Lagrangian Mean curvature flow and the Whitney sphere

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Abstract

In this notes, we are going to describe the main basic aspects regarding the mean curvature flow in codimension one as well as in higher codimension. The higher codimension case is particularly of interest for the class of Lagrangian surfaces inside a Kahler Einstein manifold since the property of being lagragian is preserved by the mean curvature flow in this setting. In the end we will discuss the evolution of a nice lagrangian surface in $\mathbb{R}^4$ namely the Whitney sphere. This surface is special by different reasons, firstly, it plays an important role in symplectic topology since it has only one double point as an immersed surface, secondly it enjoys very interesting rigidity results in the sense of differential geometry and finally it appears as a singular limit surface of a particular solution of lagrangian mean curvature flow.

1 Preliminares

Let Σ^n a manifold of dimension n and $F : \Sigma^n \to M^{n+k}$ a isometric immersion of Σ inside a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ of dimension $n+k$. The immersion induces a Riemannian metric g on Σ, we are interested on its extrinsic geometry.

Let's use $\overline{\nabla}$ to denote the Levi-Civita connection of M regarding the Riemannian metric $\langle\cdot,\cdot\rangle$, the *Levi-Civita* connection of (Σ, g) is given by the tangential part of $\overline{\nabla}$.

Definition 1.1. *The second fundamental form of the immersion is the map* $A : \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)^{\perp}$ *given by* $A(X, Y) = (\overline{\nabla}_{\overline{X}} \overline{Y})^{\perp}$. *It can be checked that this map is well defined and symmetric.*

In local coordinates, we express the metric and the second fundamental form in terms of the immersion

$$A(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = (\overline{\nabla}_{\frac{\partial F}{\partial x_i}} \frac{\partial F}{\partial x_j})^{\perp}, \quad \text{and} \quad g_{ij} = \langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle.$$

Definition 1.2. *The mean curvature vector* $\overrightarrow{H}$ *of* Σ *is the normal vector given by the trace of* A, *i.e,*

$$\overrightarrow{H} = \sum_{i=1}^{n} A(e_i, e_i),$$

where $\{e_i\}_{i=1}^n$ *is a orthonormal basis on* Σ.

Again, one should check that above definition is well defined, i.e, the vector does not depend on the choices of the orthonormal basis for $T_p\Sigma$.

Example 1.3. *Let* Σ^n *a hypersurface of* M^{n+1}, *then* $A(x, y) = \lambda(x, y)\nu$ *where* ν *is a local unit normal vector. So* $\lambda(x, y) = \langle A(x, y), \nu \rangle = \langle \overline{\nabla}_x y, \nu \rangle = \langle -\overline{\nabla}_x \nu, y \rangle$. *Therefore, we get a self-adjoint linear map* $A_\nu(x) = -\overline{\nabla}_x \nu$ *on the tangent space, also called the Second Fundamental Form. Its trace* H*,the mean curvature of* Σ, *can be computed by taking the sum of its eigenvalues, called the the principal curvatures. The mean curvature vector in this case is just* $\overrightarrow{H} = H\nu$.

Example 1.4 (Graphs)**.** *Let* $\Sigma = graph(f)$ *where* $f : \mathbb{R}^n \to \mathbb{R}$ *is a smooth function. Then* $\frac{\partial F}{\partial x_i} = (e_i, \partial_{x_i} f)$ *and hence* $\overline{\nabla}_{\frac{\partial f}{\partial x_i}} \frac{\partial f}{\partial x_j} = (0, \frac{\partial^2 f}{\partial x_i \partial x_j})$. *The unit normal vector field is*

$$\nu = \frac{(Df, -1)}{\sqrt{1 + |Df|^2}} \Longrightarrow A_{ij} = -\frac{D^2_{ij} f}{\sqrt{1 + |Df|^2}} \nu.$$

To compute the mean curvature vector we use the expressions for the metric

$$g_{ij} = \delta_{ij} + \partial_{x_i} f \partial_{x_j} f \quad \textit{and} \quad g^{ij} = \delta_{ij} - \frac{\partial_{xi} f \partial_{x_j} f}{1 + |Df|^2},$$

finally we have

$$\overrightarrow{H} = \Big(-\frac{\Delta f}{\sqrt{1+|Df|^2}} + \frac{D^2 f(Df, Df)}{\sqrt{1+|Df|^2}} \Big)\nu = -div \frac{Df}{\sqrt{1+|Df|^2}}\nu \tag{1}$$

Lemma 1.5. *If $F : (\Sigma, g) \to (M, g_0)$ is a isometric immersion then*

$$\Delta_{g,g_0} F = H.$$

Proof. Let's do the proof for the case $M = \mathbb{R}^N$, the general case are similar. The following equation describe the second fundamental form $\nabla^{\mathbb{R}^N} = \nabla^{\Sigma} + A$ and so

$$\frac{\partial^2}{\partial x_i \partial x_j} F = \Gamma^k_{i,j} \frac{\partial F}{\partial x_k} + A_{i,j}.$$

The laplacian is defined as

$$\Delta f = \mathrm{trace}(\nabla^{\Sigma})^2 f = g^{ij}(\nabla^2 f)_{i,j} = g^{ij}\Big(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k}\Big).$$

Therefore, doing this computation on each coordinate of F

$$\Delta_g F = g^{ij}\Big(\frac{\partial^2 F}{\partial x_i \partial x_j} - \Gamma^k_{ij}\Big)\frac{\partial F}{\partial x_k}\Big) = g^{ij} A_{ij} = \overrightarrow{H}.$$

□

A submanifold where $\overrightarrow{H} \equiv 0$ is called *minimal submanifold*. In particular, minimal surfaces in the Euclidean space have harmonic coordinate functions. So far it is not yet clear why the mean curvature is a very important object of Σ, the next lemma goes toward the answer of this question.

Lemma 1.6. *A smooth variation of F is smooth map $F : \Sigma^n \times (-\varepsilon, \varepsilon) \to M^{n+1}$, where $F(\cdot, 0) = F$. Let's assume the variation has compact support and let $f = \langle \frac{\partial F}{\partial t}|_{t=0}, N \rangle$ the normal velocity of the variation. Then:*

$$\frac{d}{dt} Area(\Sigma_t)|_{t=0} = -\int_{\Sigma} H \cdot f dv_g$$

2 Mean Curvature Flow

The mean curvature flow consist of a family of immersions $F_t : \Sigma\times[0,T) \to M$ satisfying the following condition:

$$\big(\frac{\partial F}{\partial t}(p,t)\big)^{\perp} = \overrightarrow{H}(F(p,t)).$$

If F_t is a solution of the mean curvature flow then $\frac{\partial F}{\partial t}(p,t) = \overrightarrow{H}(F(p,t)) + X(F(x,t))$, where X is a tangent vector to $F(\Sigma,t)$. Let $\phi : \Sigma \times [0,T) \to \Sigma$ be a 1 parameter of diffeomorphisms of Σ and composing it with our solution of the flow we obtain a new family of immersions $\varphi(p,t) = F(\phi(p,t),t)$. The evolution equation for φ_t is:

$$\frac{\partial \varphi}{\partial t}(p,t) \;=\; D(F_t)_{(\phi(p,t))} \cdot \frac{\partial \phi}{\partial t}(p,t) + \frac{\partial F}{\partial t}(\phi(p,t),t)$$

Plugging the mean curvature flow equation in above expression we obtain:

$$\frac{\partial \varphi}{\partial t}(p,t) = \overrightarrow{H}(F_t(\phi(p,t)) + X(F_t(\phi(p,t))) + D(F_t)_{(\phi(p,t))} \cdot \frac{\partial \phi}{\partial t}(p,t)$$

Now notice that the mean curvature vector of $\varphi(\cdot,t)$, witch we denote by $\overline{H}$, is just $\overrightarrow{H}(F_t(\phi(p,t)))$. Therefore,

$$\frac{\partial \varphi}{\partial t}(p,t) \;=\; \overline{H}(\varphi_t(p)) + X(F_t(\phi(p,t))) + D(F_t)_{(\phi(p,t))} \cdot \frac{\partial \phi}{\partial t}(p,t).$$

In the case where the surface is compact we can choose $\phi_t(p)$ as the solution of the following ODE:

$$\frac{\partial \phi}{\partial t}(p,t) = Y(\phi(p,t),t), \phi(p,0) = p$$

where $Y(p,t) = -D(F_t)^{-1}_{(p,t)} \cdot X(F_t(p))$ and we arrive at:

$$\frac{\partial \varphi}{\partial t}(p,t) = \overline{H}(\varphi_t(p)). \tag{2}$$

In other words, the mean curvature flow is invariant by diffeomorphism transformations. The tangential component does not change the shape of the surface along the flow and we can always change parametrization of the surface in order to get rid of this part as described above. From now on, the equation (2) will be the definition for the mean curvature flow.

The lemma (1.5) provides a interesting way to see the mean curvature flow as a type of heat equation for the immersion

$$\frac{\partial F}{\partial t}(p,t) = \Delta_{g(t)} F(p,t)$$

Indeed, this flow behaves like the heat equation in many ways. For example, while the latter is making the temperature equally distributed, the mean curvature flow is trying to make the shape of the surface nicer in terms of the curvature, which is regarded as the analogue of temperature in our context.

