lee form* is given by

$$\zeta \coloneqq -\left(1 + \frac{1}{a_{2m}}\right)\tau_1 = \frac{m-1}{m-2} \cdot \frac{m-2}{m-1}\theta_\omega = \theta_\omega$$

so, the Lee form as defined in Chapter 6 is, in fact, the Lee form in the context of almost Hermitian structures (in fact, this correspondence was the motivation to call ζ the Lee form). The partial norm using Proposition 7.4.1, we have

$$\begin{aligned}
[\![\psi]\!]^2 &= \tfrac{1}{16}\psi^{ij}{}_{kl}\psi^{kl}{}_{ab}\psi^{ab}{}_{ij} = \tfrac{2(m-2)}{16}\omega_{kl}\omega^{ab}\left(\omega_{kl}\omega^{ab} + \omega_k{}^a\omega_b{}^l - \omega_k{}^b\omega^a{}_l\right)\\
&= \tfrac{2(m-2)}{16}\left(2m\cdot 2m - \delta_{la}\delta_{la} - \delta_{lb}\delta_{lb}\right) = \tfrac{2(m-2)}{16}\left(4m^2 - 4m\right)\\
&= \tfrac{1}{2}m(m-1)(m-2)
\end{aligned}$$

Consequently, we have

$$\frac{4}{2m}|\psi|^4 - |\psi|^2 = \frac{4}{2m}\left(\frac{1}{2}m(m-1)\right)^2 - \frac{1}{2}m(m-1) = \frac{1}{2}m(m-1)(m-2) = [\![\psi]\!]^2$$

i.e., we are in the hypothesis of Theorem 6.3.6; consequently, we have the result about generalized Ricci flatness:

Theorem 7.6.1. *Let $P \to M^{2m}$ be a principal K-bundle for $(M^{2m}, \omega, g_\omega)$ a $\mathrm{U}(m)$-structure with instanton form $\psi = \frac{1}{2}\omega^2$ and suppose a connection θ satisfies the instanton condition and Bianchi identity:*

$$F_\theta \lrcorner\, \psi = -F_\theta, \qquad dH = \langle F_\theta \wedge F_\theta\rangle,$$

and that the endomorphism part of R_{∇^+} lives in $\mathfrak{su}(m) \leq \mathfrak{u}(m) \leq \Omega^2$. Then the Riemannian metric $g = g_\omega$ on M satisfy:

$$\begin{aligned}
\mathrm{Ric}^g - \tfrac{1}{4}H^2 + F_\theta \circ F_\theta + \tfrac{1}{2}\mathcal{L}_{\theta_\omega^\#}g &= 0,\\
d^*H - d\theta_\omega + i_{\theta_\omega^\#}H &= 0,\\
d^{\theta*}F_\theta - F_\theta \lrcorner\, H + i_{\theta_\omega^\#}F_\theta &= 0.
\end{aligned}\tag{7.6.1}$$

where $H = H_\omega = -d^c\omega + N$ is the flux (cf. Theorem 7.5.2) and $\theta_\omega \in \Omega^1(M)$ the Lee form. In particular,

$$\mathrm{GRic}^+_{\mathbf{G}_\omega, \mathrm{div}^\omega} = 0,$$

where $\mathbf{G}_\omega$ is obtained analogously as in Remark 4.1.3 and the divergence operator is uniquely determined by the $\mathrm{U}(m)$-structure via the explicit formula given by Remark 2.2.4:

$$\mathrm{div}^\omega = \mathrm{div}^{\mathbf{G}_\omega} - 2\langle\theta_\omega, \cdot\rangle.$$

The first two equations in (7.6.1) are the symmetric and skew-symmetric parts of the equation

$$\mathrm{Ric}^+ + F_\theta \circ F_\theta + \nabla^+\theta_\omega = 0.$$

In concluding our discussion, let's consider the coupled instanton equations in the case of $\mathrm{U}(m)$-structures, which is a specific case of Theorem 6.4.4. To apply the theorem, as we have mentioned in Lemma 6.4.3, we need to ensure that $\mathfrak{su}(m) \lrcorner^1 \mathfrak{su}(m)$. In contrast to the case of G_2 and Spin(7)-structures, where the Lie algebra was not a component of Ω^4 (so the condition would be trivial due to the imposition $\mathfrak{h} \lrcorner^1 \psi = 0$, cf.

Lemma 6.4.3), in almost Hermitian manifolds, $\mathfrak{su}(m)$ is part of Ω^4 [Sal89]. Therefore, we need to prove that the condition $\mathfrak{su}(m) \lrcorner^1 \mathfrak{su}(m)$ still holds. In fact:

$$\begin{aligned}(dz^i \wedge d\overline{z}^j) \lrcorner^1 (dz^k \wedge d\overline{z}^l) &= (e_\mu \lrcorner (dz^i \wedge d\overline{z}^j)) \wedge (e_\mu \lrcorner (dz^k \wedge d\overline{z}^l)) \\ &= ((\delta^i_\mu + \mathbf{i}\delta^i_{m+\mu})d\overline{z}^j - (\delta^j_\mu - \mathbf{i}\delta^i_{m+\mu})dz^j) \wedge ((\delta^k_\mu + \mathbf{i}\delta^k_{m+\mu})d\overline{z}^l - (\delta^l_\mu - \mathbf{i}\delta^j_{m+\mu})dz^k)\end{aligned}$$

which is clearly in $\Omega^{1,1} = \mathfrak{u}(m)$. The operation $\lrcorner^1$ preserves $\mathfrak{su}(m)$ because in the complement $\langle\omega\rangle$ it is zero:

$$\omega \lrcorner^1 \omega = \frac{1}{1!1!1!}\omega_{ij}\omega_{ik}e^{jkab} = \delta_{jk}e^{jk} = 0.$$

With this, we then have

Theorem 7.6.2. *Let E be a transitive Courant algebroid with generalized metric $\mathbf{G}$ over a manifold with* U(m)*-structure (M^{2m}, g, ω) and denote $\psi = \frac{1}{2}\omega^2$ and suppose:*

$$\nabla^+\psi = 0, \qquad F_\theta \lrcorner \psi = -F_\theta, \qquad dH - \langle F_\theta \wedge F_\theta\rangle = 0.$$

Then, the coupled U(m)*-instanton equation is satisfied, i.e., $F_D \lrcorner \psi = -F_D$.*

Bibliography

[ACDAdLHGF24] Luis Álvarez-Cónsul, Andoni De Arriba de La Hera, and Mario García-Fernández. Vertex algebras from the Hull-Strominger system, 2024. cf. pages 31 and 34.

[ACFH15] Ilka Agricola, Simon G. Chiossi, Thomas Friedrich, and Jos Höll. Spinorial description of $SU(3)$ and G_2-manifolds. *Journal of Geometry and Physics*, 98:535–555, December 2015. cf. pages 50, 51, 63, 64, and 66.

[AF03] Ilka Agricola and Thomas Friedrich. On the holonomy of connections with skew-symmetric torsion, 2003. cf. pages 42, 44, and 45.

[AFS05] Bogdan Alexandrov, Thomas Friedrich, and Nils Schoemann. Almost hermitian 6-manifolds revisited. *Journal of Geometry and Physics*, 53(1):1–30, January 2005. cf. pages 121, 130, 131, 132, 133, 149, and 154.

[Agr06] Ilka Agricola. The SRNI lectures on non-integrable geometries with torsion, 2006. cf. pages 61, 148, and 151.

[AMP24] Anthony Ashmore, Ruben Minasian, and Yann Proto. Geometric flows and supersymmetry. *Communications in Mathematical Physics*, 405(1):Paper No. 16, 50, 2024. cf. page 42.

[BG08] Charles Boyer and Krzysztof Galicki. *Sasakian Geometry*. Oxford University Press, 2008. cf. page 88.

