Mini-Project

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Pseudoholorphic Curves in Nearly Kähler Six-Manifolds

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1 Introduction

Nearly Kähler manifolds are a class of almost Hermitian manifolds which show certain similarities to Calabi-Yau or Kähler manifolds. The most important case is when the dimension is six. In this dimension, nearly Kähler manifolds are always Einstein and have positive scalar curvature. Furthermore, strictly nearly Kähler six-manifolds are related to special holonomy, as the Riemannian cone of such a space carries a torsion-free G_2 structure. The classical example of a nearly Kähler manifold is S^6 and the study of this structure has been related to the question of the existence of a complex structure on S^6.

The two main approaches to construct nearly Kähler manifolds come from the study of 3-symmetric spaces and twistor theory. In this project, we will take a look at the spaces which carry both structures. In fact, in dimension six the only two such spaces are $\mathbb{CP}^3$ and the flag manifold $\mathcal{F} = \mathrm{SU}(3)/T^2$.

To gain a better understanding of these spaces one can investigate their moduli space of pseudoholomorphic curves with the ultimate aim to construct invariants. In his thesis, Xu has constructed invariants for pseudoholomorphic curves in $\mathbb{CP}^3$ and classified those with genus 0. Furthermore, he found out that certain pseudoholomorphic curves in the nearly Kähler-$\mathbb{CP}^3$ are in 1:1 correspondence with integral submanifolds of the contact structure on $\mathbb{CP}^3$. However, he has left a gap in the proof of this theorem which we will close in section 7. This is achieved by putting his constructions on a more general basis using the language of principal bundles. We indicate how these constructions can be carried over to the flag manifold since this space shares the twistorial structure with $\mathbb{CP}^3$. To that end, we have worked out the holomorphic contact structure on the flag manifold explicitly.

2 Prerequisites

2.1 Divisors and Line Bundles

The following is a recollection of the correspondence between line bundles and divisors on a Riemann surface extracted from [7]. The set of divisors $\mathrm{Div}(M)$ is the free abelian group generated by the set of points of M. Thus we may write an element $D \in \mathrm{Div}(M)$ as

$$D = \sum_{i=1}^{n} a_i p_i$$

for integers a_i and points $p_i \in M$. In a chart U_α, each p_i is the only zero set of a holomorphic function $g_{i\alpha}$ such that p_i is a simple root of $g_{i\alpha}$. This function is well defined up to multiplication by a non-vanishing holomorphic function. Now set $f_\alpha = \Pi g_{i\alpha}^{a_i}$. So, to any divisor D we can associate a collection of functions $f_\alpha \colon U_\alpha \to \mathbb{C}$ which is unique up to a multiplication of non-vanishing holomorphic functions. Recall, that a line bundle can be defined by specifying transition functions $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbb{C}$. Two such collections $g'_{\alpha\beta}$ and $g_{\alpha\beta}$ of transition functions yield the same line bundle when there are holomorphic functions $h_\alpha \colon U_\alpha \to \mathbb{C}$ such that

$$g'_{\alpha\beta} = \frac{h_\alpha}{h_\beta} g_{\alpha\beta}. \tag{1}$$

Now, for a divisor D given by a collection $\{U_\alpha, f_\alpha\}$ define $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbb{C}^*$ by $\frac{f_\alpha}{f_\beta}$. Note, that the quotient is a well-defined holomorphic map since the restriction of $g_{i\alpha}$ and $g_{i\beta}$ to $U_\alpha \cap U_\beta$ both satisfy $\{g_{i\alpha} = 0\} = \{g_{i\beta} = 0\} = V \cap U_\alpha \cap U_\beta$ and are minimal in the sense above with that property. Hence their quotient is a non-vanishing holomorphic function on $U_\alpha \cap U_\beta$. If one chooses to represent D by a different family of functions, the associated line bundles will be isomorphic by equation 1. Hence, this construction yields a group homomorphism $\phi \colon \mathrm{Div}(M) \to \mathrm{Pic}(M)$. It descends to an isomorphism with the following inverse

$$\begin{aligned} \mathrm{Div}(M)/\{\text{principal divisors}\} &\to \{\text{line bundles which admit a global meromorphic section}\} \\ D = \{(U_\alpha, f_\alpha)\} &\mapsto [D] := \{(g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}, U_\alpha \cap U_\beta)\} \\ D_L := \sum_p \mathrm{ord}_p(s) p &\leftarrow\!\shortmid L. \end{aligned}$$

Here, s is any global meromorphic section of L. The equivalence class of the resulting divisor does not depend on the choice of s.

2.2 Holomorphic Data on Principal Bundles

Holomorphicity is a property that can be checked on the level of principal bundles. Suppose that M is a Kähler manifold and that P is a G-principal bundle over M with the property that there is a representation of G on V such that $TX = P \times_\rho V$. For example, P could be the bundle of frames of M and ρ the standard representation of Gl_n on $\mathbb{R}^n$ or if $M = K/G$ one has $K \times_{\mathrm{Ad}} \mathfrak{m} = TM$.

We may complexify this representation and get $TX_\mathbb{C} = P \times_{\rho_\mathbb{C}} V_\mathbb{C}$. The complex structure on X will give rise to $T^\vee X_\mathbb{C} = \Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X)$. Let us assume that this comes from a splitting $V_\mathbb{C}^\vee = V^{1,0} \oplus V^{0,1}$. Now, let W, τ be another representation of G and let $E = P \times_\tau W$. Since M is a Kähler manifold, the Chern connection and Levi-Civita ∇ connection coincide, i.e. $\bar\partial = \pi^{0,1} \circ \nabla$.

The connection ∇ corresponds to a connection one-form ϕ on the principal bundle. We get the following commutative diagram

$$\begin{array}{ccccc}
\Gamma(P,W)^{bas} & \longrightarrow & \Gamma(P,V^{\vee}\otimes W)^{bas} & \longrightarrow & \Gamma(P,V^{0,1}\otimes W)^{bas} \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(M,E) & \longrightarrow & \Gamma(M,T^{\vee}X_{\mathbb{C}}\otimes E) & \longrightarrow & \Gamma(M,\Lambda^{0,1}\otimes W)
\end{array} .$$

3 Nearly Kähler Manifolds

Given a manifold M with an almost complex structure J one can always find a Riemannian metric g on M such that J becomes an isometry. The resulting structure (M, J, g) is then said to be an almost Hermitian manifold. On such a manifold, one can define the two-form

$$\omega(X, Y) = g(JX, Y).$$

If ω is covariant constant with respect to the Levi-Civita connection ∇ then (M, J, g) is a Kähler manifold. The tensor $\nabla\omega$ has generally four irreducible components. By requiring only some of them to vanish one obtains different notions of an almost Hermitian manifolds which satisfy weaker integrability conditions than a Kähler manifold. One of them is the class of nearly Kähler manifolds.

Definition 1. *An almost Hermitian manifold is a nearly Kähler manifold if*

$$\nabla_X(J)(X) = 0$$

for every vector field X on M.

Equivalently, if J is considered as a section of $TM^\vee \otimes TM$ then $\nabla J \in \Gamma(\Lambda^2(TM^\vee) \otimes TM) \subset \Gamma(TM^\vee \otimes TM^\vee \otimes TM)$ if and only if M is nearly Kähler. A nearly Kähler manifold which is not Kähler is called a strictly nearly Kähler manifold. Sometimes, they are ambiguously also simply called nearly Kähler manifolds.
Generally, a condition which is easier to compute is the following.

Proposition 1. *A manifold M is nearly Kähler if and only if*

$$\mathrm{d}\omega = 3\nabla\omega.$$

If the manifold in question is a homogeneous space then it is worthwhile studying nearly Kähler manifolds within the framework of 3-symmetric spaces.

3.1 3-Symmetric Spaces

3-Symmetric spaces were introduced by Wolf and Gray in [17] when they studied homogeneous nearly Kähler manifolds. Let G be a Lie-Group and H be a closed Lie Subgroup. Then the homogeneous space G/H is called a 3-symmetric space when there is a $\sigma \in \mathrm{Aut}(G)$ with $\sigma^3 = 1$ such that

$$G_0^\sigma \subset H \subset G^\sigma.$$

Here, G^σ is the fixed point set of σ and G_0^σ denotes the connected component of the identity in this space. This implies that $\mathfrak{h} = \{\xi \in \mathfrak{g} \mid \sigma_*(\xi) = \xi\}$. In particular, σ_* is an endomorphism over $\mathfrak{g}_\mathbb{C}$ and has a decomposition into different eigenspaces, i.e.

