Mirror symmetry for V_7

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1. "Wait, what's mirror symmetry?"

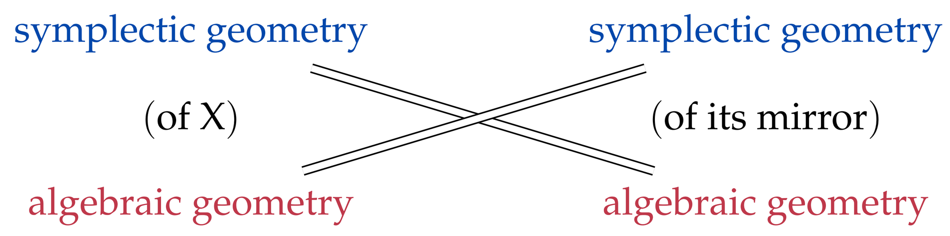
In some types of String Theory, the universe is 10-dimensional. Four of these dimensions are the standard 3+1 of spacetime, but the 6 extra are curled up—so small that we can't detect them—in a space called a Calabi–Yau (C–Y) threefold. There is a construction

C–Y threefold ⇝ physics nonsense ⇝ model of particle physics;

parameters of the universe depend on the geometry of these extra dimensions. It turns out that two completely distinct C–Ys can give rise to the same physics; this is the first hint of mirror symmetry.

Mathematically, mirror symmetry is a mysterious duality between two fields: symplectic geometry (a type of geometry where the fundamental notion is area, not length) and algebraic geometry (the study of spaces which can be described algebraically).

Conjecture. Given a space X which is simultaneously symplectic and algebraic, there should exist a mirror space, such that the symplectic geometry of X corresponds to the algebraic geometry of the mirror, and vice versa.



The conjecture has been generalised from C–Y threefolds to a wide class of spaces, including Fano threefolds, which are a bit like positively curved C–Y threefolds.

Often the mirror of a space X is another space, but not always. For example, the mirror of a Fano threefold X should be a space Y equipped with a function $Y \to \mathbb{C}$ to the complex numbers.

2. "What's your project about?"

This summer, I looked at V_7, which is a Fano threefold obtained by blowing up a 6-dimensional space $\mathbb{P}^3$ at a point (see FIGURE 5). The predicted mirror of V_7 is the function $W\colon (\mathbb{C}^\times)^3 \to \mathbb{C}$ given by

$$W(z_0, z_1, z_2) = \frac{1}{z_0 z_1 z_2} + (1+z_0)(1+z_1)(1+z_2) - 1.$$

Project Aim. There exist two data structures called triangulated categories,

$$D^b\mathsf{Fuk}(W) \quad \text{and} \quad D^b\mathsf{Coh}(V_7),$$

which organise all information about the symplectic geometry of W and the algebraic geometry of V_7, respectively. This project aimed to prove one side of mirror symmetry by showing

$$D^b\mathsf{Fuk}(W) = D^b\mathsf{Coh}(V_7).$$

(I ignored the other side of mirror symmetry, which relates the algebraic geometry of W with the symplectic geometry of V_7.)

3. "How do you even prove something like that?"

A common way to show that two algebraic structures are equal is by demonstrating that they have matching sets of generators.

Example. Given two vector spaces X and Y, one can find a basis for each one. If these bases are the same size, then X = Y.

Example. Given two groups G and H, one can find a generating set for each one. If they satisfy the same relations, then G = H.