[BH15] David Baraglia and Pedram Hekmati. Transitive Courant algebroids, string structures and T-duality. *Advances in Theoretical and Mathematical Physics*, 19(3):613–672, 2015. cf. page 16.

[Bis89] Jean-Michel Bismut. A local index theorem for non-Kähler manifolds. *Mathematische Annalen*, 284(4):681–699, 1989. cf. pages 42, 45, and 48.

[Bla10] David E. Blair. *Riemannian Geometry of Contact and Symplectic Manifolds*. Birkhäuser, 2010. cf. pages 11, 88, and 89.

[Bry87] R. L. Bryant. Metrics with exceptional holonomy. *Ann. of Math. (2)*, 126(3):525–576, 1987. cf. page 10.

[Bry05] Robert L. Bryant. Some remarks on G_2-structures, 2005. cf. pages 50, 54, 57, and 72.

[BS89] R. L. Bryant and S. M. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.*, 58(3):829–850, 1989. cf. page 10.

[BW58] W. M. Boothby and H. C Wang. On contact manifolds. *Ann. of Math*, 68:721–734, 1958. cf. page 89.

[Car98] Ramón Reyes Carrión. A generalization of the notion of instanton. *Differential Geometry and its Applications, Volume 8, Issue 1*, pages 1–20, 1998. cf. pages 101 and 102.

[CGFT22] Andrew Clarke, Mario García-Fernández, and Carl Tipler. T-dual solutions and infinitesimal moduli of the G_2-Strominger system. *Advances in Theoretical and Mathematical Physics*, 26(6):1669–1704, 2022. cf. pages 11 and 82.

[CPR24] Gil R. Cavalcanti, Jaime Pedregal, and Roberto Rubio. On the equivalence of generalized ricci curvatures, 2024. cf. pages 23 and 24.

[CSCW11] André Coimbra, Charles Strickland-Constable, and Daniel Waldram. Supergravity as generalised geometry I: type II theories. *Journal of High Energy Physics*, 11:091, 35, 2011. cf. pages 32, 34, and 42.

[CSX13] Z. Chen, M. Stienon, and P. Xu. On Regular Courant Algebroids. *J. Symplectic Geom. vol 11(1)*, pages 1–24, 2013. cf. page 19.

[Dar94] R. W. R. Darling. *Differential Forms and Connections.* Cambridge University Press, CUP, 1994. cf. pages 11 and 150.

[DK90] Simon K. Donaldson and Peter B. Kronheimer. *The geometry of four-manifolds.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990. Oxford Science Publications. cf. page 105.

[dlOG21] Xenia de la Ossa and Mateo Galdeano. Families of solutions of the heterotic G_2-system, 2021. cf. pages 12 and 82.

[dlOLS16] Xenia de la Ossa, Magdalena Larfors, and Eirik E. Svanes. Infinitesimal moduli of G_2 holonomy manifolds with instanton bundles. *Journal of High Energy Physics*, 2016(11), nov 2016. cf. page 115.

[dlOLS18a] Xenia de la Ossa, Magdalena Larfors, and Eirik E. Svanes. The infinitesimal moduli space of heterotic G_2 systems. *Communications in Mathematical Physics*, 360(2):727–775, 2018. cf. pages 6, 7, 11, 31, 36, 38, 39, 50, 54, 62, 68, 75, 115, and 145.

[dlOLS18b] Xenia de la Ossa, Magdalena Larfors, and Eirik E. Svanes. Restrictions of heterotic G_2 structures and instanton connections. In *Geometry and physics. Vol. II*, pages 503–517. Oxford Univ. Press, Oxford, 2018. cf. pages 31, 38, and 75.

[Don96] Simon Donaldson. Symplectic Submanifolds and Almost-Complex Geometry. *ReCALL*, 44:666–705, 1996. cf. page 101.

[DS11] Simon K. Donaldson and Ed Segal. Gauge theory in higher dimensions, II. *Surveys in Differential Geometry*, 16:1–41, 2011. cf. page 105.

[dSJGFLSE24] Agnaldo A. da Silva Jr., Mario Garcia-Fernandez, Jason D. Lotay, and Henrique N. Sá Earp. Coupled G_2-instantons, 2024. cf. pages 12, 13, 15, 33, and 82.

[DT98] Simon K. Donaldson and Richard P. Thomas. Gauge theory in higher dimensions. In *The geometric universe (Oxford, 1996)*, pages 31–47. Oxford Univ. Press, Oxford, 1998. cf. page 105.

[FG82] Marisa Fernández and A. Gray. A classification of riemannian manifolds with structure group G_2. *Annali di Matematica Pura ed Applicata, 132*, pages 19–45, 1982. cf. page 60.

[FI02] Thomas Friedrich and Stefan Ivanov. Parallel spinors and connections with skew-symmetric torsion in string theory. *Asian Journal of Mathematics*, pages 303–326, 2002. cf. pages 12, 50, 54, 56, 60, 61, 65, 83, 130, 132, 148, 149, 151, and 154.

[FIUV11] Marisa Fernández, Stefan Ivanov, Luis Ugarte, and Raquel Villacampa. Compact supersymmetric solutions of the heterotic equations of motion in dimensions 7 and 8. *Advances in Theoretical and Mathematical Physics*, 15(2):245–284, 2011. cf. page 82.

[FIUV15] Marisa Fernandez, Stefan Ivanov, Luis Ugarte, and Dimiter Vassilev. Quaternionic Heisenberg Group and Heterotic String Solutions with Non-Constant Dilaton in Dimensions 7 and 5. *Communications in Mathematical Physics*, 339(1):199–219, 2015. cf. page 82.

[FKMS97] Thomas Friedrich, Ines Kath, Andrei Moroianu, and Uwe Semmelmann. On nearly parallel G_2-structures. *Journal of Geometry and Physics*, pages 259–286, 1997. cf. pages 50, 51, 63, 64, and 66.

[FKS20] Robert Feger, Thomas W. Kephart, and Robert J. Saskowski. LieART 2.0 – a Mathematica application for Lie Algebras and Representation

Theory. *Computer Physics Communications*, 257:107490, December 2020. cf. page 54.

[FLMSE23] Daniel Fadel, Eric Loubeau, Andrés J. Moreno, and Henrique N. Sá Earp. Flows of geometric structures, 2023. cf. page 122.

[FMMR23] Anna Fino, Lucía Martín-Merchán, and Alberto Raffero. The twisted G_2 equation for strong G_2-structures with torsion, 2023. cf. pages 83, 84, and 85.

[Fri00] Thomas Friedrich. *Dirac Operators in Riemannian Geometry.* American Mathematical Society, 2000. cf. page 12.

[Fri02] Thomas Friedrich. On types of non-integrable geometries, 2002. cf. pages 50, 124, 125, 130, 148, 149, 151, and 152.

[Fri03] Thomas Friedrich. Spin(9)-structures and connections with totally skew-symmetric torsion. *Journal of Geometry and Physics*, 47(2–3):197–206, August 2003. cf. pages 151 and 152.

[FSE19] Daniel Fadel and Henrique N. Sá Earp. *Gauge Theory in Higher Dimensions.* IMPA (Instituto de Matemática Pura e Aplicada) - 32º Colóquio Brasileiro de Matemática, 2019. cf. pages 11, 101, 102, and 130.

[FSE23] Udhav Fowdar and Henrique N. Sá Earp. Harmonic flow of quaternion-Kähler structures, 2023. cf. pages 102 and 105.

[GF14] Mario García-Fernández. Torsion-free generalized connections and heterotic supergravity. *Communications in Mathematical Physics*, vol. 332:pag 89–115, 2014. cf. pages 11, 13, 15, 16, 23, 35, and 110.

[GF19] Mario García-Fernández. Ricci flow, Killing spinors, and T-duality in generalized geometry. *Advances in Mathematics*, 350:1059–1108, 2019. cf. pages 11, 15, 19, 24, 31, 32, 34, 42, and 45.