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \mathfrak{m}_\zeta \oplus \mathfrak{m}_{\zeta^2},$$

where $\zeta = e^{2\pi i/3}$. Since σ_* is a Lie-Algebra homomorphism it commutes with the $\mathrm{Ad}(H)$-action and as a consequence, $\mathfrak{m}_\zeta$ and $\mathfrak{m}_{\zeta^2}$ are both $\mathrm{Ad}(H)$-invariant. Denote by $M/G/H$ and by $\mathfrak{m} = \mathfrak{g} \cap (\mathfrak{m}_\zeta \oplus \mathfrak{m}_{\zeta^2})$ then we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, which is a decomposition into $\mathrm{Ad}(H)$-invariant subspaces. Hence, as for symmetric spaces, $\mathfrak{g}$ is reductive. We write $\mathfrak{m}^+ = \mathfrak{m}_\zeta$ and $\mathfrak{m}^- = \mathfrak{m}_{\zeta^2}$ and observe

$$[\mathfrak{h}_\mathbb{C}, \mathfrak{h}_\mathbb{C}] \subset \mathfrak{h}_\mathbb{C}, \quad [\mathfrak{m}^+, \mathfrak{m}^+] \subset \mathfrak{m}^-, \quad [\mathfrak{m}^-, \mathfrak{m}^-] \subset \mathfrak{m}^+, \quad [\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{h}_\mathbb{C}. \tag{2}$$

Conversely, a homogeneous space G/H is a 3-symmetric space if there is a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-$ such that the above relations hold. The key point is that $T_e(G/H) \cong \mathfrak{g}/\mathfrak{h} = \mathfrak{m}$ and $\mathrm{Ad}(H)$-invariant objects on $\mathfrak{m}$ extend to structures on M. For example, we can declare $\mathfrak{m}^+$ to be the $+i$ and $\mathfrak{m}^-$ to be the $-i$ eigenspace of an almost complex structure J, which commutes with σ_*. This extends to an almost complex structure on M and is referred to as the canonical almost complex structure J_{can} on G/H. Furthermore, an $\mathrm{Ad}(H)$-invariant metric on $\mathfrak{m}$ gives rise to a Riemannian metric on M. A crucial result is that the property of being nearly Kähler can be formulated in Lie-algebraic terms.

Theorem 1. [6, Proposition 5.6] Let $M = G/H$ with a decomposition satisfying 2, i.e. M is a 3-symmetric space. Let g be an $\mathrm{Ad}(H)$-invariant metric on $\mathfrak{m}$. Then (M, g, J_{can}) is a nearly Kähler manifold if and only if

$$g([X,Y],Z) = g([Z,X],Y) \quad \forall X,Y,Z \in \mathfrak{m}.$$

3.2 Nearly Kähler Manifolds in Dimension Six

When the dimension of M is less than 6 there are no strictly nearly Kähler manifolds. [9, Lemma 1.5] Indeed, in dimension six, nearly Kähler manifolds have the most interesting properties and are also most relevant for applications.

Proposition 2. [1, Theorem 2] Let (M, J, g) be a six-dimensional strictly nearly Kähler manifold. Then the Riemannian cone $C(M) = M \times \mathbb{R}^{>0}$ can be equipped with a three form $\phi \in \Lambda^3(C(M))$ which turns $(C(M), \phi)$ into a torsion-free G_2-manifold and induces the cone metric $\mathrm{d}r^2 + r^2 g$.

Furthermore, strictly nearly Kähler manifolds in dimension six are Einstein and their first Chern class vanishes. Another reason to focus on the case of dimension six is the following theorem by Nagy.[10]

Theorem 2. *A strict and complete nearly Kähler manifold is locally a Riemannian product of homogeneous nearly Kähler spaces, twistor spaces over quaternionic Kähler manifolds and 6-dimensional nearly Kähler manifolds.*

In dimension six there is a convenient characterisation of nearly Kähler manifolds using differential forms.

Proposition 3. *[11] Let (M, ω) be a six-dimensional almost Hermitian manifold. Then M is nearly Kähler if and only if there is a three-form $\psi \in \Lambda^{3,0}(M)$ and a constant $\mu \in \mathbb{R}$ satisfying*

$$\mathrm{d}\omega = 12\mu \mathrm{Re}(\psi)$$
$$\mathrm{d}(\mathrm{Im}(\psi)) = \mu\omega \wedge \omega.$$

In fact, the homogeneous strictly nearly Kähler 6-folds are classified.

Proposition 4. [3, Theorem 1] If $M = G/H$ is a homogeneous strictly nearly Kähler manifold of dimension six, then M is an element of the following list:

- $G = G_2$ and $H = \mathrm{SU}(3)$ such that $M = S^6$
- $G = S^3 \times S^3 \times S^3$ and $H = \{(g,g,g) \mid g \in S^3\}$ such that $M = S^3 \times S^3$
- $G = \mathrm{Sp}(2)$ and $H = S^1 \times S^3$ such that $M = \mathbb{CP}^3_{NK}$

- $G = \mathrm{U}(3)$ and $H = \mathrm{U}(1)^3$ such that M is the manifold of complete complex flags of $\mathbb{C}^3$

All of these examples carry an invariant strictly nearly Kähler structure, unique up to homothety, and arise as 3-symmetric spaces.

The 3-symmetric structure on $M = S^3 \times S^3$ comes from considering

$$\sigma \in \mathrm{Aut}(S^3 \times S^3 \times S^3), \quad \sigma(x_1, x_2, x_3) = (x_3, x_1, x_2)$$

It should be noted that possibly non-homogeneous strictly nearly Kähler manifolds have not been classified yet. Recently, Foscolo and Haskins have found nearly Kähler structures on S^6 and $S^3 \times S^3$ that are not homogeneous.[5]
The last two examples listed above be the ones which are important for this project. They arise as twistor spaces of four-dimensional manifolds.

3.3 Twistor spaces

To each Riemmanian manifold M one can associate a twistor space $Z(M)$, which is a fibre bundle over M. The fibre of this bundle over x is given by

$$\{J_x \colon T_xM \to T_xM \mid J_x^2 = -1, \quad g(J_xv, J_xw) = g(v, w), \quad J_x \text{ preserves orientation}\}.$$

If M is four-dimensional then the twistor space $Z(M)$ can be identified with elements $\Lambda^2_-(M)$ with unit length. Hence it is a sphere bundle over M which inherits a connection from the connection on $\Lambda^2(M)$ which is induced from the Levi-Civita connection. Hence $TZ(M)$ splits into a vertical and horizontal subbundle $TZ(M) = \mathcal{H} \oplus \mathcal{V}$. Now, pick a point $p \in Z(M)$ and $x = \pi(p)$, then $(\pi_*)_p \colon \mathcal{H}_p \to T_xM$ is an isomorphism of vector spaces. This gives on the one hand rise to a metric on $\mathcal{H}_p$ coming from g_x on T_xM. On the other hand, by the construction of the twistor space, p defines an orthogonal almost complex structure on T_xM. Hence there is one on $\mathcal{H}_p$. Furthermore, $\mathcal{V}_p$ is the tangent space of a sphere. So there is a canonical metric on $\mathcal{V}_p$ and two possible choices for an orthogonal almost complex structure. Denote by j_1 the orientation preserving and by j_2 the orientation reversing one. This means that one can equip $Z(M)$ in a natural manner with a metric and with two choices of almost complex structures, J_1, J_2.

Proposition 5. [13, Theorem 3.3, Proposition 3.4] Let M be a Riemannian four-manifold, then $(Z(M), J_1)$ is a complex manifold iff M is self-dual and $(Z(M), J_2)$ is never integrable.

It turns out that the twistor space of $M = S^4$ with the round metric is $\mathbb{CP}^3$. The fibration $\mathbb{CP}^3 \to S^4$ is constructed by

$$\mathbb{CP}^3 \to \mathbb{HP}^1, \quad [z_0 : z_1 : z_2 : z_3] \mapsto [z_0 + jz_1 : z_2 + jz_3]$$

and identifying $\mathbb{HP}^1$ with S^4. The twistor space of $M = \mathbb{CP}^2$ is the flag manifold $\mathrm{U}(3)/\mathrm{U}(1)^3$. There are three choices for the fibration

$$\pi_{ij} \colon \mathcal{F} = U(3)/U(1)^3 \to U(3)/U(1) \times U(2) \cong \mathbb{CP}^2,$$

induced from embedding the i-th times the j-th factor of U(1) into U(2). This yields $2^3 = 8$ possible almost complex structures corresponding to the splitting of $\mathfrak{m}$ into three one-dimensional $\mathrm{U}(1)^3$-invariant subspaces which will be described in subsection 4.4.2. If not stated otherwise, we will refer to the twistor fibration of the flag manifold the as π_{13}. On $\mathbb{CP}^3$ and $\mathcal{F}$, J_1 and J_2 will give rise to a Kähler and a nearly Kähler structure, respectively. More generally, the twistor space of quaternion-Kähler manifold with positive Ricci curvature always carries a strictly nearly Kähler structure. [14] However, when $\dim(M) = 4$ then this fact does not provide new examples.

Theorem 3. [8, Proposition 6.1] The twistor space $Z(M)$ of a compact oriented Riemmanian four-manifold is Kähler if and only if $M = \mathbb{CP}^2$ or $M = S^4$.