We see from above expression that the metric is changing with time and this makes the mean curvature flow a non-linear second order system of parabolic equations, not strictly parabolic but not strictly parabolic by the diffeomorphis invariance. Fortunately, at least in the compact case we still have existence and uniqueness result for the mean curvature flow (2) for a small time interval. The non-compact case one should make more assumptions on the surface, for example one might require a good estimates for the second fundamental form at infinity for example.

2.1 Examples

One good principle to keep in mind is that the symmetries of the surface are preserved by the flow. Spheres and cylinders have a simple evolution equation

Example 2.1. *Consider $\mathbb{S}^n(R)$ the sphere of dimension n and radius in $\mathbb{R}^{n+1}$ through the Identy map . In this case, the the normal vector field is given by $N(x) = \frac{x}{R}$ which just means that the Weingarten operator $A_N = -\frac{1}{R} Id$. The mean curvature is then given by $H = -\frac{n}{R}$ and so mean curvature vector is $\overrightarrow{H}(x) = -\frac{n}{R^2} x$. If $F(x,t) = \phi(t)x$ is a solution of the flow then*

$$\phi'(t) = -\frac{n}{\phi R^2} \Longrightarrow \phi(t) = \sqrt{1 - \frac{2n}{R^2} t}$$

From above we see that the flow is only defined on $[0, \frac{R^2}{2n})$, at the final time the sphere shrinks to a point. If we reproduce the same computation for the *cylinder* $\mathbb{S}^{n-1} \times \mathbb{R}$ *inside* $\mathbb{R}^{n+1}$ *we will see that the Identity map will evolve as* $F(x, y, t) = (\sqrt{1 - \frac{2n}{R^2}t} \cdot x, y)$. *The cylinder will shrink and collapse into the subspace* $\{0\} \times \mathbb{R}$.

The sphere are just particular case of a homothetic solutions of the mean curvature flow in the Euclidean space , i.e, solutions of the form $F(p, t) = \lambda(t)(F(p, 0)$. When $\{e_i\}$ is an orthonormal basis for F_0 $\{\frac{1}{\lambda}e_i\}$ is orthonormal for F_t and so from $B(p, t) = \lambda B(p, 0)$ we have $H_t = \frac{1}{\lambda}H$. Let's proceed to find the ODE equation for $\lambda(t)$.

$$\lambda'(t)F(p, 0)^{\perp} = H(p, t) = \frac{1}{\lambda(t)}H(p, 0) \Longrightarrow \lambda(t) = \sqrt{1 + 2\alpha t}$$

We must distinguish two cases here, $\alpha < 0$ or $\alpha > 0$. The latter is called expand solution in the sense that the flow is defined for all $t \geq 0$. The former correspond to the so called self-shrinker solution,and they play a big role in the study of singularities for the flow. In this case, the surface will shrink to a point and $T_{\max} = \frac{-1}{2\alpha}$. Therefore, a surface is a self-shrinker solution for the mean curvature flow if, and only if

$$\overrightarrow{H}(p) = -\frac{F(p)^{\perp}}{2T}. \tag{3}$$

Another example of self-shrinker surface is the Clifford Torus defined as $\Sigma = \{(z, w) \in \mathbb{C}^2; |z| = |w| = \frac{1}{\sqrt{2}}\}$. This is an example of surface with codimension 2 in $\mathbb{R}^4$. It is known that this surface is minimal as hypersurface in $\mathbb{S}^3$, as consequence we see that its mean curvature vector as a submanifold of the euclidean space is

$$H(z, w) = -(z, w) = -\frac{(z, w)^{\perp}}{2 \cdot \frac{1}{2}}.$$

The maximal time of definition of the mean curvature flow for the Clifford Torus is $T = \frac{1}{2}$.

Example 2.2 (Graphical)**.** *Let* $\Sigma = Graph(u)$*, the immersion is* $F : \mathbb{R}^n \to \mathbb{R}^{n+1}$ *given by* $F(x) = (x, u(x))$. *The property of being graph is preserved by the flow. Indeed, Consider the function* $f(x, t) = \langle N_t(x), e_{n+1}\rangle + \epsilon t$. *In the initial time this function is strictly positive(because is is a graph), so it is enough to show that stay so.*

We will see later on that $\frac{\partial N_t}{\partial t} = -\nabla H$. *Computing the laplacian of* f *using normal coordinates* $\{f_i\}_{g(t)}$*:*

$$\Delta f = f_i(f_i f = f_i\langle -\lambda_i f_i, e_{n+1}\rangle = \langle -\nabla H, e_{n+1}\rangle - |B|^2 f.$$

Assume T_1 *is the first moment where this function vanish and let* x_0 *the point of minimum for this function in the respective time. Then,*

$$\begin{aligned} 0 &\geq \frac{\partial f}{\partial t}(x_0, T_1) = \langle -\nabla H, e_{n+1}\rangle(x_0, T_1) + \epsilon \\ 0 &\geq \frac{\partial f}{\partial t}(x_0, T_1) = \Delta f(x_o, T_1) + |B|^2 f(x_0, T_1) + \epsilon \\ 0 &\geq \epsilon. \end{aligned}$$

So we can assume now that $F_t(p) = (x(p,t), u(x(p,t),t))$, *we are interested in finding the equation for* u_t.

$$\begin{aligned} \frac{\partial F}{\partial t}^{\perp} &= (\frac{\partial x}{\partial t}, D(u_t)\cdot\frac{\partial x}{\partial t} + \frac{\partial u_t}{\partial t})^{\perp} = \langle(\frac{\partial x}{\partial t}, D(u_t)\cdot\frac{\partial x}{\partial t} + \frac{\partial u_t}{\partial t}), \frac{(-Du_t, 1)}{\sqrt{1+|\nabla u|^2}}\rangle N_t \\ \frac{\partial F}{\partial t}^{\perp} &= \frac{1}{\sqrt{1+|\nabla u|^2}}\frac{\partial u}{\partial t}N_t = \overrightarrow{H} = div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})N_t \end{aligned}$$

Therefore, the evolution equation satisfied by u_t *is*

$$\frac{\partial u}{\partial t} = \sqrt{1+|\nabla u|^2}\, div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}). \tag{4}$$

2.2 Evolution equations

Evolution formulas for hypersurfaces are much simpler and easier to obtain than in higher codimension, specially when the ambient manifold is the Euclidean space. Together with maximum principles, which we shall discuss later, these formulas have been very useful in understanding how geometry evolves.

Let's begin wit the evolution of the metric. Recal that

$$g_{ij} = \langle\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\rangle \quad \text{and} \quad A_{ij} = (\frac{\partial^2 F}{\partial x_i \partial x_j})^{\perp}.$$

Now taking the derivative in time and recalling that we can commute derivatives in coordinates

$$
\begin{aligned}
\partial_t g_{ij} &= 2\langle \partial_{x_i}\partial_t F(x,t), \partial x_j F\rangle = -2\langle \partial_{x_i}\partial_{x_j} F, \overrightarrow{H}(x,t)\rangle \\
\partial_t g_{ij} &= -2\langle A_{ij}, \overrightarrow{H}\rangle \\
\partial_t g_{ij} &= = -2\langle A_{ij}N_t, H\cdot N_t\rangle = -2HA_{ij}. \quad \text{(codimension 1)}
\end{aligned}
$$

Doing similar computations we get:

$$\frac{\partial}{\partial t}N = -\nabla H.$$

The element volume can now be computed once we know the evolution of the metric. Defining $a_{ij}(x,t) = g_{ik}(x,t)\cdot g^{kj}(x,t_0)$ and using $|\cdot|$ to denote the determinant of a matrix, we have:

$$
\begin{aligned}
\frac{\partial}{\partial t}\sqrt{|a|}(x,t_0) &= \frac{1}{2\sqrt{|a|}}\partial_t|a| = \frac{1}{2\sqrt{|a|}}\text{trace}(\frac{\partial a}{\partial t}(x,t_0)) \\
\frac{\partial_t\sqrt{|g|}(x,t_0)}{\sqrt{|g|}(x,t_0)} &= \frac{1}{2}\text{trace}(-2HA_{ik}(x,t_0)\cdot g^{kj}(x,t_0)) \\
\partial_t\sqrt{|g|}(x,t_0) &= -H\sqrt{|g|}(x,t_0)(g^{ik}(x,t_0)\cdot A_{ik}(x,t_0)) \\
\frac{\partial}{\partial t}\sqrt{|g|}(x,t_0) &= -H^2\sqrt{|g|}(x,t_0)
\end{aligned}
$$

Therefore, we have found the following

$$\frac{\partial}{\partial t}dv_g = -H^2 dv_g \quad \text{and} \quad \text{Area}'(t) = -\int_\Sigma H^2 dv_g. \tag{5}$$

Lemma 2.3. *Let $F : \Sigma^n \to \mathbb{R}^{n+1}$, then:*

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4; \quad \textit{and} \quad \frac{\partial}{\partial t}H = \Delta H + H|A|^2. \tag{6}$$

2.3 Maximum Principle

Theorem 2.4. *Let $u : M\times[0,T)\to\mathbb{R}$ a smooth function solution of*

$$\frac{\partial u}{\partial t} \le \Delta_{g(t)}u + \langle X(t), \nabla u\rangle + F(u,t),$$

where $g(t)$ is a one parameter family of smooth riemannian metrics and $X(t)$ a one parameter family of smooth vector fields on a closed manifold M. $F : \mathbb{R} \times [0,T] \to \mathbb{R}$ is a smooth map, let's assume also that ϕ solves the following ODE:

$$\frac{d\phi}{dt} = F(\phi(t), t) \quad \text{and} \quad \phi(0) = \alpha.$$

If $u(x,0) \leq \alpha$, then $u(x,t) \leq \phi(t)$ for all t.