[GFGM23] Mario García-Fernández and Raul González Molina. Futaki invariants and Yau's conjecture on the Hull-Strominger system, 2023. cf. pages 6, 7, 11, 31, 36, 38, 72, 79, 82, 101, 116, and 119.

[GFGMS24] Mario García-Fernández, Raul González Molina, and Jeffrey Streets. Generalized Ricci flow and the Hull-Strominger system, 2024. cf. page 42.

[GFJS23] Mario García-Fernández, Joshua Jordan, and Jeffrey Streets. Non-Kähler Calabi-Yau geometry and pluriclosed flow. *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 177:329–367, 2023. cf. pages 31, 72, and 82.

[GFRST22] Mario García-Fernández, Roberto Rubio, Carlos Shahbazi, and Carl Tipler. Canonical metrics on holomorphic Courant algebroids. *Proceedings of the London Mathematical Society*, 125(3):700–758, 2022. cf. pages 48 and 83.

[GFRT16] Mario García-Fernández, Roberto Rubio, and Carl Tripler. Infinitesimal moduli for the Strominger system and Killing spinors in generalized geometry. *Mathematische Annalen*, 369(1-2):539–595, sep 2016. cf. pages 31, 32, 33, 34, and 47.

[GFRT20] Mario García-Fernández, Roberto Rubio, and Carl Tipler. Holomorphic string algebroids. *Transactions of the American Mathematical Society*, 373(10):7347–7382, jul 2020. cf. pages 16, 34, and 82.

[GFS20] Mario García-Fernández and Jeffrey Streets. *Generalized Ricci Flow*. American Mathematical Society (AMS), 2020. cf. pages 11, 15, 16, 18, 19, 25, 35, 42, 45, 46, 48, and 84.

[GH80] Alfred Gray and Luis M. Hervella. The sixteen classes of almost Hermitian manifolds and their linear invariants. *Annali di Matematica Pura ed Applicata, 123*, pages 35–58, 1980. cf. pages 130 and 131.

[GN95] Murat Gunaydin and Hermann Nicolai. Seven-dimensional octonionic Yang-Mills instanton and its extension to an heterotic string soliton. *Physics Letters B*, 351:169–172, 1995. [Addendum: Phys.Lett.B 376, 329 (1996)]. cf. page 82.

[Gra69] Alfred Gray. Vector cross products on manifolds. *Transactions of the American Mathematical Society*, 141:465–504, 1969. cf. pages 58 and 66.

[GS24] Mateo Galdeano and Leander Stecker. The heterotic G_2 system with reducible characteristic holonomy, 2024. cf. page 82.

[Hit10] Nigel Hitchin. Lectures on generalized geometry, 2010. cf. page 16.

[Hul86] Chistopher M. Hull. Compactifications of the heterotic superstring. *Physics Letters B*, 178(4):357–364, 1986. cf. page 10.

[HV15] G. Habib and L. Vezzoni. Some remarks on Calabi-Yau and hyper-Kähler foliations. *Differential Geom. Appl.*, 41:12–32, 2015. cf. page 89.

[II05] Petar Ivanov and Stefan Ivanov. SU(3)-instantons and G_2, Spin(7)-heterotic string solitons. *Communications in Mathematical Physics*, 259(1):79–102, 2005. cf. pages 34, 85, and 86.

[IP01] Stefan Ivanov and George Papadopoulos. Vanishing theorems and string backgrounds. *Classical and Quantum Gravity*, 18(6):1089, mar 2001. cf. page 35.

[IP23] Stefan Ivanov and Alexander Petkov. The Riemannian curvature identities for the torsion connection on $Spin(7)$-manifold and generalized Ricci solitons, 2023. cf. pages 110, 121, and 151.

[IS23a] Stefan Ivanov and Nikola Stanchev. The Riemannian Bianchi identities of metric connections with skew torsion and generalized Ricci solitons, 2023. cf. pages 110 and 151.

[IS23b] Stefan Ivanov and Nikola Stanchev. The Riemannian curvature identities of a G_2 connection with skew-symmetric torsion and generalized Ricci solitons, 2023. cf. pages 13, 67, 77, 110, and 151.

[Iva04] Stefan Ivanov. Connections with torsion, parallel spinors and geometry of Spin(7) manifolds. *Mathematical Research Letters*, 11(2-3):171–186, 2004. cf. pages 121, 125, and 154.

[Joy96] D. D. Joyce. Compact Riemannian 7–manifolds with holonomy G_2. I, II. *J. Differential Geom.*, 43(2):291–328, 329–375, 1996. cf. page 10.

[Joy00] Dominic D. Joyce. *Compact Manifolds With Special Holonomy*. Oxford University Press cf. pages 11, 54, 58, 60, 122, and 125.

[JV16] Branislav Jurco and Jan Vysoky. Courant algebroid connections and string effective actions, 2016. cf. page 23.

[Kar05] Spiro Karigiannis. Deformations of G_2 and $Spin(7)$ structures. *Canadian Journal of Mathematics*, 57(5):1012–1055, October 2005. cf. pages 121, 123, and 124.

[Kar08a] Spiro Karigiannis. Flows of G_2-structures, I. *The Quarterly Journal of Mathematics*, 60(4):487–522, jul 2008. cf. pages 50, 54, and 59.

[Kar08b] Spiro Karigiannis. Flows of $Spin(7)$-structures. In *Differential Geometry and Its Applications*. WORLD SCIENTIFIC, July 2008. cf. pages 121, 122, 123, and 124.

[Kar20] Spiro Karigiannis. Introduction to G_2-Geometry. In *Lectures and Surveys on G_2-Manifolds and Related Topics*, pages 3–50. Springer US, 2020. cf. pages 50 and 54.

[KMS93] Ivan Kolár, Peter W. Michor, and Jan Slovák. *Natural Operations in Differential Geometry*. Springer-Verlag, 1993. cf. page 148.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry Vol. 1*. Wiley Interscience Publishers, 1963. cf. pages 11 and 58.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry Vol. 2*. Wiley Interscience Publishers, 1996. cf. page 11.

[LM90] H. Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry*. Princeton University Press, 1990. cf. pages 12, 32, 43, 45, 63, 65, and 148.

[LM22] Jason D. Lotay and Thomas Bruun Madsen. Instantons on flat space: Explicit constructions, 2022. cf. page 102.

[LSE23] Jason D. Lotay and Henrique N. Sá Earp. The heterotic G_2-system on contact Calabi–Yau 7-manifolds. *Transactions of the American Mathematical Society, Series B*, 10(26):907–943, 2023. cf. pages 6, 7, 11, 12, 14, 67, 82, 87, 88, 90, 91, 92, 93, 94, 97, 98, and 99.

[McI91] Brett McInnes. Methods of holonomy theory for Ricci-flat Riemannian manifolds. *CERN-TH.*, 1991. cf. pages 105 and 120.

[MCMS94] Francisco Martín Cabrera, Maria D. Monar, and Andrew Swann. Classification of G_2-structures. *Journal of the London Mathematical Society*, 53, 09 1994. cf. page 60.

[MM18] Lucía Martín-Merchán. Spinorial classification of Spin(7)-structures, 2018. cf. pages 121 and 125.

[Mol24] Raúl Gonzalez Molina. *New approaches to the Hull-Strominger system: Futaki invariants and harmonic metrics*. PhD thesis, Universidad Autónoma de Madrid, 2024. cf. pages 35 and 47.

[MU19] Víctor Manero and Luis Ugarte. Einstein warped G_2 and $Spin(7)$ manifolds. *Communications in Mathematical Physics*, 369(2):637–673, February 2019. cf. pages 121 and 130.

[Nol12] Christoph Nolle. Homogeneous heterotic supergravity solutions with linear dilaton. *Journal of Physics A*, 45:045402, 2012. cf. page 82.

[PSE20] Luis E. Portilla and Henrique N. Sá Earp. Instantons on Sasakian 7-manifolds, 2020. cf. pages 102 and 105.