3.4 Gauss lifts

Let (M, g) be a compact Riemann surface, (N, h) be a Riemannian manifold of dimension n and $\phi\colon M \to N$ be smooth. The differential $\mathrm{d}\phi$ can be considered as an element of $\Gamma(M, TM^\vee \otimes \phi^*(TN))$. The Levi-Civita connections on M and N induce a connection on this bundle such that one can consider $\nabla(\mathrm{d}\phi) \in \Gamma(M, TM^\vee \otimes TM^\vee \otimes \phi^*(TN))$. Because the Levi-Civita connection on M is torsion-free this tensor is symmetric in the first two components, i.e. $\nabla(\mathrm{d}\phi) \in \Gamma(M, S^2(TM^\vee) \otimes \phi^*(TN))$. By contracting with $g^\vee$ one obtains the so called tension field $\tau_\phi \in \Gamma(M, \phi^*(TN))$. We say that ϕ is conformal if $\phi^*(h)(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = 0$, i.e. $\phi^*(h)$ is a conformally equivalent to g away from the zeroes of ϕ_*. Furthermore, we say that ϕ is harmonic if $\tau_\phi = 0$. This property only depends on the conformal class of g and h and is equivalent to

$$\nabla_{\frac{\partial}{\partial z}} \nabla_{\frac{\partial}{\partial \bar{z}}}(\phi) = \nabla_{\frac{\partial}{\partial \bar{z}}} \nabla_{\frac{\partial}{\partial z}}(\phi) = 0.$$

It can be shown that ϕ is a minimal branched immersion if and only if it is harmonic and conformal. [12, Theorem 1.1] Denote by $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$ the space of oriented two-planes in $\mathbb{R}^n$. An element $V \in \widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$ can be identified with $e_1 \wedge e_2$ for an oriented orthonormal basis $\{e_1, e_2\}$ of V. As a homogeneous space,

$$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^n) = \mathrm{SO}(n)/(\mathrm{SO}(2) \times \mathrm{SO}(n-2)).$$

The group $\mathrm{SO}(n)$ acts on this space and we can define the fibre bundle

$$\widetilde{\mathrm{Gr}}_2(TN) = P_{\mathrm{SO}(n)} \times_{\mathrm{SO}(n)} \widetilde{\mathrm{Gr}}_2(\mathbb{R}^n).$$

Here, $P_{\mathrm{SO}(n)}$ denotes the bundle of oriented frames of M. A fibre over a point $x \in N$ is the space oriented two-planes in T_xN. The space is constructed such that each $\phi\colon M \to N$ has a lift $\tilde{\phi}\colon M \to \widetilde{\mathrm{Gr}}_2(TN)$ which sends $x \in M$ to $\phi_*(T_xM) \in T_{\phi(x)}N$. Consider now the case $n = 4$, such that the Hodge star operator $\star$ satisfies $\star^2 = -1$ and $\Lambda^2(\mathbb{R}^4) = \Lambda^2_+(\mathbb{R}^4) \oplus \Lambda^2_-(\mathbb{R}^4)$. If $\{e_1, e_2, e_3, e_4\}$ is an oriented basis of $\mathbb{R}^4$ then

$$\{e_1 \wedge e_2 \pm e_3 \wedge e_4, \quad e_1 \wedge e_3 \pm e_4 \wedge e_2, \quad e_1 \wedge e_4 \pm e_2 \wedge e_3\}$$

is a basis of $\Lambda^2(\mathbb{R}^4)$. Declaring this to be an orthonormal basis one obtains by this splitting the double cover

$$\mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3).$$

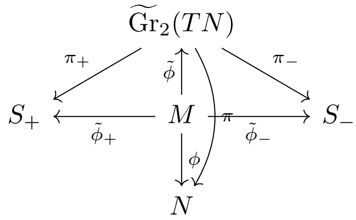
Under this map, $\mathrm{SO}(2) \times \mathrm{SO}(2) \subset \mathrm{SO}(4)$ is mapped to

$$\mathrm{Stab}_{e_1 \wedge e_2 + e_3 \wedge e_4} \times \mathrm{Stab}_{e_1 \wedge e_2 - e_3 \wedge e_4} = \mathrm{SO}(2) \times \mathrm{SO}(2)$$

by $(\exp(i\alpha), \exp(i\beta)) \mapsto (\exp(i(\alpha + \beta)), \exp(i(\alpha - \beta)))$, which is a double cover. Hence

$$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^4) = \frac{\mathrm{SO}(4)}{\mathrm{SO}(2) \times \mathrm{SO}(2)} \cong \frac{\mathrm{SO}(3) \times \mathrm{SO}(3)}{\mathrm{SO}(2) \times \mathrm{SO}(2)} \cong \frac{\mathrm{SO}(3)}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(3)}{\mathrm{SO}(2)} = S^2_+ \times S^2_-.$$

Denote by $S_\pm = P_{\mathrm{SO}(4)} \times_{\mathrm{SO}(4)} S^2_\pm$ then $\widetilde{\mathrm{Gr}}_2(TN) = S^+ \times S^-$ and there are two projections $\pi_\pm\colon \widetilde{\mathrm{Gr}}_2(TN)$ such that one can define the Gauss lifts $\tilde{\phi}_\pm = \pi_\pm \circ \tilde{\phi}$. In summary, one gets the following commutative diagram



Furthermore, $S_\pm$ are isomorphic to the twistor bundle $Z(N)$ by

$$e_i \wedge e_j \pm e_k \wedge e_l \mapsto (e_1 \mapsto e_2, \quad e_2 \mapsto -e_1, \quad e_3 \mapsto e_4, \quad e_4 \mapsto -e_3).$$

Hence, if N satisfies the necessary curvature conditions, $S_\pm$ are equipped with two almost complex structures J_1 and J_2 where J_1 is integrable and J_2 is not.

Proposition 6. [4, Corollary 5.4] Let M be a Riemann surface and N a four dimensional Riemannian manifold, $\phi\colon M \to N$ is a conformal harmonic map, i.e. a branched immersion, if and only if $\widetilde{\phi}_\pm\colon M \to (S_\pm, J_2)$ is a pseudoholomorphic non-vertical curve.

The upshot of this is that studying pseudoholomorphic curves in the nearly Kähler manifolds $\mathbb{CP}^3_{NK}$ and $\mathcal{F}$ is equivalent to study minimal branched immersions into S^4 and $\mathbb{CP}^2$.

4 Homogeneous Spaces

4.1 Connections on Homogeneous Spaces

Let G be a Lie group with a closed Lie subgroup H, denote by M the homogeneous space $M = G/H$. In the following, we will regard G as an H-principal bundle over M. Assume that there is an $\mathrm{Ad}(H)$-invariant metric b on $\mathfrak{g}$. This metric exists for example when $\mathfrak{g}$ is semisimple or when H is compact. It gives rise to a orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Furthermore, let $p_\mathfrak{m}$ and $p_\mathfrak{h}$ the corresponding orthogonal projections. Let $\omega \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan form on G and let $\alpha = p_\mathfrak{m} \circ \omega$ as well as $\phi = p_\mathfrak{h} \circ \omega$. Note that ϕ defines a connection on M.

Lemma 1. *For each isometry $\psi\colon \mathfrak{m} \to \mathbb{R}^n$ there is a bundle morphism $f_\psi\colon G \to P_{\mathrm{O}(n)}$ which is injective if and only if the adjoint action of H on $\mathfrak{m}$ is faithful.*

Proof. Fix an isometry $\psi\colon \mathbb{R}^n \to \mathfrak{m}$. Then for $g \in G$ the map $f_g\colon (L_g)_* \circ \psi$ is an isomorphism $\mathbb{R}^n \to T_{[g]}M$ which preserves the metric. So, $g \mapsto f_g$ is the desired bundle morphism. Its injectivity is equivalent to saying that $H \to \mathrm{O}(T_{[e]}M), \quad h \mapsto (L_h)_*$ is a faithful representation of H. Identifying $T_{[e]}M$ with $\mathfrak{m}$ turns this into the adjoint representation of H on $\mathfrak{m}$. ☐

Denote the adjoint representation by $\rho\colon H \to \mathrm{O}(\mathfrak{m})$ such that ρ_* gives a representation of $\mathfrak{h}$ on $\mathfrak{o}(\mathfrak{m})$ which is given by $\xi \mapsto [\xi, \cdot]$.

Lemma 2. *Let θ be the soldering form on $P_{\mathrm{O}(n)}$. Then we have*

$$(f_\psi)^*(\theta) = \psi^{-1} \circ \alpha.$$

Proof. Let $g \in G$ and $v \in T_gG$ then

$$(f_\psi)^*(\theta)(v) = (f_\psi(g))^{-1}(\pi_*(v)) = \psi^{-1} \circ (L_g)_*^{-1} \circ \pi_*(v) = \psi^{-1} \circ \pi_* \circ (L_g)_*^{-1}(v).$$

The pushforward π_* restricts on $T_eG = \mathfrak{g}$ to $p_\mathfrak{m}$ which implies the statement. ☐

This justifies to regard, for a fixed ψ, $\psi^{-1} \circ \alpha$ as a soldering form on $G \to M$. Hence, the torsion of ϕ is given by $\Theta = \mathrm{d}\theta + \phi \wedge \theta = \psi^{-1}(d\alpha + [\alpha, \phi])$.

Lemma 3. *The curvature and torsion of ϕ can be calculated in terms of α by*

$$F = -\frac{1}{2} p_\mathfrak{h}([\alpha, \alpha]) \tag{3}$$

$$\Theta = -\frac{1}{2} \psi^{-1}(p_\mathfrak{m}([\alpha, \alpha])) \tag{4}$$

Proof. The Maurer-Cartan form has vanishing curvature.

$$0 = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = \mathrm{d}\alpha + \mathrm{d}\phi + \frac{1}{2}[\alpha, \alpha] + \frac{1}{2}[\phi, \phi] + [\alpha, \phi] \tag{5}$$

Because b is $\mathrm{Ad}(H)$-invariant it satisfies $b([X, Y], Z) = -b(X, [Y, Z])$ and hence $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The above equation yields two equations by projecting onto $\mathfrak{m}$ and $\mathfrak{h}$ respectively.

$$0 = \mathrm{d}\alpha + [\alpha, \phi] + p_\mathfrak{m}\left(\frac{1}{2}[\alpha, \alpha]\right) \tag{6}$$

$$0 = \mathrm{d}\phi + \frac{1}{2}[\phi, \phi] + p_\mathfrak{h}\left(\frac{1}{2}[\alpha, \alpha]\right) \tag{7}$$

Corollary 1. *The connection ϕ is a reduction of the Levi-Civita connection if and only if M is a symmetric space.*

□

In general, $\mathrm{Ad}(H)$-invariant objects on $\mathfrak{m}$ give rise to left-invariant objects M, such as metrics and almost complex structures. Thus we can define an almost Hermitian structure on M by finding an Ad-invariant Hermitian metric on $\mathfrak{m}$.