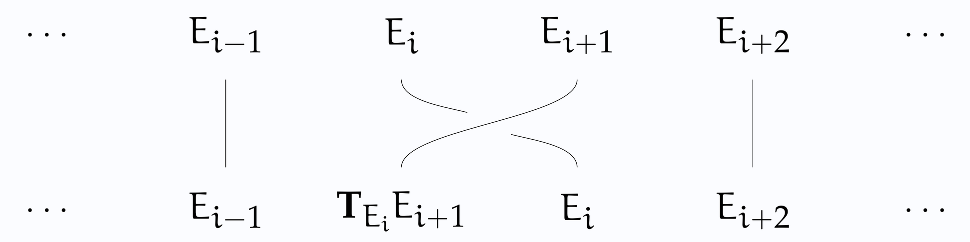
The information of a triangulated category can be encoded in a full exceptional collection (FEC), which is a collection of objects

$$(E_1, \ldots, E_n),$$

together with non-negative whole numbers $\mathcal{H}(E_i, E_j)$. In the same sense as vector spaces and groups, the objects $(E_1, \ldots, E_n)$ generate the triangulated category, and the $\mathcal{H}$-numbers act as relations.

Strategy. Find FECs of the same size for $D^b\mathsf{Fuk}(W)$ and $D^b\mathsf{Coh}(V_7)$ with matching $\mathcal{H}$-numbers. Just as in the other examples, we could then conclude $D^b\mathsf{Fuk}(W) = D^b\mathsf{Coh}(V_7)$.

An arbitrary FEC might not have the properties we want. To fix this, one can obtain new FECs from old ones by 'braiding' objects around each other:



Any combination of such twisting is called a mutation. Mutating an FEC will produce one with a different set of $\mathcal{H}$-numbers.

4. "OK, but what's $D^b\mathsf{Fuk}(W)$?"

The function W describes a family of 4-dimensional spaces, called the fibres of W and written $W^{-1}(\lambda)$, each sitting above a complex number $\lambda \in \mathbb{C}$. The fibre becomes singular (non-smooth) above six complex numbers $\{\lambda_1, \ldots, \lambda_6\}$ called the critical values of W. For each of them, take a path γ_i joining 0 to λ_i.

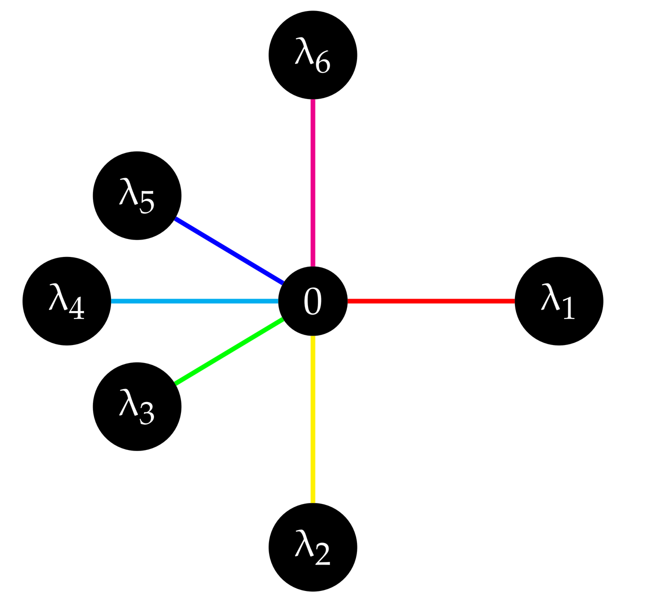


FIGURE 1: Critical values of W, joined to the origin by paths γ_i.

The fibre of W degenerates along each path γ_i. Specifically, a sphere

$$L_i \subset W^{-1}(0)$$

collapses—or vanishes—to a point along γ_i; this is the vanishing cycle associated to γ_i.