It follows from the assumptions that there exist a constant $C > 0$ such that $|F(x,t) - F(y,t)| < C|x-y|$, i.e, F is Lipchitz.

Proof. Let $f_\varepsilon = u - \varepsilon e^{2Ct} - \phi(t)$. At $t = 0$ we see $f_\varepsilon < -\varepsilon$, ie, strictly negative, supposing by contradiction that there is a moment where this function vanish. Let's call this first moment T_1 and choose x as the point of maximum for $f_\varepsilon(\cdot, T_1)$. We must have

$$\frac{\partial u}{\partial t}(x, T_1) - 2C\varepsilon e^{2CT_1} - \phi'(T_1) \geq 0.$$

Moreover, at (x, T_1) we also have $\nabla f_\varepsilon = 0$ and $\Delta f_\varepsilon \leq 0$.

$$\begin{aligned}
0 &\geq \big[\frac{\partial u}{\partial t} - \Delta u - \langle X(t), \nabla u\rangle - F(u,\cdot)\big](x, T_1) \\
&\geq \big[2C\varepsilon e^{2CT_1} + \phi'(T_1) - \Delta u - \langle X(t), \nabla u\rangle - F(u,\cdot)\big](x, T_1) \\
0 &\geq \big[2C\varepsilon e^{2CT_1} - \Delta f_\varepsilon - \langle X(t), \nabla f_\varepsilon\rangle + F(\phi,\cdot) - F(u,\cdot)\big](x, T_1) \\
0 &\geq 2C\varepsilon e^{2CT_1} - C\varepsilon e^{-2CT_1} = C\varepsilon e^{-2CT}.
\end{aligned}$$

We have found a contradiction and so this implies $f_\varepsilon < 0$ and the theorem is proved once we send ϵ to 0. □

Remark 2.5. *It is possible to prove that $|\nabla^k A|^2$ satisfies*

$$\partial_t |\nabla^k A|^2 \leq \Delta |\nabla^k A|^2 + P(|A|, \ldots, |\nabla^{k-1} A|)|\nabla^k A|^2 + Q(|A|, \ldots, |\nabla_{k-1} A|),$$

and P, Q are just polynomials. If the norm of the derivatives of the second fundamental form up to order $k-1$ are uniformly bounded then applying the maximum principle one should be able to bound $|\nabla^k A|$ as well. These bounds for the second fundamental form implies bounds on the derivatives of the immersions F_t and by Arzela-Ascoli theorem the F_t converges as $t \to T$ to a smooth map F_T, one can also prove that F_T is also a immersion. Therefore, *we get a characterization for the maximal time of definition for the mean curvature flow:*

$$T_{\max} < +\infty \Longleftrightarrow \limsup_{t\to T} \max_{\Sigma} |A| = \infty$$

By Lemma (6) we have

$$\partial_t |A|^2 \leq \Delta |A|^2 + 2|A|^4 \Longrightarrow -\frac{d}{dt}\left(\frac{1}{|A|^2_{\max}}\right) \leq 2.$$

Doing the integration in both sides on the interval $[t, T]$, and recalling that $\limsup_{s\to T} |A|^2_{\max} = \infty$, then

$$\frac{1}{|A(t)|^2_{\max}} \leq 2T - 2t$$

$$\max |A(t)|^2 \geq \frac{1}{2(T-t)}. \tag{7}$$

The above formula gives a lower bound for the blow-up rate of the curvature at the singular time. It is important to remember that the equality is attained for the case of homothetic solutions for the mean curvature flow.

Coupling the lemma (6) with the inequality $|A|^2 \geq \frac{H^2}{n}$ for hypersurfaces we get the following:

$$\frac{\partial H}{\partial t} \geq \Delta H + \frac{H^3}{n}.$$

Since $-H$ satisfies the opposite inequality then we see that the property of having non-negative mean curvature is preserved by the mean curvature flow. Indeed, just apply the maximum principle and observe that the solution of

$$h'(x) = \frac{x^3}{n}, \quad \text{and} \quad h(0) = 0$$

is the trivial one.

A hypersurface with non-negative mean curvature is called mean-convex hypersurface, this generalize the convexity property for surface.

We would like to list some more interesting properties for the mean curvature flow in codimension one, the proofs are related to the maximum principle.

Theorem 2.6 (Comparison Principle for hypersurfaces)**.** *Let $\varphi : M^n \times [0,T) \to \mathbb{R}^{n+1}$ and $\psi : N^n \times [0,T) \to \mathbb{R}^{n+1}$ mean curvature flows where M is compact. Assuming $\varphi(M,0) \cap \psi(N,0) = \emptyset$, then for all t:*

$$\varphi(M,t) \cap \psi(N,t) = \emptyset$$

Proof. By contradiction there exist a first moment os tangential intersection. At this point we write the surfaces locally as a graph over the common tangent space. We have seen that each graph will satisfy an equation of the form

$$\frac{\partial u}{\partial t} = a_{ij}(Du)(D^2)_{ij}u.$$

We claim that the difference of the graphs also satisfies a linear PDE equation

$$\begin{aligned}
\frac{\partial}{\partial t}(u-v) &= a_{ij}(Du)D^2_{ij}u - a_{ij}(Dv)D^2_{ij}v \\
&= a_{ij}(Du)\big(D^2_{ij}(u-v)\big) + a_{ij}(Du)D^2_{ij}v - a_{ij}(Dv)D^2_{ij}v \\
&= a_{ij}(Du)D^2_{ij}(u-v) + \int_0^1 \frac{d}{ds}a_{ij}(Du+s(Dv-Du))ds D^2_{ij}v \\
&= a_{ij}(Du)D^2_{ij}(u-v) \\
&\quad + \big(\int_0^1 \frac{\partial}{\partial x_k}a_{ij}(Du+s(Dv-Du))D^2_{ij}v ds\big)D_k(u-v).
\end{aligned}$$

Therefore $u-v$ satisfies an equation of type $\partial_t = a_{ij}D^2_{ij} + \beta_k D_k$. Locally the difference of the graph functions is positive and so stays positive by the parabolic maximum principle, so we have got a contradiction. □

Corollary 2.7. *Let M^n a compact hypersurface inside $B(0,R) \subset \mathbb{R}^{n+1}$ moving by mean curvature flow, then $T_{\max} \le \frac{R^2}{2n}$.*

Corollary 2.8. *If $M^n \subset \mathbb{R}^{n+1}$ is compact and embedded then it stay embedded along the Mean Curvature flow.*

2.4 Huisken's Monotonicity Formula

In this section we will prove one of the main techniques to study mean curvature flow, the monotonicity formula discovered dy G. Huisken in 1989. From now on, we are dealing with immersion of type $F : M^n \to \mathbb{R}^{n+k}$.

Let ϕ be a time dependent smooth function on $\mathbb{R}^{n+k}$. So trivially we have from equation (5)

$$\frac{d}{dt}\int_M \phi = \int_M \frac{\partial \phi}{\partial t} + \overrightarrow{H}\cdot D\phi - 2|H|^2\phi.$$

By Stoke's theorem we can also write down this in the following way, it will be useful later:

$$\frac{d}{dt}\int_M \phi = \int_M \big(\frac{d}{\partial t} + \Delta_M\big)\phi - |H|^2\phi. \tag{8}$$

An example of such ϕ is the following function known as the back heat kernel function:

$$\Phi_(x_0,t_0)(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}}e^{-\frac{|x-x_0|^2}{4(t_0-t)}},$$

the name for this function comes from the straightforward lemma

Lemma 2.9. *Let* $\Psi(x,t) = (-4\pi t)^{-\frac{k}{2}}\Phi_{(0,0)}(x,t)$*, then*

$$\big(\frac{\partial}{\partial t} + \Delta_{\mathbb{R}^{n+k}}\big)\Psi = 0.$$

In order to plug this function in (8) some computations need to be done. Let f_i and e_i local orthonormal basis for the normal bundle and tangent bundle of M respectively, then

$$\begin{aligned}
\Delta_M\Phi &= div_M(D\Phi)^\top = \sum_{i=1}^n \langle \nabla_{e_i}(D\Phi)^\top, e_i\rangle \\
\Delta_M\Phi &= \sum_{i=1}^n \langle \nabla_{e_i}(D\Phi), e_i\rangle - \langle \nabla_{e_i}(D\Phi)^\perp, e_i\rangle \\
\Delta_M\Phi &= div_M(D\Phi) - \sum_{j=1}^k \langle D\Phi, f_j\rangle \sum_{i+1}^n \langle \nabla_{e_i} f_j, e_i\rangle \\
\Delta_M\Phi &= div_M(D\Phi) + \sum_{j=1}^k \langle D\Phi, f_j\rangle\langle \overrightarrow{H}, f_j\rangle \\
\Delta_M\Phi &= div_M(D\Phi) + D\Phi\cdot\overrightarrow{H}.
\end{aligned}$$