[Puh09] Christof Puhle. *Spin*(7)-manifolds with parallel torsion form. *Communications in Mathematical Physics*, 291(2):303–320, July 2009. cf. page 124.

[Puh11] Christof Puhle. Riemannian manifolds with structure group $PSU(3)$. *Journal of the London Mathematical Society*, 85(1):79–100, November 2011. cf. page 149.

[Sal89] Simon Salamon. *Riemannian geometry and holonomy groups*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, 1989. cf. pages 121, 129, 130, and 135.

[Sis19] Mária Anna Sisák. *Heterotic Courant algebroids and T-duality*. Master dissertation, University of Amsterdan, 2019. cf. page 16.

[SSCV24] Jeffrey Streets, Charles Strickland-Constable, and Fridrich Valach. Ricci flow on Courant algebroids. *arXiv:2402.11069*, 2024. cf. page 42.

[Str86] Andrew Strominger. Superstrings with torsion. *Nuclear Phys. B*, 274(2):253–284, 1986. cf. pages 10 and 151.

[ŠV20] Pavol Ševera and Fridrich Valach. Courant algebroids, Poisson-Lie T-duality, and type II supergravities. *Communications in Mathematical Physics*, 375:307–344, 2020. cf. page 42.

[SW17] Dietmar A. Salamon and Thomas Walpuski. Notes on the octonions, 2017. cf. pages 50, 52, 53, 54, and 122.

[Tor15] Maike Torm¨ahlen. *Yang-Mills Solutions on Manifolds with G-Structure*. PhD thesis, Gottfried Wilhelm Leibniz Universitat Hannover, 2015. cf. page 77.

[Tu10] Loring W. Tu. *An Introduction to Manifolds*. Springer, 2010. cf. pages 11 and 150.

[Tu17] Loring W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes*. Springer International Publishing, 2017. cf. pages 11, 58, 148, and 150.

APPENDIX A – Conventions on differential forms and notation

A.1 Differential Forms

Let M^n be a n-dimensional Riemannian manifold. We denote $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$ the space of k-forms which standard notation in coordinates will be for $\xi \in \Omega^k$

$$\xi = \frac{1}{k!}\xi_{i_1\cdots i_k}\ e^{i_1\cdots i_k} = \frac{1}{k!}\xi_{i_1\cdots i_k}\ e^{i_1}\wedge\cdots\wedge e^{i_k} \tag{A.1.1}$$

where the repeated covariant and contravariant indices implies an implicit sum over all indices (even the repetition and permutations, and because that the correction factor $\frac{1}{k!}$ is considered on the expression), the so-called Einstein's sum convention. Note that in the standard notation, the indices are given by

$$\xi_{i_1\cdots i_k} := \xi(e_{i_1},\cdots,e_{i_k}) \tag{A.1.2}$$

and the indices are completely skew-symmetric. The wedge product $\wedge : \Omega^k \times \Omega^l \to \Omega^{k+l}$ is given in coordinates by

$$\xi\wedge\eta = \frac{1}{k!l!}\xi_{i_1\cdots i_k}\eta_{j_1\cdots j_l}\ e^{i_1\cdots i_k j_1\cdots j_l} \tag{A.1.3}$$

this formula holds, however it not in the standard form (A.1.1) because the indices don't satisfy (A.1.2). To find the standard form for the wedge product, we have to reorganize the indices on the (linearly dependend) generator set e^{ij} to obtain

$$\begin{aligned}\xi\wedge\eta &= \xi_{[i_1\cdots i_k}\eta_{i_{k+1}\cdots i_{k+l}]}\ e^{i_1\cdots i_k i_{k+1}\cdots i_{k+l}}\\ &:= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(-1)^{|\sigma|}\xi_{i_{\sigma(1)}\cdots i_{\sigma(k)}}\eta_{i_{\sigma(k+1)}\cdots i_{\sigma(k+l)}}\ e^{i_2\cdots i_{k+l}}.\end{aligned} \tag{A.1.4}$$

We will also use the Hodge star operator $* = *_g$ for induced by a Riemannian metric g on an oriented manifold (M^n, vol_M). The Hodge star operators are isomorphisms $* : \Omega^k \to \Omega^{n-k}$ between the space of k-forms defined by the relation

$$\xi\wedge *\xi = |\xi|^2\mathrm{vol}_M. \tag{A.1.5}$$

It satisfies $**\xi = (-1)^{k(n-k)}\xi$, in particular $** = \mathrm{Id}$ for odd-dimensional manifolds and $** = (-1)^k\,\mathrm{Id}$ for even-dimensional manifolds. In coordinates, for $\xi \in \Omega^k$, the Hodge star operator is given by

$$(*\xi)_{i_{k+1}\cdots i_n} = \frac{\sqrt{g}}{k!}\xi^{i_1\cdots i_k}\varepsilon_{i_1\cdots i_n}, \qquad \text{where } *\,\xi = \frac{1}{(n-k)!}(*\xi)_{i_{k+1}\cdots i_n}e^{i_{k+1}\cdots i_n} \tag{A.1.6}$$

In this context, we have the co-differential $d^* = (-1)^{n(k+1)+1} * d* : \Omega^k \to \Omega^{k-1}$, which is $d^* = - * d*$ in even dimensional manifolds and $d^* = (-1)^k * d*$ for odd ones.

A.2 Contractions of forms

A Riemannian metric g (or more generally, any non-degenerate bilinear form) on the tangent space TM defines a natural identification $TM \cong T^*M$ by

$$X \in TM \mapsto g(X, \cdot) \in T^*M$$

this identification is called the musical isomorphism, and we normally denote it and its inverse by $\flat : X \in TM \mapsto X^\flat \in T^*M$ and $\# : \xi \in T^*M \mapsto \xi^\# \in TM$. In coordinates, if we set the vector field $X = X^j e_j$ and the 1-form $\xi = \xi_j e^j$ with covariant indices, the musical isomorphisms are given by

$$X^\flat = X_j e^j := X^k g_{kj}\, e^j \qquad \text{and} \qquad \xi^\# = \xi^j e_j := \xi_k g^{kj}\, e_j. \tag{A.2.1}$$

Let $\xi \in \Omega^k$ and a fixed vector field X, we define the contraction operator $X \lrcorner = i_X : \Omega^k \to \Omega^{k-1}$ normally denoted by $X \lrcorner \xi$ or $i_X \xi$ and defined by

$$(X \lrcorner \xi)(Y_1, \cdots, Y_{k-1}) := \xi(X, Y_1, \cdots, Y_{k-1}) \tag{A.2.2}$$

in coordinates, we have the coefficients of this operation is given by $(e_j \lrcorner \xi)_{i_1 \cdots i_{k-1}} = \xi_{j i_1 \cdots i_{k-1}}$. So we write the expression for the $(k-1)$-form $i_X \xi = X \lrcorner \xi$ by

$$X \lrcorner \xi = \frac{1}{(k-1)!} X^j \xi_{j i_1 \cdots i_{k-1}}\, e^{i_1 \cdots i_{k-1}} \tag{A.2.3}$$

Naturally we can contract a 1-form with a k-form defining by $\alpha \lrcorner \xi := \alpha^\# \lrcorner \xi$. We can generalize this operation by taking $\eta \in \Omega^k$ and $\xi \in \Omega^{k+p}$ and defining $\eta \lrcorner \xi \in \Omega^p$ by:

$$\eta \lrcorner \xi := \frac{1}{k!p!} \eta^{i_1 \cdots i_k} \xi_{i_1 \cdots i_k i_{k+1} \cdots i_{k+p}}\, e^{i_{k+1} \cdots i_{k+p}} \tag{A.2.4}$$

Some interesting property of contraction of forms is the relation of it with the Hodge star operator given by the lemma below.