4.2 $\mathbb{CP}^3$ and $\mathcal{F}$ as Homogeneous Spaces

On $\mathbb{H}^n$, there is a standard quaternion valued inner product given by $\langle v, w\rangle = \sum v_i \bar{w}_i$. The group of $\mathbb{H}$-linear endomorphisms of $\mathbb{H}^n$ which preserve this inner product is denoted by $\mathrm{Sp}(n)$. Here, only the group $\mathrm{Sp}(2)$ will be of interest. By picking the standard $\mathbb{H}$ basis on $\mathbb{H}^2$ one can identify $\mathrm{Sp}(2)$ with pairs of vectors (v_1, v_2) which are an ONB of $\mathbb{H}^2$. Hence, for $i = 1, 2$ one can consider $\mathrm{Sp}(2) \to S^7 \subset \mathbb{H}^2, \quad (v_1, v_2) \mapsto v_i$. Both of these maps give $\mathrm{Sp}(2)$ the structure of a S^3 fibre bundle over S^7. By canonically identifying $\mathbb{H}^2$ with $\mathbb{C}^4$ one obtains by composing with the projection to $\mathbb{CP}^3_{NK}$ two maps $e_1, e_2 \colon \mathrm{Sp}(2) \to \mathbb{CP}^3_{NK}$. Thus, $\mathbb{CP}^3_{NK}$ may be regarded as a homogeneous space as either $\mathrm{Sp}(2)/S^3 \times S^1$ or $\mathrm{Sp}(2)/S^1 \times S^3$. The homogeneous structure will be used to define different almost complex structures and metrics on $\mathbb{CP}^3_{NK}$. To distinguish them, we will write $\mathrm{Sp}(2)/S^3 \times S^1 = \mathbb{CP}^3_K$ and $\mathrm{Sp}(2)/S^3 \times S^1 = \mathbb{CP}^3_{NK}$. By quotienting out $\mathrm{Sp}(2)$ just by $S^1 \times S^1$ one obtains the manifold F. In summary,

$$\begin{array}{ccccc} & & \mathrm{Sp}(2) & & \\ & & \downarrow & & \\ & & F & & \\ & \swarrow & & \searrow & \\ \mathbb{CP}^3_{NK} & & & & \mathbb{CP}^3_K \end{array} .$$

On the flag manifold, we will work with the homogeneous space structure $\mathrm{U}(3)/\mathrm{U}(1)^3$ or $\mathrm{SU}(3)/T^2$ which arises from equipping $\mathbb{C}^3$ with a Hermitian inner product.

4.3 Homogeneous Vector Bundles

A vector bundle $\pi \colon E \to M$ over a Homogeneous space $M = G/H$ is called homogeneous if G acts on E by bundle homomorphisms and $gE_x \subset E_{gx}$ (which implies $gE_x = E_{gx}$). A morphism of homogeneous vector bundles is a G-equivariant bundle morphism. Given a representation $\rho \colon H \to \mathrm{End}(V)$ the associated bundle $E_\rho = G \times_{(H,\rho)} V$ is homogeneous with respect to the action $g'[(g, v)] = [(g'g, v)]$. Given a homogeneous bundle $E \to M$ one sees that H acts on $E_{[e]}$, i.e. one obtains a representation of H. Both operations are inverse to each other and equivariant homomorphisms between two representations correspond to bundle morphisms. I.e. the category of finite dimensional representations of H is equivalent to the category of homogeneous vector bundles over M. The model example for a homogeneous vector bundle is the tangent bundle on G/H which corresponds to the adjoint action of H on $\mathfrak{m}$, assuming that homogeneous space is reductive with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.
Similarly, unitary representations will correspond to Hermitian line bundles. For more details, we refer the reader to [16].
We can use this to prove that different bundles over $\mathbb{CP}^3_{NK}$ are isomorphic by showing that the

corresponding representations are equivalent. One can not argue in the other direction as an isomorphism between homogeneous bundles is stronger than just a bundle isomorphism.
Note that the tautological bundle $\mathcal{O}(-1)$ corresponds to the standard representation of S^1 on $\mathbb{C}$.

4.4 Representations

4.4.1 $\mathbb{CP}^3_{NK}$

The Lie-algebra $\mathfrak{sp}(2)$ splits into

$$\mathfrak{sp}(2) = \mathfrak{h} \oplus \underbrace{\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in j\mathbb{C} \right\}}_{\mathfrak{m}_1} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & h \\ -\bar{h} & 0 \end{pmatrix} \mid h \in \mathbb{H} \right\}}_{\mathfrak{m}_2}.$$

If we identify

$$\mathbb{C} \to \mathfrak{m}_1, \quad z \mapsto \begin{pmatrix} jz & 0 \\ 0 & 0 \end{pmatrix}; \qquad \mathbb{H} \to \mathfrak{m}_2, \quad h \mapsto \begin{pmatrix} 0 & h \\ -\bar{h} & 0 \end{pmatrix}$$

then an element $(\lambda, q) \in S^1 \times S^3$ acts by multiplication of λ^2 on $\mathbb{C}$ and on $\mathbb{H}$ by $h \mapsto qh\lambda^{-1}$. The tautological bundle $\mathcal{O}(-1)$ lives inside the trivial bundle $\mathbb{CP}^3 \times \mathbb{C}^4 \subset \mathbb{CP}^3 \times \mathbb{H}^2$. Thus we consider the bundle $j\mathcal{O}(-1) \subset \mathbb{CP}^3 \times \mathbb{H}^2$. Since $\lambda j = j\bar{\lambda}$ for $\lambda \in \mathbb{C}$ this bundle corresponds to the representation $z \mapsto \lambda^{-1} z$, so $j\mathcal{O}(-1) \cong \mathcal{O}(1)$ as Hermitian line bundles. This splitting gives rise to a splitting of $T\mathbb{CP}^3_{NK}$ which corresponds to the splitting into a vertical and horizontal component

$$\begin{aligned} T\mathbb{CP}^3_{NK} &= \mathrm{Sp}(2) \times_{S^1 \times S^3} \mathfrak{m} = \mathrm{Sp}(2) \times_{S^1 \times S^3} \mathfrak{m}_1 \oplus \mathfrak{m}_2 \\ &= \mathrm{Sp}(2) \times_{S^1 \times S^3} \mathfrak{m}_1 \oplus \mathrm{Sp}(2) \times_{S^1 \times S^3} \mathfrak{m}_2 = \mathcal{H} \oplus \mathcal{V}. \end{aligned}$$

4.4.2 Flag manifold

Since $\mathcal{F} = \mathrm{U}(3)/\mathrm{U}(1)^3$, we need to decompose the representation of $\mathrm{U}(1)^3$ on $\mathfrak{u}(3)$ into irreducible ones.

$$\mathfrak{u}(3) = \mathfrak{u}(1)^3 \oplus \underbrace{\left\{ \begin{pmatrix} 0 & -\bar{z} & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}}_{\mathfrak{m}_{12}} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & 0 & -\bar{z} \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \right\}}_{\mathfrak{m}_{13}} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{z} \\ 0 & z & 0 \end{pmatrix} \right\}}_{\mathfrak{m}_{23}}.$$

This notation makes clear how to identify each $\mathfrak{m}_i$ with $\mathbb{C}$. Then $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ acts on $\mathfrak{m}_{ij}$ by multiplication with $\lambda_i^{-1}\lambda_j$. Hence, the tangent bundle splits into a sum of three line bundles.

$$T\mathcal{F} = \mathrm{U}(3) \times_{\mathrm{U}(1)^3} (\mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}) = L_{12} \oplus L_{13} \oplus L_{23}.$$

The horizontal distribution for the fibration π_{13} is then given by $L_{12} \oplus L_{23}$ while the vertical distribution is equal to L_{13}.

4.5 The Maurer-Cartan form in exponential coordinates

In the following, we will work with differential forms which come from projections of the Maurer-Cartan form. To find explicit examples of certain types of pseudoholomorphic curves it is of use to be able to express these differential forms in coordinates. As before, regard G as an H principal bundle over M. The restriction of exp to $\mathfrak{m}$ gives rise to a chart $\phi\colon \mathfrak{m} \supset U \to V \subset M$.