Using this information we compute

$$
\begin{aligned}
\big(\frac{d}{dt} + \Delta_{M_t}\big)\Phi &= \frac{\partial}{\partial t}\Phi + \overrightarrow{H} \cdot D\Phi + div_{M_t}(D\Phi) + \overrightarrow{H} \cdot D\Phi \\
&= \frac{\partial}{\partial t}\Phi + div_{M_t}(D\Phi) + \frac{|D\Phi^{\perp}|^2}{\Phi} - |\overrightarrow{H} - \frac{D\Phi^{\perp}}{\Phi}|^2\Phi + |\overrightarrow{H}|^2\Phi.
\end{aligned}
$$

The following is a key property of the back heat function, the proof is based on lemma (2.9) and the formula relating the ambient and the intrinsic laplacian.

Lemma 2.10.

$$
\frac{\partial}{\partial t}\Phi + div_{M_t}(D\Phi) + \frac{|D\Phi^{\perp}|^2}{\Phi} = 0.
$$

Therefore, we get

$$
\big(\frac{d}{dt} + \Delta_{M_t}\big)\Phi = -|\overrightarrow{H} - \frac{D\Phi^{\perp}}{\Phi}|^2\Phi + |\overrightarrow{H}|^2\Phi.
$$

Now observe that,

$$
D\Phi = \Phi\big(-\frac{(x-x_0)}{2(t_0-t)}\big),
$$

and this implies

$$
\frac{D\Phi^{\perp}}{\Phi} = -\frac{(x-x_0)^{\perp}}{2(t_0-t)}.
$$

Plugging this in the formula (8) we have proved the Huisken's Monotonicity Formula:

Theorem 2.11 (Huisken)**.** *Let* $(x_0, t_0) \in \mathbb{R}^{n+k}$*, then for all* $t < t_0$

$$
\frac{d}{dt}\int_{M_t} \Phi_{x_0,t_0} dv_{g_t} = -\int_{M_t} \Phi_{x_0,t_0} \big|\overrightarrow{H} + \frac{(x-x_0)^{\perp}}{2(t_0-t)}\big| dv_{g_t}. \tag{9}
$$

Corollary 2.12. *Let* f_t *a 1-parameter family of time-dependent smooth functions on* M_t *then*

$$
\begin{aligned}
\frac{d}{dt}\int_{M_t} f_t \Phi_{x_0,t_0} dv_{g_t} &= \int_{M_t} \big(\frac{df_t}{dt} - \Delta f_t\big)\Phi_{x_0,t_0} dv_{g_t} \\
&- \int_{M_t} f_t \Phi_{x_0,t_0} \big|\overrightarrow{H} + \frac{(x-x_0)^{\perp}}{2(t_0-t)}\big| dv_{g_t}.
\end{aligned}
$$

The first observation from the theorem is that in a self-shrinker solution of the mean curvature flow the right hand side in above equation is identically zero. This also follows from the fact that the quantity on left hand side is invariant by parabolic rescaling. Precisely if we rescale time by $s = \lambda^2(t - t_0)$ and space by $x' = \lambda(x - x_0))$. Then $\widetilde{M}_s = \lambda M_{T-\frac{s}{\lambda^2}}$ also satisfies the mean curvature flow and

$$\int_{M_t} \Phi_{(x_0,t_0)} dv_{g_t}(x) = \int_{\widetilde{M}_s} \Phi_{(0,0)} dv_{g_s}(x').$$

This suggest that self-shrinker should play a role in the study of singularities. Recalling that

$$\max |A|(t) \geq \frac{1}{2\sqrt{T-t}},$$

one might ask if there exist a upper bound for $\max |A_t|$ like in above expression up to a constant, for self-shrinker surfaces we have equality on that. This motivates:

Definition 2.13. *We say a solution M_t of the mean curvature flow develops type I singularity if*

$$\limsup_{t \to T} \max_M |A|(t)\sqrt{T-t} < \infty. \tag{10}$$

Otherwise, we say it develops type II singularity.

For type I singularities we have a beautiful theorem from Huisken. Before we enounce it let's consider

$$\widetilde{M}_s^{\lambda_i} = \lambda_i M_{T-\frac{s}{\lambda^2}},$$

where λ_i is a sequence of positive numbers going to infinity. Then we have

Theorem 2.14 (Huisken)**.** *If $M_t \subset \mathbb{R}^{n+1}$ develops type I singularity at the origin then there exist a subsequence λ_{i_k} where the sequence $\widetilde{M}_s^{\lambda_{i_k}}$ converge smoothly to a solution of the mean curvature flow N_t. Moreover N_t is a self-similar shrinking solution.*

It is important to remark that there is no uniqueness of limit in above theorem.

Theorem 2.15 (Huisken, Gage-Hamilton)**.** *Let M^n a closed convex hypersurface in $\mathbb{R}^{n+1}$. Then along the flow this surface will shrink to a round point.*

In other words, if you rescale the surface before it collapse to a point you will see something asymptotic to a round sphere.

Definition 2.16. *Let $(x_0, T) \in \mathbb{R}^{n+k} \times \mathbb{R}$, we define*

$$\theta(x_0, T, t) = \int_{M_t} \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{4(T-t)}} dv_{g_t}. \tag{11}$$

The Gaussian density at (x_0, T) is defined by the following limit

$$\theta(x_0, T) = \lim_{t \to T} \theta(x_0, T, t).$$

It is clear that such limit exist because the right hand side in (11) is monotone non-increasing by monotonicity formula.

It is important to remark that if we compute density at (x_0, T_1) where $x_0 = F(p, T_1), \quad T_1 < T$ then

$$\theta(x_0, T_1) = \lim_{t \to T_1} \int_{M_t} \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{4(T-t)}} dv_{g_t} = 1.$$

Indeed, the gaussian density is invariant by parabolic rescale and because the maximum of the curvature is bounded the rescale flow will converge to flat space which has to be a plane. Now, do a change of coordinates and check that the density of plane is 1.

Theorem 2.17 (B.White)**.** *Let $F : M^n \to \mathbb{R}^N$ an embedding of a compact manifold. There exist a constant $\varepsilon > 0$ such that if (x_0, T) is a space time point satisfying $\theta(x_0, T) < 1 + \varepsilon$ then there exist a space time open neighborhood $\mathcal{U}$ of (x_0, T) where $|A|(x, t)$ is uniformly bounded.*

3 Lagrangian Submanifolds

This section is devoted to mean curvature flow for a special class of submanifolds with higher codimension, namely, lagrangian submanifolds.

A symplectic manifold is a pair (M^{2n}, ω) of a manifold with a non-degenerate closed two form. The classical example is $(\mathbb{R}^{2n}, \omega)$ where $\omega =$

$\sum_{i=1}^n dx^i \wedge dy^i$. Because ω is non-degenerate there exist a almost complex structure J, i.e, it is endomorphism on the tangent space and it satisfies $J^2 = -Id$.

Let (M^{2n}, g, J) a complex manifold with a Riemannian metric g. The metric induces a non-degenerate two form $\omega = g(\cdot, J\cdot)$ on M, when this form is closed then the complex manifold is called Kähler manifold. Therefore, any Kähler manifold is a symplectic manifold.

A good example to keep in mind is the complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric $\omega = \frac{i}{2\pi}\partial\overline{\partial}\log|z|^2$.

Definition 3.1. *A submanifold L of a symplectic manifold (M^{2n}, ω) is called Lagrangian if $\dim(L) = n$ and $\omega|_L \equiv 0$.*

The subspace $L = \{(x_1, 0, \ldots, x_n, 0) \in \mathbb{C}^n\}$ is a lagrangian plane and because the complex structure is invariant by $\mathrm{SU}(n)$ the set of lagrangian planes with fixed orientation in $\mathbb{C}^n$ is isomorphic to $\mathrm{SU}(n)/\mathrm{SO}(n)$. More examples can be found by taking the graph of symplectomorphisms. More precisely, let $f : (M_1, \omega_1) \to (M_2, \omega_2)$ a map such that $f^*\omega_2 = \omega_1$, then the graph $\{(x, f(x)); x \in M_1\}$ is a lagrangian submanifold of $(M_1 \times M_2, \omega_1 - \omega_2)$. Actually, every manifold can be seen as a lagrangian submanifold if we consider it inside its cotangent bundle. Indeed, let $\{x_i, y_i\}$ a local coordinate system for T^*M, we define ω in this coordinates by $\omega = d(y_i dx^i)$. It can be checked that that is well defined and defines a globally non-degenerate two form on T^*M. We look M as the zero section of T^*M and so it follows immediately that ω is identically zero on it.