Lemma A.2.1. *Let $\eta \in \Omega^k$ and $\xi \in \Omega^{k+p}$ forms on an oriented Riemannian manifold M^n, then we have that*

$$\eta \lrcorner \xi = (-1)^{p(n-p-k)} * (\eta \wedge *\xi) \tag{A.2.5}$$

Which writes as

$$\eta \lrcorner \xi = (-1)^{pk} * (\eta \wedge *\xi) \qquad n \text{ odd.} \tag{A.2.6}$$

$$\eta \lrcorner \xi = (-1)^{p(k+1)} * (\eta \wedge *\xi) \qquad n \text{ even.} \tag{A.2.7}$$

Proof. Cf. [dlOLS18a, Appendix A]. □

Another important property of contractions is some kind of commutativity when three contractions are considered. To be precise, the following result holds:

Lemma A.2.2. *Let $\alpha \in \Omega^k, \beta \in \Omega^l, \gamma \in \Omega^p$ (where k, l, p are integers such that the contractions below make sense), then the following identity holds:*

$$\alpha \lrcorner (\beta \lrcorner \gamma) = (-1)^{kl} \beta \lrcorner (\alpha \lrcorner \gamma) \tag{A.2.8}$$

Proof. By direct computation, we just have to translate the indices of γ:

$$\begin{aligned}
\alpha \lrcorner (\beta \lrcorner \gamma) &= \frac{1}{k!l!(p-l-k)!} \alpha^{i_1 \cdots i_k} \beta^{j_1 \cdots j_l} \gamma_{j_1 \cdots j_l i_1 \cdots i_k i_{k+1} \cdots i_{p-l-k}} e^{i_{k+1} \cdots i_{p-k-l}} \\
&= \frac{(-1)^{kl}}{k!l!(p-l-k)!} \beta^{j_1 \cdots j_l} \alpha^{i_1 \cdots i_k} \gamma_{i_1 \cdots i_k j_1 \cdots j_l i_{k+1} \cdots i_{p-l-k}} e^{i_{k+1} \cdots i_{p-k-l}} \\
&= (-1)^{kl} \beta \lrcorner (\alpha \lrcorner \gamma)
\end{aligned}$$

as desired. □

The contraction of $\eta \in \Omega^k$ in $\xi \in \Omega^{k+p}$ consists in contracting the indices of η in ξ. We can make this process partially, contracting just some indices and defining a map

$$\lrcorner^q : \Omega^k \times \Omega^{k+p} \to \Omega^{2k+p-2q}$$

by the formula (where the second line is the case of orthonormal basis)

$$\begin{aligned}
\eta \lrcorner^q \xi &:= \frac{1}{q!(k-q)!(k+p-q)!} \eta^{a_1 \cdots a_q}{}_{i_1 \cdots i_{k-q}} \xi_{a_1 \cdots a_q j_1 \cdots j_{k+p-q}} \; e^{i_1 \cdots i_{k-q} j_1 \cdots j_{k+p-q}} \\
&= \frac{1}{q!} \eta_{i_1 \cdots i_q} \wedge \xi_{i_1 \cdots i_q} = \sum_{i_1 < \cdots < i_q} \eta_{i_1 \cdots i_q} \wedge \xi_{i_1 \cdots i_q}
\end{aligned} \tag{A.2.9}$$

where the notation used above for an arbitrary form $\beta \in \Omega^r$ is defined by

$$\beta_{i_1 \cdots i_q} := i_{e_q} \cdots i_{e_1} \beta = \frac{1}{(r-q)!} \beta_{i_1 \cdots i_q i_{q+1} \cdots i_r} e^{i_{q+1} \cdots i_r} \in \Omega^{r-q}(M).$$

Note that the partial contractions generalize the 'usual' contraction and the wedge product, in the sense that

$$\eta \lrcorner \xi = \eta \lrcorner^k \xi \qquad \text{and} \qquad \eta \wedge \xi = \eta \lrcorner^0 \xi. \tag{A.2.10}$$

also note that the sign correction for commutation: $\eta \lrcorner^q \xi = (-1)^{k-q} \xi \lrcorner^q \eta$. The partial contractions are important because, like the contractions, Hodge dual and wedge product, they can define G-invariant maps (when taken using the forms preserved by such structure) and appear on countless occasions. The equivalences (A.2.10) suggest a relation between the q-contraction $\lrcorner^q$ and the $(k-q)$-contraction $\lrcorner^{k-q}$ via the Hodge star operator. In fact, we have

Proposition A.2.3. *Let $\alpha \in \Omega^k$, $\beta \in \Omega^{k+p}$, then the relation below holds*

$$\alpha \lrcorner^q \beta = (-1)^{(k-q)(k-q+n)+p(n-p-q)} * (\alpha \lrcorner^{k-q} * \beta). \tag{A.2.11}$$

which writes as

$$\alpha \lrcorner^q \beta = (-1)^{pq} * (\alpha \lrcorner^{k-q} *\beta) \qquad n \text{ odd.} \tag{A.2.12}$$

$$\alpha \lrcorner^q \beta = (-1)^{p(q+1)+k-q} * (\alpha \lrcorner^{k-q} *\beta) \qquad n \text{ even.} \tag{A.2.13}$$

Proof. By definition of partial contractions, we have

$$*(\alpha \lrcorner^q \beta) = \frac{1}{q!} * (\alpha^{i_1\cdots i_q} \wedge \beta_{i_1\cdots i_q}) = \frac{1}{q!(k-q)!}\alpha^{i_1\cdots i_q}{}_{i_{q+1}\cdots i_k} * (e^{i_{q+1}\cdots i_k} \wedge \beta_{i_1\cdots i_q})$$

now, working with the Hodge dual on this expression, we then have

$$\begin{aligned}
*(e^{i_{q+1}\cdots i_k} \wedge \beta_{i_1\cdots i_q}) &= *(e^{i_{q+1}\cdots i_k} \wedge (e^{i_1\cdots i_q} \lrcorner \beta)) \\
&= (-1)^{(k+p-q)(n-k-p)} * (e^{i_{q+1}\cdots i_k} \wedge *(e^{i_1\cdots i_q} \wedge *\beta)) \\
&= (-1)^{(k-q)(n-k)+p(n-q+1)+p(n-p)} e^{i_{q+1}\cdots i_k} \lrcorner (e^{i_1\cdots i_q} \wedge *\beta) \\
&= (-1)^{(k-q)(n-k)+p(p+q+1)} e^{i_k} \lrcorner \cdots \lrcorner e^{i_{q+1}} \lrcorner (e^{i_1\cdots i_q} \wedge *\beta)
\end{aligned}$$

performing the first contraction, we then obtain

$$(-1)^{(k-q)(n-k)+p(p+q+1)} e^{i_k} \lrcorner \cdots \lrcorner e^{i_{q+2}} \lrcorner \Big(\sum_{a=1}^{q} (-1)^{a+1} \delta^{i_a i_{q+1}} e^{i_1\cdots \hat{i_a} \cdots i_q} \wedge *\beta + (-1)^q e^{i_1\cdots i_q} \wedge (*\beta)_{i_{q+1}} \Big)$$