On the other hand, the form $\alpha \in \Omega^1(G, \mathfrak{g})$ corresponds to a form $\bar{\alpha} \in \Omega^1(M, TM)^{\mathrm{bas}}$ since $G \times_{\mathrm{Ad}} \mathfrak{m} = TM$. Because $\pi^*(\bar{\alpha}) = \alpha$ and the following diagram commutes

$$\begin{array}{ccc} & \overset{\exp}{\nearrow} & G \\ & & \downarrow \pi \\ U & \overset{\phi}{\longmapsto} & V \end{array}$$

one has $\phi^*(\bar{\alpha}) = \exp^* \alpha$. To compute $\exp^* \omega$, pick a base vector $\xi \in U$ and $\eta \in T_\xi U$ which is identified with $\frac{d}{dt}\big|_{t=0} \xi + t\eta$ and recall Duhamel's formula.

$$\frac{d}{dt}\bigg|_{t=0} \exp(\xi + t\eta) = \exp(\xi) \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\mathrm{ad}_\xi)^n(\eta).$$

This series is absolutely convergent in a neighbourhood of $(0, 0) \in \mathfrak{g} \times \mathfrak{g}$. Hence,

$$(\exp^* \omega)_\xi(\eta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\mathrm{ad}_\xi)^n(\eta).$$

Now let $J \in \mathrm{End}(\mathfrak{m})$ with $J^2 = -1$ such that $J \circ \mathrm{Ad}_H = \mathrm{Ad}_H \circ J$. Then J gives rise to an almost complex structure on M. Assume that this structure is integrable. This means that the atlas $\{((L_g)_*(U), (L_g)_* \circ \exp\}_{g \in G}$ where U is a neighbourhood of the identity has holomorphic transition functions. By picking a basis one can identify via J the vector space $\mathfrak{m}$ with $\mathbb{C}^n$. Then α is a holomorphic form if and only if $\exp^* \alpha$ is holomorphic as a map $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$. From the above expression one can read off, that $(\exp^* \omega)_\xi(\eta)$ is a convergent power series and thus holomorphic. Thus we have.

Lemma 4. *Let G/H be a homogeneous space and J be an $\mathrm{Ad}(H)$-invariant almost complex structure on $\mathfrak{m}$ which gives rise to an integrable complex structure on M. Then $\tilde{\alpha}$ is holomorphic. Furthermore, $\mathrm{pr} \circ \alpha$ is holomorphic when pr is a projection onto a complex subspace.*

5 Structure Equations

5.1 Of $\mathbb{CP}^3_{NK}$

Note that the Lie-Algebra $\mathfrak{sp}(2)$ is given by

$$\{A \in M_{2\times 2}(\mathbb{H}) \mid A + \bar{A}^T = 0\},$$

so the Maurer-Cartan form on Sp(2) is of the form

$$\omega = \begin{pmatrix} i\rho_1 + j\overline{\omega_3} & -\frac{\overline{\omega_1}}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} \\ \frac{\omega_1}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} & i\rho_2 + j\tau \end{pmatrix}.$$

Here, $\omega_1, \omega_2, \omega_3$ and τ are complex valued and ρ_1 as well as ρ_2 are real valued one forms on Sp(2). Now, with Equations (6) and (7) one can derive structure equations for $\mathbb{CP}^3_{NK}$ and $\mathbb{CP}^3_K$. In the first case, $H = S^1 \times S^3$ and

$$\phi = \begin{pmatrix} i\rho_1 & 0 \\ 0 & i\rho_2 + j\tau \end{pmatrix}, \quad \alpha = \begin{pmatrix} j\overline{\omega_3} & -\frac{\overline{\omega_1}}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} \\ \frac{\omega_1}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} & 0 \end{pmatrix}.$$

From this, one can derive the structure equations

$$\mathrm{d}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = -\begin{pmatrix} i(\rho_2 - \rho_1) & -\bar{\tau} & 0 \\ \tau & -i(\rho_1 + \rho_2) & 0 \\ 0 & 0 & 2i\rho_1 \end{pmatrix} \wedge \begin{pmatrix} \overline{\omega_2 \wedge \omega_3} \\ \overline{\omega_3 \wedge \omega_1} \\ \overline{\omega_1 \wedge \omega_2} \end{pmatrix}$$

Let furthermore,

$$\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} = \begin{pmatrix} i(\rho_2 - \rho_1) & -\bar{\tau} \\ \tau & -i(\rho_1 + \rho_2) \end{pmatrix}$$

So one obtains

$$\mathrm{d}\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} = -\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \wedge \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} + \begin{pmatrix} \omega_1 \wedge \overline{\omega_1} - \omega_3 \wedge \overline{\omega_3} & \omega_1 \wedge \overline{\omega_2} \\ \omega_2 \wedge \overline{\omega_1} & \omega_2 \wedge \overline{\omega_2} - \omega_3 \wedge \overline{\omega_3} \end{pmatrix}$$

Note that $\omega_1, \omega_2, \omega_3$ are horizontal but not equivariant forms but their span is a $S^1 \times S^3$ invariant subspace. In other words, the tensor

$$\omega_1 \otimes \overline{\omega_1} + \omega_2 \otimes \overline{\omega_2} + \omega_3 \otimes \overline{\omega_3}$$

is a basic tensor and thus corresponds to a complex valued two-tensor h on $\mathbb{CP}^3_{NK}$ which defines its Hermitian metric. This defines the nearly Kähler structure on $\mathbb{CP}^3_{NK}$. The associated two-form is

$$\sigma = i(\omega_1 \wedge \overline{\omega_1} + \omega_2 \wedge \overline{\omega_2} + \omega_3 \wedge \overline{\omega_3}).$$

Locally, one can regard ω_1, ω_2 and ω_3 as unitary $(1,0)$-forms. In contrast, one obtains $\mathbb{CP}^3_K$ by quotienting out $S^3 \times S^1$. That means, that ω_3 will not be a horizontal form anymore but κ_{21} is. Analogously, the tensor

$$\frac{1}{2}\overline{\omega_1} \otimes \omega_1 + \frac{1}{2}\omega_2 \otimes \overline{\omega_2} + \kappa_{21} \otimes \overline{\omega_2}.$$

is basic and gives rise to the Fubini-Study metric on $\mathbb{CP}^3_K$. Locally, one can regard $\frac{\overline{\omega_1}}{\sqrt{2}}, \frac{\omega_2}{\sqrt{2}}, \kappa_{21}$ as unitary $(1,0)$ forms. This is the explicit description of J_1 and J_2 on $\mathbb{CP}^3$. In the light of diagram

4.2, it is also useful to equip F with an almost complex structure. The forms $\bar{\omega_1}, \omega_2, \omega_3, \kappa_{21}$ are all basic forms of the bundle $\mathrm{Sp}(2) \to F$. Hence, they correspond to forms $\bar{\omega_1}^F, \omega_2^F, \omega_3^F, \kappa_{21}^F$ with values in complex line bundles. The Hermitian structure on F is defined by requiring that these forms are unitary $(1,0)$-forms. It turns out that this almost complex structure is integrable and π_2 is holomorphic as

$$\pi_2^*(T^{1,0}\mathbb{CP}^3_K) = \pi_2^*\mathrm{span}(\overline{\omega_1}^{\mathbb{CP}^3_K}, \omega_2^{\mathbb{CP}^3_K}, \kappa_{21}^{\mathbb{CP}^3_K}) = \mathrm{span}(\overline{\omega_1}^F, \omega_2^F, \kappa_{21}^F) \subset T^{(1,0)}F$$

5.2 Of the Flag Manifold

To reduce the number of variables involved in the structure equations it is worthwhile to view $\mathcal{F}$ as the homogeneous space G/H with $G = \mathrm{SU}(3)$ and $H = \mathbb{T}^2 \subset G$. We can decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and denote

$$\omega = \begin{pmatrix} \rho_1 & -\bar{\omega_1} & \omega_2 \\ \omega_1 & \rho_2 & -\bar{\omega_3} \\ -\bar{\omega_2} & \omega_3 & \rho_3 \end{pmatrix},$$

with $\rho_1 + \rho_2 + \rho_3 = 0$. And so α is given by

$$\alpha = \begin{pmatrix} 0 & -\bar{\omega_1} & \omega_2 \\ \omega_1 & 0 & -\bar{\omega_3} \\ -\bar{\omega_2} & \omega_3 & 0 \end{pmatrix}.$$

By picking a basis of $\mathfrak{m}$, we may write this as a vector $(\omega_1, \omega_2, \omega_3)$. A direct calculation shows that

$$p_{\mathfrak{m}}([\alpha, \alpha]) = (-2\bar{\omega_2} \wedge \bar{\omega_3}, 2\bar{\omega_1} \wedge \bar{\omega_3}, -2\bar{\omega_1} \wedge \bar{\omega_2}).$$

Furthermore,

$$[\alpha, \phi] = (\omega_1 \wedge (\rho_1 - \rho_2), \omega_2 \wedge (\rho_1 - \rho_3), \omega_3 \wedge (\rho_2 - \rho_3))$$

Hence, the structure equations are given by

$$\mathrm{d}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \rho_1 - \rho_2 & 0 & 0 \\ 0 & \rho_1 - \rho_3 & 0 \\ 0 & 0 & \rho_2 - \rho_3 \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} \bar{\omega_2} \wedge \bar{\omega_3} \\ \bar{\omega_3} \wedge \bar{\omega_1} \\ \bar{\omega_1} \wedge \bar{\omega_2} \end{pmatrix} \tag{8}$$

together with

$$\mathrm{d}\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} -\omega_1 \wedge \bar{\omega_1} + \omega_2 \wedge \bar{\omega_2} \\ \omega_1 \wedge \bar{\omega_1} - \omega_3 \wedge \bar{\omega_3} \\ -\omega_2 \wedge \bar{\omega_2} + \omega_3 \wedge \bar{\omega_3} \end{pmatrix}$$

Define

$$\sigma = \omega_1 \wedge \bar{\omega_1} + \omega_2 \wedge \bar{\omega_2} + \omega_3 \wedge \bar{\omega_3}$$