Let's look the case of $\mathbb{C}^n$ more closely. If we use global coordinates $\{x_1, y_1, \ldots, x_n, y_n\}$ then the complex structure is given by

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i} \quad \text{and} \quad J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}.$$

The symplectic form is $\omega(\cdot,\cdot) = \langle \cdot, J\cdot \rangle$. There is also a important complex valued $n-$form $\Omega = dz^1 \wedge \cdots \wedge dz^n$, this form plays a big role in the study of lagrangian submanifolds.

Lemma 3.2.

$$\Omega|_L = e^{i\theta} vol_L.$$

Proof. Let $\{e_1, \ldots, e_n\}$ the canonical orthonormal basis for $\mathbb{R}^n$ and $\{f_1, \ldots, f_n\}$ a orthonormal basis for T_xL. There exist a $A \in GL_n(\mathbb{C})$ such that $A(e_i) = f_i$.

Now notice that $\Omega(e_1,\dots,e_n)=1$ and so $\Omega(f_1,\dots,f_n)=\det_{\mathbb{C}} A$. The determinants $\det_{\mathbb{C}} A$ and $\det_{\mathbb{R}} A$ are related by $|\det_{\mathbb{C}} A|^2=\det_{\mathbb{R}} A$, hence

$$|\Omega(f_1,\dots,f_n)|^2=|\det_{\mathbb{C}} A|^2=\det_{\mathbb{R}} A=|f_1|\cdots|f_n|\cdot|Jf_1|\cdots|Jf_n|=1.$$

We used that A, as a real matrix, sends $\{e_1,\dots,e_n,Je_1,\dots,Je_n\}$ to $\{f_1,\dots,f_n,Jf_1,\dots,Jf_n\}$ and that L is lagrangian to conclude $\{f,Jf\}$ is in fact a orthonormal basis for $\mathbb{R}^{2n}$ and so the volume generated by them is one. □

$\mathbb{C}^n$ is an example of a very important class of Kahler manifolds, namely the Calabi-Yau manifolds. These are Kahler manifolds with flat Ricci curvature. On each Calabi-Yau manifold (M,g,J) there exist a canonical holomorphic $(n,0)$ form Ω which is also parallel with respect to the riemannian metric g, i.e $\overline{\nabla}\Omega=0$. The parallel property implies that $\Omega|_L=e^{i\theta}vol_L$ in each lagrangian $L\subset M$.

Proposition 3.3. *Let (M,g,J,Ω) a Calabi-Yau manifold and L a lagrangian submanifold. Then*

$$\overrightarrow{H}=J(\nabla\theta).$$

Proof. Let $\{e_i\}_{i=1}^n$ a orthonormal basis for TL around $p\in L$, obtained by parallel transport of a orthonormal basis at T_pL and $\{f^i\}_{i=1}^n$ its dual basis. It follows that $\{g_i=-f^i\circ J\}$ is the dual basis for $\{Je_i\}$. Now notice that at $p\in L$

$$\Omega_L=e^{i\theta}\bigwedge_j(f^j+ig^j).$$

Using $\nabla_X\Omega=0$, we get

$$0=iX(\theta)\Omega+e^{i\theta}\sum_{k=1}^n(f^1+ig^1)\wedge\cdots\wedge\nabla_X(f^k+ig^k)\wedge\cdots\wedge(f^n+ig^n).$$

This implies,

$$iX(\theta)\bigwedge_j(f^j+ig^j)=-\sum_{k=1}^n(f^1+ig^1)\wedge\cdots\wedge\nabla_X(f^k+ig^k)\wedge\cdots\wedge(f^n+ig^n).$$

Let's compute $\nabla_X(f^k + ig^k)$ by applying it on $\alpha = \alpha_i \frac{e_i - Je_i}{2}$ obtained by parallel transport like before, so

$$\begin{aligned}
\nabla_X(f^k+ig^k)(\alpha) &= \sum_{i=1}^n \alpha_i X(f^k+ig^k)(\frac{e_i-Je_i}{2}) - \alpha_i(f^k+ig^k)(\nabla_X\frac{e_i-Je_i}{2}) \\
&= \sum_{i=1}^n \nabla_X(f^k+ig^k)(\frac{e_i-iJe_i}{2})(f^i+ig^i)\alpha \\
&= \sum_{i=1}^n \nabla_X(f^k+ig^k)(\frac{e_i-iJe_i}{2})(f^i+ig^i)\alpha.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
iX(\theta)\bigwedge_j(f^j+ig^j) &= -\sum_{k=1}^n \nabla_X(f^k+ig^k)(\frac{e_k-iJe_k}{2})\bigwedge_j(f^j+ig^j) \\
iX(\theta) &= -\sum_{k=1}^n \nabla_X(f^k+ig^k)(\frac{e_k-iJe_k}{2})
\end{aligned}$$

From $(f^k+ig^k)(e_k - iJe_k) = 1$ we have

$$iX(\theta) = \sum_{k=1}^n (f^k+ig^k)\nabla_X(\frac{e_k-iJe_k}{2}).$$

Now we compute the right hand side and making use of $\nabla J = 0$:

$$\begin{aligned}
2iX(\theta) &= \sum_{k=1}^n -f^k(\nabla_X(iJe_k)) + ig^k(\nabla_X e_k) \\
&= -\langle \nabla_X(iJe_k), e_k\rangle + i\langle \nabla_X e_k, Je_k\rangle \\
&= \langle iJe_k, \nabla_X e_k\rangle + i\langle \nabla_X e_k, Je_k\rangle \\
&= \langle X, e_j\rangle\langle iJe_k, \nabla_{e_j} e_k\rangle + i\langle X, e_j\rangle\langle \nabla_{e_j} e_k, Je_k\rangle \\
&= 2i\langle X, e_j\rangle\langle Je_k, \nabla_{e_j} e_k\rangle = 2i\langle X, e_j\rangle\langle Je_k, \nabla_{e_k} e_j\rangle \\
&= -2i\langle X, e_j\rangle\langle \nabla_{e_k} Je_k, e_j\rangle = -2i\langle X, e_j\rangle\langle J\nabla_{e_k} e_k, e_j\rangle \\
&= 2i\langle -J(\overrightarrow{H}), X\rangle.
\end{aligned}$$

Hence, $\langle X, \nabla\theta\rangle = \langle -J(\overrightarrow{H}), X\rangle$ which implies $\overrightarrow{H} = J(\nabla\theta)$.

□

3.1 Special Lagrangian and Calibrations

Let α a closed $k-$form on M, we say α is a *calibration* if for all $p \in M$ and any k dimensional subspace $V^k \subset T_pM$ we have $\alpha \leq vol_V$. A submanifold N of M is said to be *calibrated by* α if $\alpha_N = vol_N$. The theory of calibration was invented by Harvey and Lawson in their seminal paper *Calibrated Geometries.* The key idea behind this notion is give by the following proposition

Proposition 3.4. *If $N \subset M$ is a closed submanifold calibrated by the closed $k-$ form α then for any submanifold N' with $[N'] = [N]$ in $H_k(M, \mathbb{R})$*

$$vol(N) \leq vol(N').$$

Proof.

$$\begin{aligned} vol(N) &= \int_N dvol_N = \text{by the calibrated assumption} = \int_N \alpha \\ &= \text{by Stoke's theorem} = \int_{N'} \alpha + \int_{[N-N']} d\alpha \\ &= \int_{N'} \alpha \leq \int_{N'} dvol_{N'} = vol(N'). \end{aligned}$$

□

The proposition is true without assuming that the submanifolds are closed. A calibrated submanifold is therefore volume minimizing on its homology class and minimal.

In a Kähler manifold M with a Kähler form ω is true that $\frac{\omega^n}{n!} = vol_M$. In particular, if N^k is a complex submanifold of dimension k then it is calibrated by $\frac{\omega^k}{k!}$, so they are volume minimizing on its homology class.

Let's discuss briefly the example of calibrated lagrangians in a Calabi-Yau manifold. From lemma (3.2) we see that $Re(\Omega)$ is a calibration and the submanifolds calibrated by $Re(\Omega)$ are the lagrangian submanifolds in which the lagrangian angle is constant and $Im(\Omega) = 0$. When the lagrangian angle is constant we can always choose $Re(e^{-i\theta}\Omega)$ to get a new calibration. Therefore, any minimal lagrangian is in fact area-minimizing on its homology class, observe that this is true for minimal submanifolds in general. A *Special Lagrangian* is a lagrangian which is calibrated by $Re(\Omega)$.

Special Lagrangians are very important objects in the theory of Calabi-Yay manifolds, an interesting and hard problem is to prove the existence of minimal lagrangians in each homology or Hamiltonian Isotopy class.