Note that the only term in the parentheses which has no δ's is $(-1)^q e^{i_1\cdots i_q} \wedge (*\beta)_{i_{q+1}}$. When we perform the second contraction, now with $e^{i_{q+2}}$, the only term with no δ's will be $(-1)^{2q} e^{i_1\cdots i_q} \wedge (*\beta)_{i_{q+1}i_{q+2}}$. Performing all the $k-q$ contractions, we finally have that the only term with no δ's will be $(-1)^{q(k-q)} e^{i_1\cdots i_q} \wedge (*\beta)_{i_{q+1}\cdots i_k}$. Denoting the terms which contain some δ involving the indices $i_1 \cdots i_q$ and $i_{q+1}, \cdots i_k$ by $\Sigma\delta$, we then have

$$*(e^{i_{q+1}\cdots i_k} \wedge \beta_{i_1\cdots i_q}) = (-1)^{(k-q)(n-k)+p(p+q+1)} \Big(\Sigma\delta + (-1)^{q(k-q)} e^{i_1\cdots i_q} \wedge (*\beta)_{i_{q+1}\cdots i_k} \Big)$$

substituting in the initial expression for $*(\alpha \lrcorner^k \beta)$ the term $\Sigma\delta$ cancel with the skew-symmetry of indices in γ (which are the ones which appear in the terms of $\Sigma\delta$), we finally have

$$\begin{aligned}
(\alpha \lrcorner^k \beta) &= \frac{1}{q!(k-q)!} (-1)^{(k-q)(n-k)+p(p+q+1)+q(k-q)} \alpha^{i_1\cdots i_q}{}_{i_{q+1}\cdots i_k} e^{i_1\cdots i_q} \wedge (\beta)_{i_{q+1}\cdots i_k} \\
&= (-1)^{(k-q)(n-k)+p(p+q+1)+q(k-q)} \frac{1}{(k-q)!} \alpha_{i_{q+1}\cdots i_k} \wedge (*\beta)_{i_{q+1}\cdots i_k} \\
&= (-1)^{(k-q)(n-k)+p(p+q+1)+q(k-q)} \alpha \lrcorner^{k-q} *\beta
\end{aligned}$$

now, applying the Hodge star dual on both sides, and using that $\alpha \lrcorner^k \beta \in \Omega^{2k+p-2q}$, we then have the expression below

$$\alpha \lrcorner^q \beta = (-1)^{(k-q)(n-k)+p(p+q+1)+q(k-q)+p(n-p)} = (-1)^{(k-q)(n-k-q)+p(n-p-q)} * (\alpha \lrcorner^{k-q} *\beta)$$

as desired. □

APPENDIX B – Intrinsic torsion

This discussion is found in [FI02] and can also be found in [Fri02, Agr06].

B.1 Intrinsic torsion and classes of G-structures

Let (M^n, g) be an oriented Riemannian manifold and $\mathrm{Fr}(M)$ its frame bundle (already considering the reduction to an $\mathrm{SO}(n)$-bundle), consequently the Levi-Civita connection[1] is a 1-form

$$Z^g \in \Omega^1(\mathrm{Fr}(M), \mathfrak{so}(n))$$

with values in the Lie algebra $\mathfrak{so}(n)$.

Suppose the manifold admits a G-structure (for some $G \subset \mathrm{SO}(n)$), which is encoded in the G-subbundle $\mathcal{R} \leq \mathrm{Fr}(M)$. We can restrict the Levi-Civita connection to $\mathcal{R}$ as a form $Z^g|_{\mathcal{R}} \in \Omega^1(\mathcal{R}, \mathfrak{so}(n))$ and decompose it with respect to the orthogonal decomposition of the Lie algebra $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$:

$$Z^g|_{\mathcal{R}} := \tilde{Z} \oplus \Gamma, \tag{B.1.1}$$

where $\tilde{Z} \in \Omega^1(\mathcal{R}, \mathfrak{g})$ is a connection form for $\mathcal{R}$ and $\Gamma \in \Omega^1(\mathcal{R}, \mathfrak{m})$ and can be identified as $\Gamma \in \Omega^1(M, \mathcal{R} \times_G \mathfrak{m})$ (such identification is possible since the form is G-invariant by the existence of the G-structure, [KMS93, Section 10.12]).

Definition B.1.1. *On a manifold (M, g) with G-structure $\mathcal{R} \leq \mathrm{Fr}(M)$, the 1-form $\Gamma \in \Omega^1(M, \mathcal{R} \times_G \mathfrak{m})$ defined in (B.1.1) is called the* intrinsic torsion *of the G-structure $\mathcal{R}$. The G-structure is called* integrable *or* torsion-free *if $\Gamma \equiv 0$.*

We can classify types of G-structure analysing how the intrinsic torsion Γ decomposes into irreducible representations of G. To understand this, first see that the intrinsic torsion is a 1-form with vector values is a section

$$\Gamma \in \Omega^1(M, \mathcal{R} \times_G \mathfrak{m}) = \Gamma\left(T^*M \otimes (\mathcal{R} \times_G \mathfrak{m})\right) \cong \Gamma\left(TM \otimes (\mathcal{R} \times_G \mathfrak{m})\right)$$

i.e., Γ is a section of $TM \otimes (\mathcal{R} \times_G \mathfrak{m})$, which fibres are isomorphic to $\mathbb{R}^n \otimes \mathfrak{m}$. So, the classes of G-structures are the irreducible components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$.

The theory of intrinsic torsion discussed above is very general and extensively examined in Friedrich's seminal work [Fri02] for many types of geometries. However,

[1] Considering the tangent bundle as an associated bundle of frame bundle, cf. [LM90], a connection on the frame bundle is equivalent to connections on the associated bundles, cf. [Tu17, LM90]. For the details of such equivalence see specifically [Tu17, Theorem 29.10].

a specific scenario holds particular significance for our discussion: when the group G represents the isotropy or stabilizer group of a tensor Φ_0. This case is notable, as many well-known G-structures fall within its scope, including almost Hermitian structures (or $\mathrm{U}(m)$-structures), G_2-structures, and Spin(7)-structures.

Let's discuss more this class of examples, i.e., the group G is the stabilizer of some tensor. Let (M^n, g) be an oriented Riemannian manifold and suppose that there is a faithful representation in a vector space V, $\rho : \mathrm{SO}(n) \to \mathrm{SO}(V)$ for the group $G \subset \mathrm{SO}(n)$ and an element $\Phi_0 \in V$ such that

$$G = \Big\{ g \in \mathrm{SO}(n) : \rho(g)\Phi_0 = \Phi_0 \Big\},$$

then a G-structure is a triple (M^n, g, Φ) consisting of a Riemannian manifold equipped with an additional tensor field $\Phi \in \Gamma(V) \coloneqq \Gamma(\mathrm{Fr}(M) \times_\rho V)$ which takes the form of Φ_0 punctually. In such cases, the intrinsic torsion of the structure is directly related to the Riemannian covariant derivative of Φ (Cf. [Fri02, Puh11, AFS05, FI02]).

Proposition B.1.2. *Let (M^n, g, Φ) be a G-structure when $G \subset \mathrm{SO}(n)$ is the stabilizer of the punctual form of Φ via representation $\rho : \mathrm{SO}(n) \to \mathrm{SO}(V)$. Then the intrinsic torsion $\Gamma \in \Omega^1(M, \mathfrak{m})$ of this structure satisfies:*

$$\nabla^g_X \Phi = \rho_*(\Gamma(X))(\Phi) \tag{B.1.2}$$

where $\rho_* : \mathfrak{so}(n) \to \mathfrak{so}(V)$ is the differential of the representation. $\nabla^g\Phi$ is an element of the space $\mathbb{R}^n \otimes V$ (which is a rough way to denote $\Gamma(TM \otimes V)$). In particular, if $V = \Lambda^k(\mathbb{R}^n)^*$ is a space of forms, we can identify $\mathfrak{so}$ with two forms and the intrinsic torsion satisfies [Puh11, Section 2]:

$$\nabla^g_X \Phi = \Gamma(X) \lrcorner^1 \Phi = \sum_{j=1}^n i_{e_j}\Gamma(X) \wedge i_{e_j}\Phi.$$

where $\Gamma(X) \in \mathfrak{m} \leq \mathfrak{so}(n) = \Lambda^2(\mathbb{R}^n)^*$ and Φ is in $V = \Lambda^k$.