Let us calculate $\mathrm{d}\sigma$. Since $\rho_i \in i\mathbb{R}$ only the torsion part of the structure equations will contribute for the the following reason: Let τ be a complex-valued differential form with $\mathrm{d}\tau = \gamma \wedge \tau$ where τ is $i\mathbb{R}$ valued. Then

$$\mathrm{d}(\tau \wedge \bar{\tau}) = \mathrm{d}\tau \wedge \bar{\tau} - \tau \wedge \mathrm{d}\bar{\tau} = \gamma \wedge \tau \wedge \bar{\tau} - \tau \wedge (-\gamma) \wedge \bar{\tau} = 0.$$

Hence,

$$\mathrm{d}\sigma = -6(\omega_1 \wedge \omega_2 \wedge \omega_3 + \bar{\omega_1} \wedge \bar{\omega_2} \wedge \bar{\omega_3}) = -12\mathrm{Re}(\omega_1 \wedge \omega_2 \wedge \omega_3)$$

One can also check that $\mathrm{d}(\mathrm{Im}(\omega_1 \wedge \omega_2 \wedge \omega_3)) = -\sigma \wedge \sigma$. Hence, Proposition 3 implies that $(\omega_1, \omega_2, \omega_3)$ define a nearly Kähler structure on $\mathcal{F}$ which is J_2. We may also choose another basis for $\mathfrak{m}$ which amounts to replacing ω_2 by $\tau = \bar{\omega_2}\sqrt{2}$. Define now

$$\sigma' = \omega_1 \wedge \bar{\omega_1} + \tau \wedge \bar{\tau} + \omega_3 \wedge \bar{\omega_3} = \omega_1 \wedge \bar{\omega_1} - 2\omega_2 \wedge \bar{\omega_2} + \omega_3 \wedge \bar{\omega_3}.$$

And then we obtain

$$\mathrm{d}\sigma' = 0.$$

In fact, $(\omega_1, \bar{\omega_2}\sqrt{2}, \omega_3)$ is the integrable structure on $\mathcal{F}$ defined by J_1. From now on, we will refer to $(\mathcal{F}, J_2)$ as $\mathcal{F}_{NK}$ or if no confusion arises simply as $\mathcal{F}$ and to $(\mathcal{F}, J_1)$ as $\mathcal{F}_K$.

6 Contact Structures

Salamon showed that the twistor space of a quaternionic-Kähler manifold with positive scalar curvature admits a holomorphic contact structure.[15] The horizontal bundle described above will define the contact structure in these cases.

6.1 On $\mathbb{CP}^3_K$

More explicitly, the contact structure on $\mathbb{CP}^3_K$ is defined by the form κ_{12}. This is a basic form on $\Omega^1(\mathrm{Sp}(2), \mathbb{C})^{S^1 \times S^3}$ so it corresponds to a form θ on $\mathbb{CP}^3_K$ with values in a line bundle L. Note that $\mathrm{d}\theta$ is not in general a well-defined object, but $\mathrm{d}\theta \wedge \theta$ is a well defined form in $\Omega^3(\mathbb{CP}^3_K, L)$. The condition $\mathrm{d}\theta \wedge \theta \neq 0$ can be checked more conveniently on $\mathrm{Sp}(2)$. Using the structure equations,

$$\mathrm{d}\kappa_{12} \wedge \kappa_{12} = \omega_1 \wedge \overline{\omega_2} \wedge \kappa_{12} \neq 0.$$

In the affine chart $\mathbb{CP}^3_K \supset \mathbb{A}_0 = \{[1 : z_1 : z_2 : z_3]\}$, the contact form is given by

$$\kappa_{12} = \mathrm{d}z_1 - z_3\mathrm{d}z_2 + z_2\mathrm{d}z_3.$$

In fact, Bryant has found an explicit description of the integral submanifolds of the contact structure.
Let M be a connected Riemann surface and let f, g be meromorphic functions on M with g being non-constant. Define $\Phi(f, g) \colon M \to \mathbb{A}_0 \subset \mathbb{CP}^3_K$ by

$$\Phi(f, g) = [1 : f - \frac{1}{2}g\Big(\frac{\mathrm{df}}{\mathrm{dg}}\Big), g, \frac{1}{2}\Big(\frac{\mathrm{d}f}{\mathrm{d}g}\Big)].$$

Note that $\frac{\mathrm{d}f}{\mathrm{d}g}$ is a well-defined meromorphic function on M because $\mathrm{d}f$ and $\mathrm{d}g$ are both complex-linear, so $\frac{\mathrm{d}f(v)}{\mathrm{d}g(v)}$ does not depend on the choice of a non-zero $v \in TM$.

Theorem 4. [2, Theorem F] Each $\Phi(f, g)$ defines an integral submanifold of the holomorphic contact structure on $\mathbb{CP}^3_K$. Conversely, every such submanifold is of that form or lies in some $\mathbb{CP}^1 \subset \mathbb{CP}^3_{NK}$.

Theorem 5. [2, Theorem G] Let M be a compact Riemann surface then there is a holomorphic embedding $M \to \mathbb{CP}^3_K$, integral to the contact distribution.

6.2 On the Flag Manifold

As before, since $\bar{\omega_2}$ is a basic form on U(3), it corresponds to a form $\theta \in \Omega^1(\mathcal{F}_K, L)$ for some line bundle on $\mathcal{F}_K$. The kernel of θ is exactly the horizontal distribution coming from the twistor fibration. Indeed, θ defines the holomorphic contact structure on $\mathcal{F}_K$. By Lemma 4, $\bar{\omega_2}$ is holomorphic because it is the projection of α onto a complex subspace. This is because the integrable almost complex structure on $\mathcal{F}_K$ is chosen such that

$$(z_1, z_2, z_3) \mapsto \begin{pmatrix} 0 & -\bar{z_1} & -\bar{z_2} \\ z_1 & 0 & -\bar{z_3} \\ z_2 & z_3 & 0 \end{pmatrix}.$$

identifies $\mathbb{C}^3$ with $\mathfrak{m}$. Furthermore,

$$\bar{\omega_2} \wedge \mathrm{d}\bar{\omega_2} = \bar{\omega_2} \wedge \omega_1 \wedge \omega_3$$

which vanishes nowhere.

7 Pseudoholomorphic Curves

7.1 In $\mathbb{CP}^3_{NK}$

We will now investigate pseudoholomorphic curves in $\mathbb{CP}^3_{NK}$, i.e. a Riemann surface X and a nonconstant map $\gamma\colon X \to \mathbb{CP}^3_{NK}$ such that γ_* commutes with the respective almost complex structures. All bundles over $\mathbb{CP}^3_{NK}$ can be pulled back to bundles over X, we will denote them by a superscript X. The connection defined on $\mathrm{Sp}(2)$ pulls back to a connection ϕ^X on $\mathrm{Sp}(2)^X \to X$. Note that α is a basic form on $\mathrm{Sp}(2)$ and hence corresponds to a vector field on $\mathbb{CP}^3_{NK}$. According to the splitting of $\mathfrak{m}$ into $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ we obtain two forms α_1 and α_2, pulled back to $\mathrm{Sp}(2)^X$, they correspond to sections $I_1\colon X \to T^\vee X \otimes \mathcal{H}_X$ and $I_2\colon X \to T^\vee X \otimes \mathcal{V}_X$. Let us define the constant maps $f_i\colon \mathrm{Sp}(2) \to \mathfrak{m}$ given by

$$f_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}.$$

If the forms ω_i are considered as sections $\Gamma(\mathrm{Sp}(2), \mathfrak{m}^\vee \otimes \mathbb{C})$ then f_i are the dual sections to ω_i. Furthermore, we have

$$I_1 = f_1 \otimes \omega_1 + f_2 \otimes \omega_2, \quad I_2 = f_3 \otimes \omega_3.$$

When pulled back to X, the sections I_1, I_2 serve as invariants for $X \subset \mathbb{CP}^3_{NK}$. For example, when I_1 vanishes, then X lies in a twistor-fibre, so it is a submanifold of $\mathbb{CP}^1$. On the other hand, recall that the horizontal distribution defines the holomorphic contact structure on $\mathbb{CP}^3_K$. Furthermore, the almost complex structures J_1 and J_2 are equal on the horizontal distribution. So, pseudoholomorphic curves in $\mathbb{CP}^3_{NK}$ on which I_2 vanishes identically are the integral holomorphic curves of the contact structure on $\mathbb{CP}^3_K$. So assume that neither I_1 nor I_2 vanish identically. To use statements from complex analysis, we establish that these sections are in fact holomorphic.