3.2 Lagrangian mean curvature flow

The previous section showed that the mean curvature on a lagrangian is locally a function when the ambient is Calabi-Yau manifold and this suggests that the mean curvature flow should behave well for such submanifolds. One natural question at this point is if the lagrangian property is preserved by the mean curvature flow. The next theorem gives a positive answer for this question.

Theorem 3.5 (K. Smoczyk)**.** *Let (M, J, ω) a Kahler Einstein manifold and L a compact Lagrangian submanifold. If L_t is a solution of the mean curvature flow starting on L then L_t is Lagrangian for all t.*

Proof. The proof consists in showing that there exist a positive constant C such that

$$\frac{\partial}{\partial t}|\omega_L|^2 \leq \Delta|\omega_L|^2 + C|\omega_L|^2.$$

Since the solution of the ODE

$$\phi' = C\phi(t), \quad \text{and} \quad \phi(0) = 0$$

is just the trivial one, application of the maximum principle gives $|\omega_L|^2(t) \leq 0$ for all t, and this implies $\omega_L = 0$.

Let's prove that the above constant do exist, let start computing the evolution of ω_t

$$\frac{\partial \omega}{\partial t} = \mathfrak{L}_{\overrightarrow{H}}\omega = d\iota(\overrightarrow{H})\omega + \iota(\overrightarrow{H})d\omega = d\iota(\overrightarrow{H})\omega.$$

It was used Cartan's Formula and that ω is closed, now we evaluate the right hand side using normal coordinates in a neighborhood of a point $p \in L_t$

$$\begin{aligned} \iota(\overrightarrow{H})\omega(X) &= \omega(\overrightarrow{H}, X) = \omega(\overline{\nabla}_{e_i}e_i, X) \\ &= -(\overline{\nabla}_{e_i}\omega) + e_i\omega(e_i, X) - \omega(e_i, \overline{\nabla}_{e_i}X) \\ &= e_i\omega(e_i, X) - \omega(e_i, \nabla_{e_i}X) - \omega(e_i, A(e_i, X)) \\ &= (\nabla_{e_i}\omega)(e_i, X) - \omega(e_i, A(e_i, X)) \\ &= -d^*\omega(X) - \omega(e_i, A(e_i, X). \end{aligned}$$

It was used that ω is closed and compatible with the ambient metric, i.e Kahler . Let $\{e_i\}_{i=1}^n$ a local orthonormal frame on L_t and $\{e^i\}_{i=1}^n$ its dual basis, then by *Weitzenböck Formula*

$$-dd^*\omega = \Delta\omega + e^i \wedge \iota_{e_i}(R(e_i, e_j, \omega)).$$

Therefore,

$$\frac{\partial \omega}{\partial t} = \Delta\omega + e^i \wedge R(e_i, e_j, \omega) - d\omega(e_i, A(e_i, \cdot)).$$

Let's deal now with the last term in above right hand side, since the quantities are tensors we can assume that $[X, Y] = 0$ (the coordinate basis for example) so

$$d\omega(e_i, A(e_i, \cdot))(X, Y) = X\omega(e_i, A(e_i, Y)) - Y\omega(e_i, A(e_i, X))$$

$$\begin{aligned}
&= \omega(\overline{\nabla}_X e_i, A(e_i, Y)) - \omega(\overline{\nabla}_Y e_i, A(e_i, X)) + \omega\big(e_i, \overline{\nabla}_X\big(\overline{\nabla}_Y e_i - \nabla_Y e_i\big)\big) \\
&\quad -\omega\big(e_i, \overline{\nabla}_Y\big(\overline{\nabla}_X e_i - \nabla_X e_i\big)\big) \\
&= 2\omega(A(X, e_i), A(Y, e_i)) + \omega(\nabla_X e_i, A(e_i, Y)) - \omega(\nabla_Y e_i, A(e_i, X)) \\
&\quad +\omega\big(e_i, \overline{\nabla}_X\big(\overline{\nabla}_Y e_i - \nabla_Y e_i\big)\big) - \omega\big(e_i, \overline{\nabla}_Y\big(\overline{\nabla}_X e_i - \nabla_X e_i\big)\big) \\
&= 2\omega(A(X, e_i), A(Y, e_i)) + \omega(e_i, \overline{R}(X, Y, e_i)) + \omega(e_i, R(X, Y, e_i)) \\
&\quad +\omega(e_i, A(Y, \nabla_X e_i)) - \omega(e_i, A(X, \nabla_Y e_i)) \\
&= 2\omega(A(X, e_i), A(Y, e_i)) + \omega(e_i, \overline{R}(X, Y, e_i)) + \omega(e_i, R(X, Y, e_i)).
\end{aligned}$$

Now observe that $T_pL \cap JT_pL = \{0\}$ at least for time very close to zero, just use the fact L_0 is lagrangian and continuity argument. This implies there exist $x, y \in T_pL$ such that $Jx^\perp = A(X, e_i)$ and $Jy^\perp = A(Y, e_i)$ and so very use to see

$$\omega(x, y) = \omega(Jx, Jy) = \omega(Jx^\top, Jy^\top) + \omega(A(X, e_i), A(Y, e_i)).$$

The next term which have vectors which does not belong to the tangent space is $\omega(e_i, \overline{R}(X, Y, e_i))$. For this we use the first *Bianchi Identity* and $\nabla J = 0$ to express it as

$$-\omega(e_i, \overline{R}(e_i, X, Y) + R(e_i, Y, X)) = g(e_i, \overline{R}(e_i, X, JY)) - g(Je_i, \overline{R}(e_i, Y, JX))$$

$$= 2g(e_i, \overline{R}(e_i, X, JY) = g(e_i, \overline{R}(e_i, X, JY) + g(Je_i, \overline{R}(Je_i, X, JY).$$

Although the set $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ is not an orthonormal the above sum is indeed the trace of $\overline{R}(\cdot, X, JY, \cdot)$, which is $\overline{Ric}(X, JY)$. The reason relies in the fact that the expression is actually a $2-$form. Here is the moment to use the assumption that M is *Einstein*, and so $\omega(e_i, \overline{R}(X, Y, e_i)) = \lambda\omega(X, Y)$.

We have proved

$$\frac{\partial \omega}{\partial t} = \Delta\omega + \omega(C, C),$$

where we C is a tensor which depends on the curvature tensor of the Lagrangian, the curvature tensor on the manifold and also on the second fundamental form of the lagrangian.

$$\frac{\partial}{dt}|\omega_L|^2 = \frac{\partial g}{\partial t}(\omega, \omega) + 2g(\frac{\partial}{\partial t}\omega, \omega).$$

Using $\partial_t g_{ij} = -2\langle A_{ij}, \overrightarrow{H}\rangle$ and $g(\Delta\omega, \omega) = \Delta g(\omega, \omega) - 2|\nabla\omega|^2$ we get

$$\frac{\partial}{\partial t}|\omega_L|^2 = \Delta\omega - 2|\nabla\omega|^2 + C_{ij}\omega_{im}\omega_{jn} \leq \Delta|\omega_L|^2 + C|\omega_L|^2$$

□

From now on we will be working on a Calabi-Yau manifold where the above theorem apply naturally and the mean curvature flow on lagrangian submanifolds will be called Lagrangian mean curvature flow.

It is important to point out that Smoczyk's theorem was a starting point for the study of Lagrangian mean curvature flow and gave people interested in Calabi-Yau manifold the hope to use this flow to prove existence of special lagrangians as described before.

Recall $J(\overrightarrow{H})$ is a tangent vector field on L and using the metric g we can take its dual form $\sigma = g(J\overrightarrow{H}, \cdot) = \omega(\overrightarrow{H}, \cdot)$. From (3.3) we have $\sigma = d\theta$. It follows that σ is closed $1-$form, and so it represent a cohomology class in $H^1_{dR}(L)$. When this class is trivial we say L has *zero maslov class*, and the lagrangian angle θ can be lift to a single valued function on L.