Proof. [Fri02]. □

B.2 Connections with skew-symmetric torsion

Let ∇ be an arbitrary affine connection in the tangent bundle TM of some manifold M^n. The torsion of the connection is defined by

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

and it is a (2,1)-tensor. If the manifold is Riemannian and the connection compatible with the metric, then the torsion is skew-symmetric on its first two-components, i.e.,

$T_\nabla \in \Gamma(\Lambda^2(T^*M) \otimes TM)$. If we contract and identify T_∇ as a (3,0) tensor, we can ask when $T_\nabla \in \Omega^3(M)$, i.e., when it is skew-symmetric on all components. In this case, the connection is said to have *totally skew-symmetric torsion*. In particular, connections with skew-symmetric torsion are metric-compatible.

Since the space of connections is modelled on an affine space of (2,1)-tensors (cf. [Tu10, Tu17]), then the difference between two connections is a (2,1)-tensor, and we have:

$$\nabla_X Y = \nabla^g_X Y + A(X,Y)$$

It is immediate to compute the torsion and see that $T_\nabla(X,Y) = A(X,Y) - A(Y,X)$. In particular, ∇ compatible with the metric, we have A skew-symmetric and $T_\nabla = 2A$. So, we have the following result:

Lemma B.2.1. *Let (M,g) be a Riemannian metric and ∇ be a metric connection on M (i.e., $\nabla g = 0$), then*

$$\nabla = \nabla^g + \frac{1}{2}T_\nabla.$$

where $T_\nabla \in \Gamma(\Lambda^2 \otimes TM)$ is its torsion and ∇^g the Levi-Civita connection for g. This means that metric connections are completely characterized by its torsion. Conversely, if $T \in \Gamma(\Lambda^2 T^*M \otimes TM)$, then

$$\nabla := \nabla^g + \frac{1}{2}T$$

is a metric compatible connection and its torsion is $T_\nabla = T$.

If $T \in \Omega^3(M)$, we can define a metric compatible connection with totally skew-symmetric torsion $T \in \Omega^3(M)$ by the formula

$$\nabla := \nabla^g + \frac{1}{2}g^{-1}T \tag{B.2.1}$$

For the Levi-Civita connection (c.f. [Tu17, Dar94]), we have the (first algebraic) *Bianchi identity* given by

$$R(V,X,Y,Z) + R(V,Y,Z,X) + R(V,Z,X,Y) = 0,$$

which in coordinates reads

$$R_{a[bcd]} = 0 \Rightarrow R_{abcd} + R_{acdb} + R_{adbc} = 0.$$

We have the generalization of such identity for metric connection with totally skew-symmetric torsion $T \in \Omega^3(M)$.

Proposition B.2.2 (Bianchi identity)**.** *Consider a metric affine connection with skew-symmetric torsion T on a Riemannian manifold (M,g) given by $\nabla = \nabla^g + \frac{1}{2}g^{-1}T$, then it* *satisfies the following first Bianchi identity:*

$$R(X,Y,Z,V)+R(Y,Z,X,V)+R(Z,X,Y,V)=-\frac{1}{2}dT(X,Y,Z,V)+(\nabla_V T)(X,Y,Z) \tag{B.2.2}$$

which in coordinates, can be expressed as[2]

$$R_{ijkl}+R_{jkil}+R_{kijl}=-\frac{1}{2}(dT)_{ijkl}+(\nabla_l T)_{ijk}.$$

Proof. [IS23a, FI02]. □

B.3 Existence of compatible connections with skew-symmetric torsion

In our study, connections with totally skew-symmetric torsion hold a distinguished role. In physics, as discussed in [Str86], these connections serve as the foundation for formulating equations in string theory. In the study of geometrical structures, such connections are occasionally unique, under the hypothesis of compatibility with the structure (which will be our case for G_2-structures). Therefore, they represent a natural choice for investigating geometric phenomena in these structures [Agr06].

Suppose the G-structure is defined by some k-form $\xi \in \Omega^k(M)$ as we have discussed before. If a G-structure in this context admits a connection compatible with the structure, i.e., $\nabla\xi = 0$, it will satisfy the endomorphism part of the curvature lives in the Lie algebra:

$$\langle R_\nabla(X,Y)\cdot,\cdot\rangle \in \mathfrak{g} \leq \mathfrak{so}(n) = \Lambda^2. \tag{B.3.1}$$

In particular, if this connection has totally skew-symmetric torsion $T \in \Omega^3(M)$, it will be given by

$$\nabla^+ = \nabla^g + \frac{1}{2}g^{-1}T.$$

when this connection is unique, it is normally called *Bismut connection* or *characteristic connection*. The existence of such connection in special geometrical was deeply studied by Friedrich et al. [FI02, Fri02, Fri03]. A special result by Friedrich is given below.

2 Here, we have to be careful with the conventions. We are using

$$R(X,Y,Z,W) = g(R(Z,W)Y,X),$$

(normally used in general relativity), but some authors we are following (e.g., [IS23a, IS23b, IP23]) use the convention

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The general relativity convention is useful to represent the curvature in coordinates as being

$$R(X,Y,Z,W) = g(R(Z,W)Y,X) = X_aY^bZ^cW^dR^a{}_{bcd}$$

with the upper index being the first one.

Theorem B.3.1 ([Fri02, Fri03])**.** *Let M be a Riemannian manifold endowed with a G-structure (which splits $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$), let us introduce the map*

$$\Theta: \quad \Lambda^3 \quad \to \quad \mathbb{R}^n \otimes \mathfrak{g}^\perp; \qquad \Theta(T) = \sum_j (\sigma_j \lrcorner T) \otimes \sigma_j$$

where σ_i is an orthonormal basis of $\mathfrak{g}^\perp \leq \Lambda^2$. In this context, the G-structure admits a compatible connection with totally skew-symmetric torsion if and only if the intrinsic torsion 1-form Γ belongs to the image of Θ and in this case, the torsion satisfies $2\Gamma = -\Theta(T)$.

On A G-structure which admits a connection with skew-symmetric torsion $\nabla^+ = \nabla^g + \frac{1}{2}T^\#$, we define ∇^+-parallel tensor Φ if it satisfies $\nabla^+\Phi = 0$. This has an important consequence for the particular case of ∇^+-parallel differential forms:

Lemma B.3.2. *Let $\eta \in \Omega^p(M)$ be a ∇^+-parallel form on a manifold with G-structure admitting connection $\nabla^+ = \nabla^g + \frac{1}{2}T^\#$ with skew-symmetric torsion $T \in \Omega^3$, then it satisfies*

$$d\eta = T \lrcorner^1 \eta = \sum_j i_{e_j} T \wedge i_{e_j}\eta. \tag{B.3.2}$$

Proof. The identity is obtained by the property $\nabla^+\eta = 0$ and the fact that we can write the differential $d\eta = e^j \wedge \nabla^g_{e_j}\eta$ and $\nabla^+ = \nabla^g + \frac{1}{2}T^\#$. Evaluating, we have

$$d\eta = e^j \wedge \nabla^g_{e_j}\eta = e^j \wedge \left(\cancelto{0}{\nabla^+_{e_j}\eta} + \frac{1}{2}(T\eta)_{e_j}\right) = \frac{1}{2}e^j \wedge (T\eta)_{e_j}$$

where the form $T\eta \in T^*M \otimes \Omega^k$ acts as

$$(T\eta)_{e_j}(X_1, \cdots, X_p) = \eta(T(e_j, X_1), \cdots, X_p) + \cdots + \eta(X_1, \cdots, T(e_j, X_p))$$

each of these terms, in coordinates, is written as (taking $X_k = e_k$)

$$\begin{aligned}(T\eta)_{e_j}(e_{k_1}, \cdots, e_{k_p}) &= \eta(T(e_j, e_{k_1}), \cdots, e_{k_p}) + \cdots + \eta(e_{k_1}, \cdots, T(e_j, e_{k_p}))\\ &= T_{jk_1}{}^q\eta_{qk_2\cdots k_p} + \cdots + T_{jk_p}{}^q\eta_{k_1\cdots k_{p-1}q}\end{aligned}$$