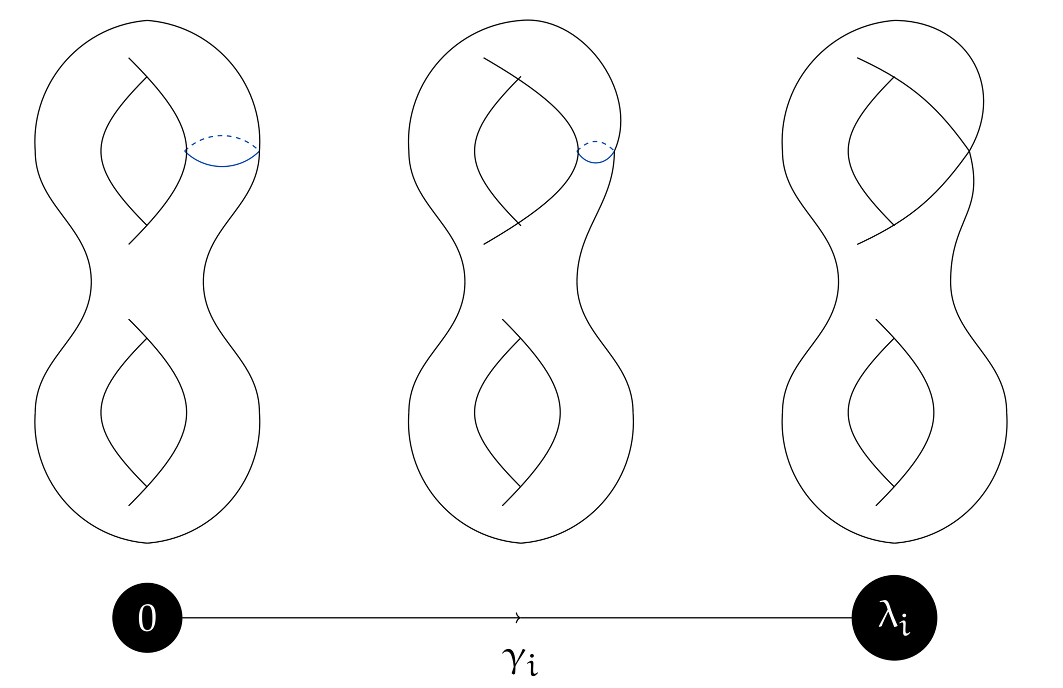


FIGURE 2: The vanishing cycle associated to the path γ_i.

Warning. This is not a faithful picture! In reality, the fibre is 4-dimensional, and the vanishing cycle is a 2-dimensional sphere.

Now we can finally say what $D^b\mathsf{Fuk}(W)$ is: it's the triangulated category with FEC given by

$$(L_1, \ldots, L_6) \quad \text{and} \quad \mathcal{H}(L_i, L_j) = |L_i \cap L_j|.$$

For example, if L_1 and L_2 intersect at five points then $\mathcal{H}(L_1, L_2) = 5$. If they don't intersect at all, then $\mathcal{H}(L_1, L_2) = 0$.

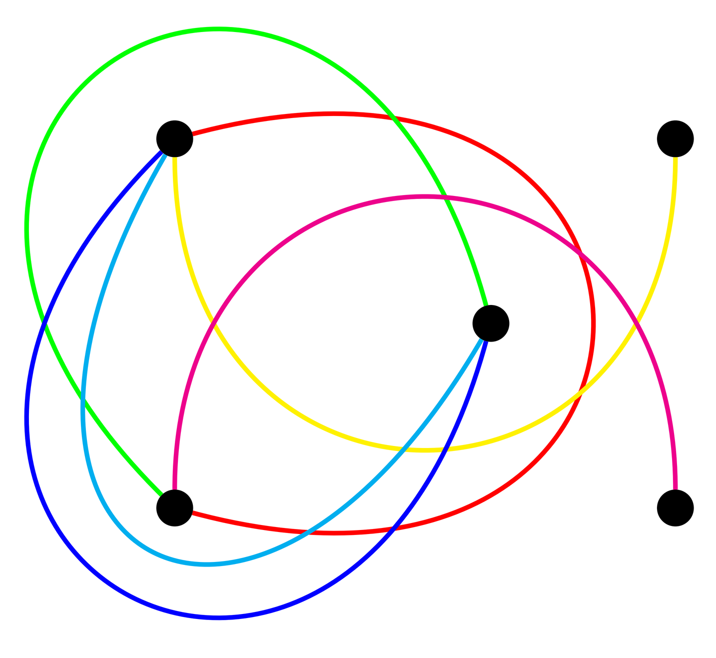


FIGURE 3: A crude picture of the vanishing cycles

$L_1, L_2, L_3, L_4, L_5, L_6.$

From this kind of diagram, one can compute the intersections $L_i \cap L_j$, which are the $\mathcal{H}$-numbers of the corresponding FEC.