Lemma 3.6. *For a zero maslov class Lagrangian L we have*

$$\frac{\partial\theta}{\partial t} = \Delta\theta \quad and \quad \frac{\partial\theta^2}{\partial t} = \Delta\theta^2 - 2|\nabla\theta|^2.$$

Proof. Let's get started by computing $\frac{\partial\Omega}{\partial t}$.

$$\begin{aligned}\frac{\partial\Omega}{\partial t} = \mathcal{L}_{\overrightarrow{H}}\Omega &= \quad \text{by Cartan's Magic Formula} \quad = d(\iota_{\overrightarrow{H}}\Omega) = d(i\iota_{\nabla\theta}\Omega) \\ &= d(e^{i\theta}i\iota_{\nabla\theta}vol_L) = d(ie^{i\theta} * d\theta) = -e^{i\theta}d\theta \wedge *d\theta + ie^{i\theta}d * d\theta.\end{aligned}$$

In other hand, we have

$$\frac{\partial \Omega}{\partial t} = ie^{i\theta}\frac{\partial \theta}{\partial t}vol_L + e^{i\theta}\frac{\partial}{\partial t}vol_L.$$

Comparing the two identities we get

$$ie^{i\theta}\frac{\partial \theta}{\partial t} = *(ie^{i\theta}\frac{\partial \theta}{\partial t}vol_L) = *(ie^{i\theta}d * d\theta) = -ie^{i\theta}\frac{\partial \theta}{\partial t}(- * d * d\theta) = -ie^{i\theta}d^* d\theta.$$

Because $\Delta = -d^* d$, the first identity is proved. For the second one just use that $\Delta f^2 = 2f\Delta f + 2|\nabla f|^2$ for any smooth function. □

Theorem 3.7. *In $\mathbb{C}^n$ the only zero-maslov class Lagrangians which are self-shrink are the Lagrangian planes.*

Proof. A self-shrinker is given by the equation $\overrightarrow{H} = -\frac{(F-x_0)^\perp}{2T}$. We notice that any lagrangian plane satisfies that trivially, since $F^\perp = \overrightarrow{H} = 0$. That equation implies that the submanifold evolves by $L_t - x_0 = \lambda_t(L_0 - x_0)$, where $\lambda(t) = \sqrt{\frac{T-t}{T}}$. Now we define

$$\rho(t) = \int_{L_t} \theta_t^2 \Phi_{x_0,T} dvol_{L_t}.$$

The first observation is that this function is constant in time by the scale invariance. Indeed,

$$\rho(t) = \int_L \theta^2 \frac{e^{-\frac{|F_0 - x_0|^2}{4T}}}{(4\pi(T-t))^{\frac{n}{2}}}\frac{(T-t)^{\frac{n}{2}}}{T^{\frac{n}{2}}}dvol_L = \int_L \theta^2 \frac{e^{-\frac{|F_0 - x_0|^2}{4T}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{T^{\frac{n}{2}}}dvol_L.$$

So applying the corollary of the Monotonicity Formula and lemma (3.6) we have:

$$0 = \frac{d\rho}{dt} = \int_{L_t} -2|\nabla\theta|^2 \Phi_{x_0,T} - \int_{L_t} \theta_t \Phi_{x_0,T}\Big|\overrightarrow{H} + \frac{(F_t - x_0)^\perp}{2(T-t)}\Big|.$$

The second term in the right hand side is zero because the lagrangian is self-shrinker and so we get $\nabla\theta = 0$ which implies $\overrightarrow{H} = 0$ and $(F - x_0)^\perp = 0$. Therefore, L is a lagrangian plane. □

A trivial consequence of above theorem is that there exist no compact zero-maslov class self-shrinker lagrangian submanifold in $\mathbb{C}^n$. In particular, a zero-maslov class lagrangian in $\mathbb{C}^n$ does not develop type I singularity.

Singularities are a quite common phenomenon for Lagrangian submanifolds, and if one wants to understand them we need to look first at lagrangian in $\mathbb{C}^n$. The idea is that to deal with singularity is through a rescale process and so after some kind of limit we will end up with a lagrangian in the Euclidean space.

The zero-maslov class is a natural class to work with once all special lagrangian are zero maslov class. The next theorem help us to have a better understanding of singularities in this setting,in fact it shows that the tangent flow is a union of special lagrangian cones.

Theorem 3.8 (Neves)**.** *Let L_0 a zero maslov class lagrangian with bounded lagrangian angle in $\mathbb{C}^n$. Let $(L^i_s)_{s\leq 0}$ the rescaled flow at (x_0, T), then there exist a finite set $\{\overline{\theta_1}, \ldots, \overline{\theta_N}\}$ and integral Special Lagrangian Cones*

$$\{L_1, \ldots, L_N\}$$

such that, after passing to a subsequence, we have that for every smooth compacted support function ϕ, for every $f \in C^2(\mathbb{R})$ and every $s < 0$

$$\lim_{i\to+\infty} \int_{L^i_s} f(\theta_{i,s})\phi d\mathcal{H}^n = \sum_{j=1}^{N} m_j f(\overline{\theta_j})\mu_j(\phi), \tag{12}$$

where m_j and μ_j denote the multiplicity and the Radon measure of the support of L_j respectively.

4 The Lagrangian Whitney sphere

The previous section some examples of lagrangian submanifolds of $\mathbb{C}^n$ were presented. Most of the examples were non-compact submanifolds, the Clifford torus is an example of genus 1 lagrangian submanifold in $\mathbb{C}^2$. Actually, it is a surprising fact that there exist no closed orientable surface of genus g embedded as lagrangian in $\mathbb{C}^2$ except for torus.

This means that there exist obstructions for the existence of lagrangian submanifolds with a particular topology. Gromov has produced beautiful theorems giving obstructions for the existence of embedded lagrangian, the next theorem is just a particular case of his work

Theorem 4.1 (Gromov)**.** *Let L a compact embedded Lagrangian in $\mathbb{C}^n$. Then there exist at least one holomorphic disk $u : \mathbb{D}^2 \to \mathbb{C}^n$ whose boundary, $\partial u := u(\partial\mathbb{D})$, is contained in L.*

Corollary 4.2. *There exist no embedded lagrangian sphere in $\mathbb{C}^n$.*

Indeed, recall that $\omega = d\lambda$ where $\lambda = \sum_{i=1}^{n} x_i dy_i - y_i dx_i$. Because u is holomorphic then $\omega(u) := \int_{\mathbb{D}} u^*\omega$ is a positive number called the symplectic area. By Stoke's theorem, $\omega(u) = \int_{\partial u} \lambda_L = \int_{\gamma} \lambda_L$ for every loop γ homotopic to ∂u, this follows just by the Lagrangian property and Stoke's theorem again. This finish the proof of the corollary because the sphere is simply connected.

One of the main problems in symplectic topology is to understand the relationship between invariants like the intersection number of lagrangian and topology itself, and many developments in the theory came up guided by questions on this regard. This makes the study of immersed lagrangian submanifolds very interesting.

There are a great amount of immersed spheres in $\mathbb{C}^n$, but between them the Whitney sphere stands out as the immersed sphere with nice geometric and topologic properties as we will se below.

Definition 4.3. *Consider the unit sphere $\mathbb{R}^{n+1}$*

$$\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}; x_1^2 + \dots + x_{n+1}^2 = 1\}$$

and the map

$$F(x_1, \dots, x_{n+1}) = \frac{r}{1 + x_{n+1}^2}(x_1, x_1 x_{n+1}, \dots, x_n, x_n x_{n+1}).$$

F is called the Whitney Sphere of radius r.

Let's show that F is an immersion in the case $n = 2$, the other cases are the same with more notation. As a map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{2n}$ the jacobian at (x, y, z) is:

$$\begin{pmatrix} 1 & 0 & x(\frac{1}{1+z^2})_z \\ z & 0 & x(\frac{z}{1+z^2})_z \\ 0 & 1 & y(\frac{1}{1+z^2})_z \\ 0 & z & y(\frac{z}{1+z^2})_z \end{pmatrix}$$

It is clear that the rank is at least 2, now if $DF_{(x,y,z)}(a,b,c) = 0$ then

$$a + cx(\frac{1}{1+z^2})_z = 0 = az + cx(\frac{z}{1+z^2})_z = az + cx\frac{1}{1+z^2} + cxz(\frac{1}{1+z^2})_z$$

and so $cx\frac{1}{1+z^2} = 0 = cy\frac{1}{1+z^2}$. If $x = y = 0$ then the rank is two and the kernel is generated by $\langle(0,0,1)\rangle$ which is orthogonal to $T_{(0,0,\pm 1)}\mathbb{S}^n$. Otherwise the rank is three. Therefore, F is in fact a immersion. It follows immediately from the definition of F that $F(x) = F(y)$ if, and only if, $x = y$ or $x = -y = (0,0,1)$. So the Whitney sphere fails to be an embedding only at the poles where it has a double point.

Lemma 4.4. *$F^*\omega = 0$, i.e F is an lagrangian immersion.*

Proof. The proof is just computation, firs check that in the first complex coordinate we have: $x_1 = \frac{u_1}{1+v^2}$ and $y_1 = \frac{u_1 v}{1+v^2}$. So

$$dx_1 = \frac{1}{1+v^2}du_1 + u_1(\frac{1}{1+v^2})_v dv$$

$$dy_1 = \frac{v}{1+v^2}du_1 + u_1(\frac{v}{1+v^2})_v dv$$

$$dx_1 \wedge dy_1 = \frac{u_1}{(1+v^2)^2}du_1 \wedge dv$$

Doing the same for each complex coordinate we get

$$F^*\omega = \frac{1}{(1+v^2)^2}(u_1 du_1 + \cdots + u_n du_n) \wedge dv.$$

Because $u_1^2 + \cdots + u_n^2 + v^2 = 1$ we have $u_1 du_1 + \cdots + u_n du_n = -vdv$. □

The Whitney sphere is nice topologically since there is only one point of self-intersection. Under the differential geometry point of view they share more interesting properties, for example its second fundamental form is given by

$$B(x,y) = \frac{1}{n+2}\langle x,y\rangle\overrightarrow{H} + \langle Jx,\overrightarrow{H}\rangle Jy + \langle Jy,\overrightarrow{H}\rangle Jx.$$

In particular we get that the scalar curvature R of the Whitney sphere is

$$R = \frac{n-1}{n+2}|\overrightarrow{H}|^2.$$

It was proved by A. Ros and F.Urbano that the above expressions are sufficient to characterize the Whitney sphere and Lagrangian planes as the only lagrangian submanifolds enjoying any of those properties.