Lemma 5. *Let $\gamma\colon X \to \mathbb{CP}^3_{NK}$ be a pseudoholomorphic curve, then $\mathrm{d}_{\phi^X}\alpha^X = 0$.*

Proof. Note that $\mathrm{d}_{\phi^X}\alpha^X = \gamma^*(\mathrm{d}_{\phi^X}\alpha^X) = -p_{\mathfrak{m}}(\frac{1}{2}[\alpha^X, \alpha^X])$. But we have explicitly calculated that this is a $(0,2)$-form on $\mathbb{CP}^3_{NK}$, i.e. it vanishes on X. □

This result has two consequences. It says that ϕ^X is torsion-free as a connection on X. Hence ϕ^X gives rise to the Levi-Civita connection on X. Furthermore, since X is a Kähler manifold it is equal to the Chern connection. Hence α_1 and α_2 are covariant constant with respect to the Chern connection and hence holomorphic. In particular they will only have isolated zeroes as they do not vanish identically. This implies that there is a holomorphic line bundle $L \subset \mathcal{H}_X$ on X such that I_1 is a non-zero section of $T^\vee X \otimes L$. In a point $x \in X$ where I_1 is non-zero, L_x is given by $\{(I_1)_x(v_x) \in \mathcal{H}_x | v_x \in T_x X\}$. With the help of L one can define a reduction of $\mathrm{Sp}(2)^X$ by

$$Q = \{p \in \mathrm{Sp}(2)^X \mid [p, f_2(p)] \in L_x\}.$$

Note that if $p \in Q$ then ph is in Q if and only if $h \in S^1 \times S^1$. To see that the is indeed bundle Q is a $S^1 \times S^1$ bundle over X one needs to prove

Lemma 6. *Let $P \to M$ be a principal G bundle and ρ be a representation of G on V with the property that $\rho(G)(v) \cap \mathrm{span}(w) \neq \emptyset$ for each two non-zero vectors v and w. Let $W \subset V$ be a one-dimensional subspace of V and H be the stabiliser subgroup of V, i.e. $H = \{g \in G \mid \rho_g(W) \subset W\}$. Furthermore let $L \subset P \times_\rho V$ be a line bundle on M, then there is a unique H-subbundle of P such that*

$$L = Q \times_{\rho|_H} W.$$

Proof. The condition $L = Q \times_H W$ requires to define

$$Q = \{p \in P \mid [p, w]_G \in L \quad \forall w \in W\}.$$

First note, that $Q \cap \pi^{-1}(\{x\}) \neq \emptyset$ for all $x \in M$. This is because L lies inside $P \times_\rho V$, so for a non-zero $l_x \in L_x$ there are $p \in P_x$ and $v \in V$ such that $[p, v] \in L_x$. Since $\rho(G)(v) \cap W \neq \emptyset$ we find $g \in G$ such that $\rho(g)(v) \in W$, so $p.g \in Q_x$. Since $\mathrm{rk}(L) = 1 = \dim(E)$ it follows that

$$Q \times_G W = L.$$

This however implies that one has

$$q.g \in Q \Leftrightarrow gW = W.$$

Hence, Q is a H-bundle and

$$Q \times_H W = L.$$

□

So, one obtains Q by applying this lemma to $G = S^1 \times S^3$, $V = \mathfrak{m}_1$ and $W = \mathrm{span} f_2$ such that $H = S^1 \times S^1$.
The metric on $\mathfrak{g}$ defines in a natural manner a connection on Q. One can restrict ϕ^X to Q and project its values orthogonally onto $\mathfrak{u}(1) \times \mathfrak{u}(1)$. We will denote this connection by ϕ_Q. In the structure equation, the projection onto $\mathfrak{u}(1) \times \mathfrak{u}(1)$ amounts to $\tau = 0$. Hence,

$$\mathrm{d}_{\phi_Q}(f_1) = \mathrm{d}f_1 + \phi_Q \wedge f_1 = \kappa_{11} f_1, \quad \mathrm{d}_{\phi_Q}(f_2) = \mathrm{d}f_2 + \phi_Q \wedge f_2 = \kappa_{22} f_2 \tag{9}$$

With respect to the $U(1) \times U(1)$-action, $\mathfrak{m}_1$ splits again into

$$\mathfrak{m}_1 = \underbrace{\left\{ \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\}}_{\mathfrak{k} = \mathrm{span}(f_1)} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & jz \\ jz & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\}}_{\mathfrak{l} = \mathrm{span}(f_2)}$$

We may identify $\mathfrak{m}_1$ with $\mathbb{C}$ where an element $(\lambda_1, \lambda_2) \in S^1 \times S^1$ acts by multiplication by $\lambda_1 \lambda_2^{-1}$. The corresponding action on $\mathfrak{m}_2$ is multiplication by $\lambda_1 \lambda_2$. Note that

$$L = Q \times_{U(1) \times U(1)} \mathfrak{l}.$$

So, ϕ_Q induces a connection on L. Similarly we can form the line bundle

$$K = Q \times_{U(1) \times U(1)} \mathfrak{k}.$$

This means, that ϕ_Q induces a connection on K as well. Let $f_2^\vee$ be the dual of f_2, i.e. $f_2 \colon \mathrm{Sp}(2)^X \to \mathfrak{l}^\vee$ and $f_2^\vee(f_2) = 1$.

Lemma 7. *The one-form* $\sigma = f_1 \otimes f_2^\vee \otimes \kappa_{12} \in \Omega^1_{\mathbb{C}}(Q, \mathfrak{k} \otimes \mathfrak{l}^\vee)$ *is a basic form and covariant constant with respect to* ϕ_Q*.*

Proof. First, observe that κ_{12} is horizontal because vertical vector fields on Q are generated by the action of $\mathrm{U}(1) \times \mathrm{U}(1)$ on Q and κ_{12} is the projection of ϕ_Q onto an orthogonal subspace of $\mathfrak{u}(1) \times \mathfrak{u}(1)$. Now let $h = (\lambda_1, \lambda_2) \in S^1 \times S^1$. Then ϕ^Q satisfies $(R_h)^*(\phi_Q) = \mathrm{Ad}_{h^{-1}}(\phi_Q)$. Note that $(R_h)^*(\kappa_{12}) = (R_h)^*(-\bar{\tau}) = \lambda_2^{-2} \kappa_{12}$. On the other hand, multiplication by $\lambda_2^{-2} = \lambda_1 \lambda_2^{-1} \overline{\lambda_1 \lambda_2}$ is how (λ_1, λ_2) acts on $\mathfrak{k} \otimes \mathfrak{l}^\vee$. Note, that from the structure equations we get

$\mathrm{d}\kappa_{12} = -(\kappa_{11} - \kappa_{22}) \wedge \kappa_{12}$. Furthermore, Equation (9) implies that $\mathrm{d}_{\phi_Q} f_2^\vee = -\kappa_{22} f_2^\vee$. So we obtain

$$\begin{aligned}\nabla(f_1 \otimes f_2^\vee \otimes \kappa_{12}) &= \nabla f_1 \otimes f_2^\vee \otimes \kappa_{12} + f_1 \otimes \nabla f_2^\vee \otimes \kappa_{12} + f_1 \otimes f_2^\vee \otimes \mathrm{d}\kappa_{12} \\ &= \kappa_{11}\sigma - \kappa_{22}\sigma - (\kappa_{11} - \kappa_{22})\sigma \\ &= 0.\end{aligned}$$

□

Theorem 6. *Pseudo-holomorphic curves in $\mathbb{CP}^3_{NK}$ on which σ vanishes are in one-to-one correspondence with integral curves of the holomorphic contact structure on $\mathbb{CP}^3_K$.*

Proof. Let $\gamma\colon X \to \mathbb{CP}^3_{NK}$ with $\sigma^X = 0$, as seen above there is a line bundle L on X which gives rise to an $S^1 \times S^1$ reduction of $\mathrm{Sp}(2)^X$ to Q. Pulling diagram 4.2 back to X yields

$$\begin{array}{ccc} & Q \subset \mathrm{Sp}(2)^X & \\ & \big\downarrow \pi^X & \\ & \pi(Q) \subset F^X & \\ \swarrow^{\pi_1^X} & & \searrow^{\pi_2^X} \\ X & & \mathbb{CP}^3_K \end{array}$$

It shall be noted that the map π_2^X is given by

$$\pi(Q) \subset F^X \subset F \times X \to F \xrightarrow{\pi_2} \mathbb{CP}^3_K.$$

Since Q is a $S^1 \times S^1$ bundle over X, the bundle $\pi(X)$ over $\mathbb{CP}^3_{NK}$ has trivial structure group. This means that the map $\pi_1\colon \pi(Q) \subset F \to X$ is invertible and holomorphic as ω_1 vanishes on Q.

$$\begin{aligned}\pi_1^*(T^{1,0}\mathbb{CP}^3_{NK}) &= \pi_1^*\mathrm{span}({\omega_1}^{\mathbb{CP}^3_{NK}}, \omega_2^{\mathbb{CP}^3_{NK}}, \omega_3^{\mathbb{CP}^3_{NK}}) = \mathrm{span}({\omega_1}^{\pi(Q)}, \omega_2^{\pi(Q)}, \omega_3^{\pi(Q)}) \\ &= \mathrm{span}(\omega_2^{\pi(Q)}, \omega_3^{\pi(Q)}) \subset T^{(1,0)}F\end{aligned}$$

So, both π_1^X and π_2^X are injective and holomorphic, hence biholomorphic. As a result, $(\pi_1^X)^{-1} \circ \pi_2^X : X \to \mathbb{CP}^3_K$ is a holomorphic curve in $\mathbb{CP}^3_K$.
On the one hand,

$$\sigma|_X = 0 \quad \Leftrightarrow \quad \kappa_{12}|_Q = 0.$$

On the other hand, κ_{12} gives rise to the contact structure on $\mathbb{CP}^3_K$. Hence, X is an integral to the contact structure if and only if $\kappa_{12}|_{\mathrm{Sp}(2)^X} = 0 \Leftrightarrow \kappa_{12}|_Q = 0$, due to the equivariance of κ_{12}. Hence, $\sigma^X = 0$ if and only if X is an integral curve of the holomorphic contact structure on $\mathbb{CP}^3_K$.
To reverse the construction consider now a holomorphic curve $\Gamma\colon Y \to \mathbb{CP}^3_K$, one can again consider I_1 as a section of $T^\vee Y \otimes \mathcal{H}$ and define $L \subset \mathcal{H}$ to be the unique line bundle over Y such that I_1 has in fact values L.