Mutations in $D^b\mathsf{Fuk}(W)$ are described by the following fact.

Theorem (By the work of Seidel). The braided object

$$T_{L_i} L_{i+1}$$

is the vanishing cycle associated with the path

$$\gamma_{i+1} \text{ twisted around } \gamma_i.$$

We illustrate this with an example.

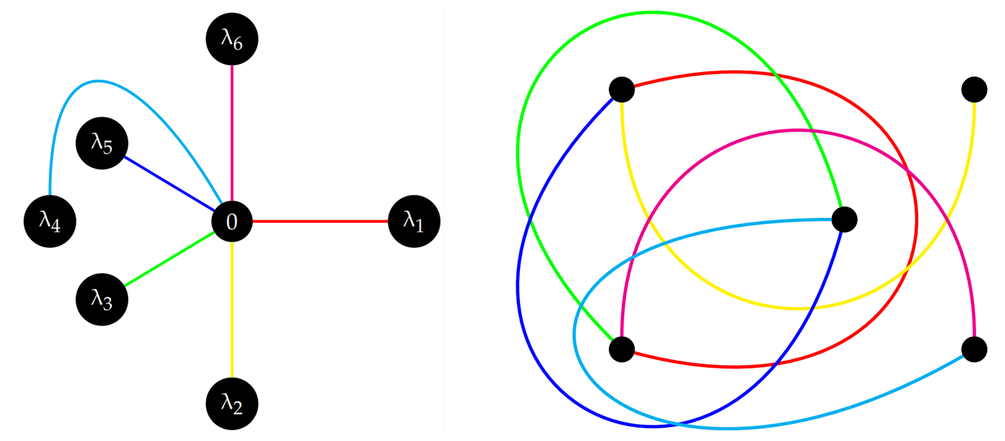


FIGURE 4: The old path γ_4 is twisted around the straight line γ_5. This has the effect of replacing L_4 with a different cycle: $T_{L_5}L_4$.

This means that mutations in $D^b\mathsf{Fuk}(W)$—which are algebraic at first sight—can just be seen as twisting paths around each other.

5. "What about $D^b\mathsf{Coh}(V_7)$?"

Instead of giving an FEC, it is easier to say what objects $D^b\mathsf{Coh}(V_7)$ contains: all 'twisted functions' on V_7 and its subspaces. For example, the set of all functions $V_7 \to \mathbb{C}$, denoted $\mathcal{O}_{V_7}$, is an object of $D^b\mathsf{Coh}(V_7)$. It is part of a family of spaces of 'twisted functions'

$$\ldots, \mathcal{O}_{V_7}(-3), \mathcal{O}_{V_7}(-2), \mathcal{O}_{V_7}(-1), \mathcal{O}_{V_7}, \mathcal{O}_{V_7}(1), \mathcal{O}_{V_7}(2), \mathcal{O}_{V_7}(3), \ldots$$

which are all objects in $D^b\mathsf{Coh}(V_7)$, too.