and the form is written as

$$\begin{aligned}(T\eta)_{e_j} &= \frac{1}{p!}\Big(T_{jk_1}{}^q\eta_{qk_2\cdots k_p} + \cdots + T_{jk_{n-4}}{}^q\eta_{k_1\cdots k_{p-1}q}\Big)\, e^{k_1\cdots k_p}\\ &= \frac{1}{p!}\Big((-1)^0 T_{qjk_1}\eta_{qk_2\cdots k_p}e^{k_1\cdots k_p} + \cdots + (-1)^{p-1}T_{qjk_p}\eta_{qk_1\cdots k_{p-1}}e^{k_1\cdots k_p}\Big)\\ &= \frac{1}{p!}\Big(T_{qjk_1}\eta_{qk_2\cdots k_p}e^{k_1} \wedge e^{k_2\cdots k_p} + \cdots + T_{qjk_p}\eta_{qk_1\cdots k_{p-1}}e^{k_p} \wedge e^{k_1\cdots k_{p-1}}\Big)\end{aligned}$$

taking the wedge product with e_j and summing over j (note that $\eta_q \in \Omega^{p-1}$ and $T_q \in \Omega^2$), we have

$$\begin{aligned}\frac{1}{2}e^j \wedge (T\eta)_{e_j} &= \frac{1}{2!p!}\Big(T_{qjk_1}\eta_{qk_2\cdots k_p}e^{jk_1} \wedge e^{k_2\cdots k_p} + \cdots + T_{qjk_p}\eta_{qk_1\cdots k_{p-1}}e^{jk_p} \wedge e^{k_1\cdots k_{p-1}}\Big)\\ &= \frac{1}{p}\Big(\underbrace{T_q \wedge \eta_q + \cdots + T_q \wedge \eta_q}_{p \text{ times}}\Big) = T \lrcorner^1 \eta\end{aligned}$$

and we can conclude that $d\eta = T \lrcorner^1 \eta$. $\square$

To establish a rigorous methodology for exploring compatible connections within a prescribed G-structure accompanied by totally skew-symmetric torsion, we introduce pivotal concepts. Our groundwork involves G-structures defined as the stabilizers of differential forms, as previously discussed. The main theorem here is the one which gives us a way to compute the torsion $T \in \Omega^3(M)$ of a metric compatible connection when its exists.

Definition B.3.3 (**H**-Operator)**.** *Let (M^n, g) be a Riemannian manifold endowed with a G-structure defined by some differential form $\psi \in \Omega^4(M)$. Consider the G-invariant operator $\mathbf{H} : \Omega^3 \to \Omega^3$*

$$\mathbf{H}(\gamma) = \gamma \lrcorner^2 \psi = (-1)^{n+1} * (\gamma \lrcorner^1 *\psi). \tag{B.3.3}$$

This operator is called **H**-operator *associated with the G-structure.*

Due to the invariance of this map, it decomposes into eigenvalues, and each irreducible component resides within some eigenspace. This is a consequence of eigenspaces being G-representations, which decompose into irreducible representations. Let's denote by a_{k_j} the eigenvalue corresponding to the irreducible component $\Omega^3_{k_j}$; then, for $\gamma_{k_j} \in \Omega^3_{k_j}(M)$, we have

$$\mathbf{H}(\gamma_{k_j}) = a_{k_j}\gamma_{k_j}.$$

Our case of interested is when all eigenvalues are zero (and this will be related to the existence of a *unique* compatible connection with totally skew-symmetric torsion). For instance, consider $G \subset \mathrm{SO}(n)$.

Definition B.3.4 (Flux)**.** *Let (M^n, g) be a manifold endowed with a G-structure defined by some 4-form $\psi \in \Omega^4(M)$. If the eigenvalues of the operator $\mathbf{H} : \Omega^3 \to \Omega^3$ are non-zero, we define the 3-form $H \in \Omega^3(M)$, sometimes called* flux*, by the expression*

$$H = \sum_j \frac{1}{a_{k_j}} \tau^{k_j} \tag{B.3.4}$$

where $\tau_{k_j} \in \Omega^3_{k_j}(M) \leq \Omega^3(M)$ is defined by $\tau^{k_j} = \pi_{k_j}(d^*\psi)$ and called the torsion forms*.*

Remark B.3.5 (General Definition of Flux)*.* When the operator $\mathbf{H}_\xi : \Omega^3 \to \Omega^3$ has a non-trivial kernel (i.e., $a_{k_j} = 0$ for some k_j), which is given by

$$\ker \mathbf{H} := \bigoplus_{a_{k_j}=0} \Omega^3_{k_j}(M) \leq \Omega^3(M)$$

we define the space of fluxes as the affine space $\mathcal{H} := H_0 + \ker \mathbf{H}_\xi$, where $H_0 \in \Omega^3$ is the (so-called) primary flux defined by

$$H_0 = \sum_{a_{k_j} \neq 0} \frac{1}{a_{k_j}} \tau^{k_j}$$

In this way, a flux is an element $H \in \mathcal{H}$. Note that this definition reduces to the anterior, when the eigenvalues are all non-zero (i.e., $\ker \mathbf{H} = 0$ and the affine space has just one element $H = H_0$). ◯

It happens that for all our interested cases (G_2, Spin(7), U(m) structures), $\ker \mathbf{H} = 0$ and the flux is unique. Let's compute them now. These results were computed before in [AFS05, FI02, Iva04], respectively. Below, using the method described above.

Let's consider the context where the G-structure which admits a (not necessarily unique) connection ∇^+ with skew-symmetric torsion T. The following result relating the torsion and the flux is relevant: the torsion is the flux.

Theorem B.3.6 (Flux's Theorem)**.** *Let M^n with a G-structure defined via $\psi \in \Omega^4(M)$ and suppose the structure admits compatible metric connection ∇^+ with skew-symmetric torsion $T \in \Omega^3(M)$. Then*

$$T = H$$

where H is the flux defined via the operator $\mathbf{H} : \Omega^3 \to \Omega^3$. In particular, the compatible connection ∇^+ is unique.

Proof. Since ψ is ∇^+-parallel, so does $\eta = *\psi \in \Omega^{n-4}(M)$, we have by the lemma above that $d\eta = T \lrcorner^1 \eta$, but on the other hand, we have

$$d^*\psi = (-1)^{n+1} * d\eta = (-1)^{n+1} * (T \lrcorner^1 \eta) = \mathbf{H}(T)$$

now, using the invariance of the flux operator, we have

$$\pi_{k_j}(T) = \frac{1}{a_{k_j}} \mathbf{H}(\pi_{k_j}(T)) = \frac{1}{a_{k_j}} \pi_{k_j} \mathbf{H}(T) = \frac{1}{a_{k_j}} \pi_{k_j}(d^*\psi) = \frac{1}{a_{k_j}} \tau^{k_j} = \pi_{k_j}(H)$$

and the result follows: $T = H$. □

Remark B.3.7. We can revisit the flux theorem when the flux operator has a non-trivial kernel, in this case, we have

$$H = H_0 + \gamma; \qquad \mathbf{H}(\gamma) = 0$$

proceeding in the same way, using again $d^*\psi = \mathbf{H}(T)$, what we obtain is that $\pi_{k_j}(T) = \pi_{k_j}(H_0)$ for all k_j such that $a_{k_j} \neq 0$. This means that the torsion $T \in \mathcal{H}$ lives within fluxes and the flux theorem can be stated: the skew-torsion of a compatible metric connection is a flux. In the general case, the connection ∇^+ is not unique because the space of fluxes is not unique. ◯

We have established that the flux corresponds to the torsion in the presence of the ∇^+ connection within the framework of stabilizing G-structures. Remarkably, the flux still makes sense even in cases where the G-structure does not admit such a connection. This concept becomes particularly valuable in subsequent discussions when we introduce coupled equations. There, the flux emerges, even in scenarios where the G-structure does not fall into the category which admits a connection with skew-symmetric torsion.