Then with the help of Lemma 6, L defines a reduction $Q \subset \mathrm{Sp}(2)^Y$ and we get the following diagram

$$\begin{array}{ccccc} & & Q \subset \mathrm{Sp}(2)^Y & & \\ & & \downarrow{\scriptstyle \pi^Y} & & \\ & & \pi(Q) \subset F^Y & & \\ & \swarrow{\scriptstyle \pi_1^Y} & & {\scriptstyle \pi_2^Y}\searrow & \\ \mathbb{CP}^3_{NK} & & & & Y \end{array} .$$

Now, $\pi_1^Y \circ (\pi_2^Y)^{-1} \colon Y \to \mathbb{CP}^3_{NK}$ defines a pseudoholomorphic curve of $\mathbb{CP}^3_{NK}$ on which σ vanishes. Since the splitting $T\mathbb{CP}^3 = \mathcal{H} \oplus \mathcal{V}$ is independent of the almost complex structure chosen on $\mathbb{CP}^3$ both constructions are inverse to each other. □

The next question which arises is whether there are obstructions to realise a compact Riemannian surface as a pseudoholomorphic curve in $\mathbb{CP}^3_{NK}$. It follows from the theorem above together with Theorem 5 that there is none when σ or I_2 vanishes.

Theorem 7. *Every compact Riemann surface can be realised as a pseudoholomorphic curve in* $\mathbb{CP}^3_{NK}$.

If however, we ask for those curves where I_1, I_2 and σ do not identically vanish, there is an obstruction involving the genus of M.

Theorem 8. *Let X be a pseudoholomorphic curve of genus g on which I_1,I_2 and σ do not vanish identically. Then*

$$8(g-1) = 2\deg([D_{I_1}]) + \deg([D_{I_2}]) + \deg([D_\sigma])$$

Proof. By the correspondence of line bundles and divisors we have that

$$[D_{I_1}] = L \otimes TX^\vee, \quad [D_{I_2}] = \mathcal{V}^X \otimes TX^\vee, \quad [D_\sigma] = K \otimes L^\vee \otimes TX^\vee.$$

Observe that the induced action on $\mathfrak{m}_1 \otimes \mathfrak{k} \otimes \mathfrak{l}$ of $S^1 \times S^1$ is trivial. Hence,

$$K \otimes L \otimes \mathcal{V}^X \cong \underline{\mathbb{C}}.$$

Combining both equations yields

$$\underline{\mathbb{C}} \cong (TX)^4 \otimes [2D_{I_1} + D_{I_2} + D_\sigma]$$

Furthermore, since X is a compact Riemann surface

$$\deg(TX) = 2(1-g).$$

Hence, taking the degree yields the statement. □

Because D_{I_1}, D_{I_2} and D_σ are all effective divisors one has

Corollary 2. *If $X = \mathbb{CP}^1$ is a pseudoholomorphic curve in $\mathbb{CP}^3_{NK}$, then X either I_1, I_2 or σ vanishes identically on X.*

7.2 In the Flag Manifold

The spaces $\mathcal{F}$ and $\mathbb{CP}^3$ share many important properties due to their twistor description. However, the preceeding analysis of pseudoholomorphic curves in $\mathbb{CP}^3$ did not exploit the twistor space but rather the homogeneous structure and the resulting structure questions. Anyways, some ideas from the previous section can be used to investigate pseudoholomorphic curves in $\mathcal{F}$ since it is the fibration which gives rise to the splitting of the tangent bundle and the definition of a holomorphic contact structure.

We have seen that $T\mathcal{F} = L_{12} \oplus L_{23} \oplus L_{13}$. The line bundle L_{ij} is the vertical component of the fibration $\pi_{ij} \colon \mathcal{F} \to \mathbb{CP}^2$. So each pseudoholomorphic curve $X \colon M \to \mathcal{F}$, where $X_*(TM) \subset L_{ij}$ is in fact a holomorphic curve in one twistor fibre $\cong \mathbb{CP}^1$.

Let us now fix the fibration π_{12} as we have derived the structure equations for this case. This means that

$$\mathcal{H} = L_{12} \oplus L_{23}, \quad \mathcal{V} = L_{13}.$$

As argued for $\mathbb{CP}^3$, we can define sections

$$I_1 \in \Gamma(X, TX^\vee \otimes \mathcal{H}^X), \quad I_2 \in \Gamma(X, TX^\vee \otimes \mathcal{V}^X).$$

They will be holomorphic as from Equation (8) one can read off that $\mathrm{d}_{\phi^X}(\alpha^X)$ is a $(0,2)$-form and thus vanishes. Also, the almost complex structures J_1 and J_2 are identical on $\mathcal{H}$. Hence, horizontal pseudoholomorphic curves in $\mathcal{F}_{NK}$ are integral submanifolds of the contact structure on $\mathcal{F}_K$. Suppose now, that $\gamma \colon X \to \mathcal{F}_{NK}$ is a pseudoholomorphic curve on which neither I_1 nor I_2 vanishes identically. Again, I_1 only has isolated zeroes and one obtains a line bundle L on X such that I_1 is a section of $TX^\vee \otimes L$. Now, applying 6 to $V = \mathfrak{m}_{12} \oplus \mathfrak{m}_{23}$ and $W = \mathfrak{m}_{23}$ gives rise to a reduction of $\mathrm{SU}(3)^X$ to Q, which is a $\mathrm{Stab}(\mathfrak{m}_{23}) = \{\mathrm{diag}(1, \lambda, \lambda^{-1}) \mid \lambda \in \mathrm{U}(1)\} \cong \mathrm{U}(1)$ bundle over X. When $\rho_1 = -\rho_2 - \rho_3$ is restricted to Q it becomes a basic form. This suggests that ρ_1 could play a similar role for the classification of pseudoholomorphic curves in $\mathcal{F}_{NK}$ as σ does for $\mathbb{CP}^3_{NK}$.

8 Outlook

It would be desirable to have a characterisation of pseudoholomorphic curves in $\mathcal{F}_{NK}$ as there is for those in $\mathbb{CP}^3_{NK}$. The first obstruction to this is that the integrals of the holomorphic contact structure are not well understood yet, i.e. there is no analogue of Theorem 4. Trying to use a similar technique to the one Bryant has used for $\mathcal{F}$ would require to find suitable charts on $\mathcal{F}$. One candidate for such charts might be the one coming from the homogeneous structure on $\mathcal{F}$, i.e. the charts used in subsection 4.5. Geometrically, the most natural atlas to consider on $\mathcal{F}$ is the one given by the Bruhat cell decomposition. However, for both choices $\bar{\omega}_2$ has a very complicated expression, in other words they do not seem to be the best choices for charts in which the PDE for integral holomorphic curves could be solved explicitly or a Cartan-Kähler analysis would seem tractable.

It is a further question whether the pseudoholomorphic curves on which ρ_1 vanishes correspond to another geometric structure on $\mathcal{F}_K$.

References

[1] Christian Bär. "Real Killing spinors and holonomy". In: *Communications in mathematical physics* 154.3 (1993), pp. 509–521.

[2] Robert L Bryant et al. "Conformal and minimal immersions of compact surfaces into the 4-sphere". In: *Journal of Differential Geometry* 17.3 (1982), pp. 455–473.

[3] Jean-Baptiste Butruille. "Homogeneous nearly Kähler manifolds". In: *Handbook of pseudo-Riemannian geometry and supersymmetry* (2010), pp. 399–423.

[4] James Eells and Simon Salamon. "Twistorial construction of harmonic maps of surfaces into four-manifolds". In: *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 12.4 (1985), pp. 589–640.

[5] Lorenzo Foscolo and Mark Haskins. "New G2 holonomy cones and exotic nearly Kähler structures on the 6-sphere and the product of a pair of 3-spheres". In: *arXiv preprint arXiv:1501.07838* (2015).

[6] Alfred Gray et al. "Riemannian manifolds with geodesic symmetries of order 3". In: *Journal of Differential Geometry* 7.3-4 (1972), pp. 343–369.

[7] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. John Wiley & Sons, 2014.

[8] NJ Hitchin. "Kählerian twistor spaces". In: *Proceedings of the London Mathematical Society* 3.1 (1981), pp. 133–150.

[9] Dave Morris. "Nearly Kahler geometry in six dimensions". In: (2014).

[10] Paul-Andi Nagy. "Nearly Kähler geometry and Riemannian foliations". In: *arXiv preprint math/0203038* (2002).

[11] Ramon Reyes Carrion. "Some special geometries defined by Lie groups." PhD thesis. University of Oxford, 1993.

[12] Jonathan Sacks and Karen Uhlenbeck. "Minimal immersions of closed Riemann surfaces". In: *Transactions of the American Mathematical Society* 271.2 (1982), pp. 639–652.

[13] Simon Salamon. "Harmonic and holomorphic maps". In: *Geometry Seminar "Luigi Bianchi" II-1984*. Springer. 1985, pp. 161–224.

[14] Simon Salamon. *Riemannian geometry and holonomy groups*. Longman Scientific and Technical, 1989.

[15] Simon M Salamon. "Differential geometry of quaternionic manifolds". In: *Annales scientifiques de l'Ecole normale supérieure*. Vol. 19. 1. Elsevier. 1986, pp. 31–55.

[16] Nolan R Wallach. "Harmonic analysis on homogeneous spaces". In: (1973).

[17] Joseph A Wolf, Alfred Gray, et al. "Homogeneous spaces defined by Lie group automorphisms. I". In: *Journal of Differential Geometry* 2.1 (1968), pp. 77–114.