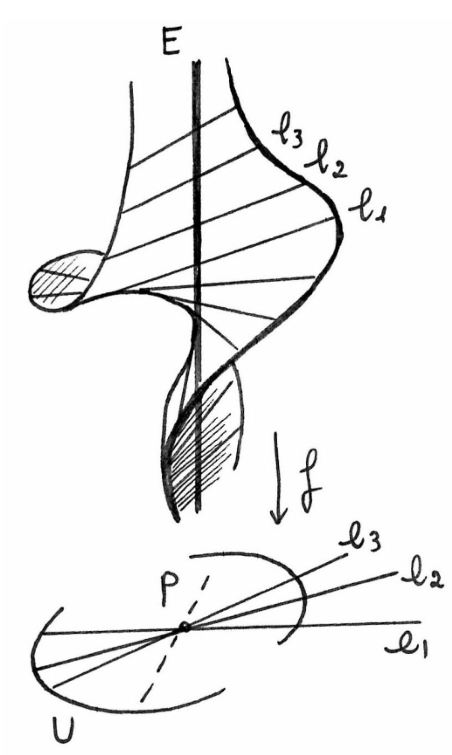


FIGURE 5: An illustration of V_7 (= $\mathbb{P}^3$ blown up at a point p). Rough idea: force all the lines through p to become parallel, by replacing p with E = {all the 'directions' pointing out of p}.

Similarly, the collection of (twisted) functions on the subspace E,

$$\ldots, \mathcal{O}_E(-2), \mathcal{O}_E(-1), \mathcal{O}_E, \mathcal{O}_E(1), \mathcal{O}_E(2), \ldots$$

are also objects of $D^b\mathsf{Coh}(V_7)$. There are many more.

The question is: can we find an FEC in $D^b\mathsf{Coh}(V_7)$?

One of the first FECs discovered in the wild was for $D^b\mathsf{Coh}(\mathbb{P}^3)$:

$$(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(2), \mathcal{O}_{\mathbb{P}^3}(3)).$$

Idea. Since V_7 is obtained by blowing up $\mathbb{P}^3$ at a point, we could hope it also has a similarly nice set of generators.

To do this, we extend the function $f\colon V_7 \to \mathbb{P}^3$ (from FIGURE 5) to a square, and take $D^b\mathsf{Coh}(-)$ of everything.

$$\begin{array}{ccc} E & \xrightarrow{j} & V_7 \\ \downarrow{\scriptstyle f|_E} & & \downarrow{\scriptstyle f} \\ p & \xrightarrow{i} & \mathbb{P}^3 \end{array} \qquad \begin{array}{ccc} D^b\mathsf{Coh}(E) & \xrightarrow{j_*} & D^b\mathsf{Coh}(V_7) \\ \uparrow{\scriptstyle (f|_E)^*} & & \uparrow{\scriptstyle f^*} \\ D^b\mathsf{Coh}(p) & \xrightarrow{i_*} & D^b(\mathbb{P}^3) \end{array}$$

Together, the images of f^* and $j_*(f|_E)^*$ do not generate the whole of $D^b\mathsf{Coh}(V_7)$. We need to add in a few twisted versions of the latter:

$$j_*(\mathcal{O}_E(-k) \otimes (f|_E)^*(-))\colon D^b\mathsf{Coh}(p) \to D^b\mathsf{Coh}(V_7)$$

for k = 1, 2. Using the FEC for $\mathbb{P}^3$, we get the following.

Proposition. The collection

$$\sigma = (\mathcal{O}_E(-2),\ \mathcal{O}_E(-1),\ \mathcal{O}_{V_7},\ f^*\mathcal{O}_{\mathbb{P}^3}(1), f^*\mathcal{O}_{\mathbb{P}^3}(2),\ f^*\mathcal{O}_{\mathbb{P}^3}(3))$$

is a FEC in $D^b\mathsf{Coh}(V_7)$.

6. "That's wonderful, but what about the aim?"

All this reduces the **Project Aim** to finding a set of paths to $\{\lambda_i\}$, and a mutation of σ, such that the resulting FECs in $D^b\mathsf{Fuk}(W)$ and $D^b\mathsf{Coh}(V_7)$ have the same $\mathcal{H}$-numbers. This is easier for $\mathbb{P}^2$ blown up at a point, where the diagram (compare FIGURE 3) is simpler. I tried by hand for a while; maybe you can write a program?

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