When $n = 2$ the scalar curvature is just twice the gaussian curvature and so by the Gauss-Bonnet theorem we have for the Whitney sphere Σ

$$\mathcal{W}(\Sigma) := \int_\Sigma \frac{1}{4}|\overrightarrow{H}|^2 = 8\pi.$$

The functional $\mathcal{W}$ is called the *Wilmore Functional* and critical points for this functional are called *Wilmore surfaces*. It is known that the Wilmore energy of any closed surface is greater than 4π and for each self-intersection this energy increases by 4π. Therefore, the minimum energy for immersed surfaces is 8π and this means that the Whitney sphere is a critical point for the Wilmore functional and moreover it minimizes the energy on its homotopic class. Moreover, it is also proved that the Whitney sphere is the only lagrangian sphere with $\mathcal{W}(\Sigma) = 8\pi$.

Remark 4.5. *It is constructed in [7] a non-compact zero-maslov class lagrangian L in $\mathbb{C}^2$ with bounded lagrangian angle and in the same Hamiltonian isotopy class of a lagrangian plane that nevertheless develops a singularity at the origin in finite time. At the singular time the limit surface pictures like a union of a smooth lagrangian (diffeomorphic to a lagrangian plane) and a immersed sphere, we regard the latter as a Whitney sphere.*

Consider $X(u, v) = (\sin(u)\cos(\alpha), \sin(u)\sin(\alpha), \cos(u))$, $u \in (0, \pi)$, $\alpha \in (0, 2\pi)$, spherical coordinates on $\mathbb{S}^2$. In this coordinates the Whitney sphere is expressed as:

$$\begin{aligned} F(u,\alpha) = \frac{1}{1+\cos^2(u)}\big(\sin(u)\cos(\alpha) \quad &, \quad \sin(u)\cos(u)\cos(\alpha), \\ \sin(u)\sin(\alpha) \quad &, \quad \sin(u)\cos(u)\sin(\alpha)\big) \end{aligned}$$

$$F(u, \alpha) = (\gamma(u)\cos(\alpha), \gamma(u)\sin(\alpha)).$$

Where $\gamma : (0, \pi) \to \mathbb{R}^2$ given by

$$\gamma(u) = \big(\frac{\sin(u)}{1 + \cos^2(u)}, \frac{\sin(u)\cos(u)}{1 + \cos^2(u)}\big).$$

The curve is invariant by the antipodal map in $\mathbb{R}^2$ and so the curve itself is can be defined from $[0, 2\pi]$, it is an immersion of the unit circle,

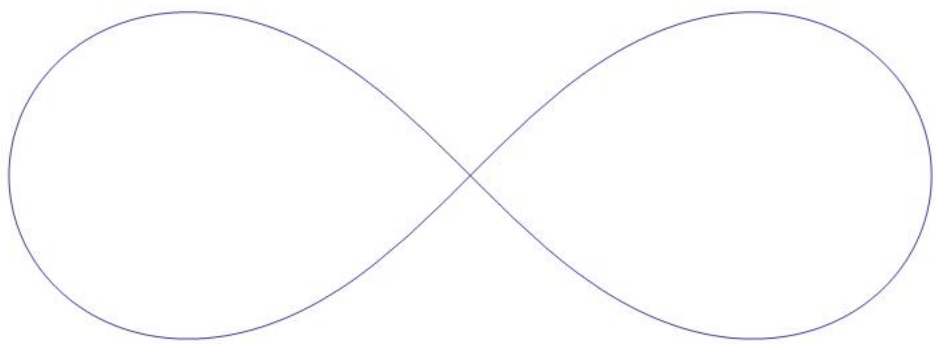

Figure 1: Whitney sphere

We would like to compute the evolution of the Whitney sphere along the Lagrangian Mean Curvature flow or at least understand its behaviour during the flow and what happens at the singular time. By symmetries we expect to reduce the mean curvature flow equation on the surface to a evolution equation for γ.

4.1 Equivariant Flow

Let γ a regular smooth curve in $\mathbb{R}^2$ then

$$L = \{(\gamma \cos(\alpha), \gamma \sin(\alpha)), \alpha \in \mathbb{R}/2\pi\mathbb{Z}\}$$

is a Lagrangian submanifold of $\mathbb{C}^2$. After choosing a parametrization of γ we have

$$\Omega_L = \frac{\gamma}{|\gamma|} \cdot \frac{\gamma'}{|\gamma'|} vol_L. \tag{13}$$

Let L_t a solution of the mean curvature flow with starting condition L. The rotationally symmetries of L are preserved by the flow and so L_t must be given by

$$L_t = \{\gamma_t \cos(\alpha), \gamma_t \sin(\alpha)), \alpha \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

The evolution equation for γ_t is given in the following lemma:

Lemma 4.6.

$$\frac{dz}{dt} = \overrightarrow{k} - \frac{z^\perp}{|z|^2}.$$

Proof. First notice that $\nabla\theta_t = \frac{\theta_t'}{|\gamma_t'|^2}\partial u$ and $\partial u = (\gamma_t'\cos(\alpha), \gamma_t'\sin(\alpha))$, where $\gamma_t(u)$ is a parametrization of γ_t. So by lemma (3.3) we get

$$\overrightarrow{H_t} = i\frac{\theta_t'}{|\gamma_t'|^2}(\gamma_t'\cos(\alpha), \gamma_t'\sin(\alpha)).$$

This implies $\frac{d\gamma_t}{dt} = \frac{i\theta_t'}{|\gamma_t'|^2}\gamma_t'$, now we need to expand the right hand side of this equation using (13)

$$i\theta_t'\frac{\gamma_t}{|\gamma_t|}\frac{\gamma_t'}{|\gamma_t'|} = \big(\frac{\gamma_t}{|\gamma_t|}\big)'\frac{\gamma_t'}{|\gamma_t'|} + \frac{\gamma_t'}{|\gamma_t'|}\big(\frac{\gamma_t}{|\gamma_t|}\big)'.$$

Recall the definition of the curvature vector of a curve $\gamma(s)$ in $\mathbb{R}^2$,

$$\overrightarrow{k}(s) = \frac{1}{|\gamma'(s)|}\frac{d}{ds}\frac{\gamma'(s)}{|\gamma'(s)|}.$$

So we get

$$i\theta_t'\frac{\gamma_t}{|\gamma_t|}\frac{\gamma_t'}{|\gamma_t'|} = |\gamma_t'|\overrightarrow{k}\frac{\gamma_t}{|\gamma_t|} + \frac{\gamma_t'}{|\gamma_t'|}\big(\frac{\gamma_t}{|\gamma_t|}\big)'$$

Putting the curvature term on the left hand side and dividing both sides by $|\gamma_t'|$

$$\begin{aligned}\big(i\theta_t'\frac{\gamma_t'}{|\gamma_t'|^2} - \overrightarrow{k}\big)\frac{\gamma_t}{|\gamma_t|} &= \frac{\gamma_t'}{|\gamma_t'|^2}\big(\frac{\gamma_t}{|\gamma_t|}\big)' \\ &= \frac{\gamma_t'}{|\gamma_t'|^2}\frac{\gamma_t'}{|\gamma_t|} + \frac{\gamma_t'}{|\gamma_t'|^2}\gamma_t\big(- \frac{1}{|\gamma_t|^2}\frac{\langle\gamma_t', \gamma_t\rangle}{|\gamma_t|}\big)\end{aligned}$$

Dividing both sides by $\frac{\gamma_t}{|\gamma_t|}$

$$\begin{aligned}
i\theta'_t \frac{\gamma'_t}{|\gamma'_t|^2} - \overrightarrow{k} &= \frac{\gamma'_t}{|\gamma'_t|^2}\frac{\gamma'_t}{|\gamma_t|}\frac{|\gamma_t|}{\gamma_t} - \frac{\gamma'_t}{|\gamma'_t|^2}\frac{\langle\gamma'_t,\gamma_t\rangle}{|\gamma_t|^2} \\
&= \frac{\gamma'_t}{|\gamma'_t|^2}\frac{\gamma'_t}{|\gamma_t|^2}\overline{\gamma_t} - \frac{\gamma'_t}{|\gamma'_t|^2}\frac{\langle\gamma'_t,\gamma_t\rangle}{|\gamma_t|^2} \\
&= \frac{\gamma'_t}{|\gamma'_t|}\frac{\gamma'_t}{|\gamma'_t|}\frac{\overline{\gamma_t}}{|\gamma_t|^2} - \frac{1}{|\gamma_t|^2}\langle\gamma_t, \frac{\gamma'_t}{|\gamma'_t|}\rangle\frac{\gamma'_t}{|\gamma'_t|} \\
&= \frac{\gamma'_t}{|\gamma'_t|^2|\gamma_t|^2}\left(\frac{d}{dt}|\gamma_t|^2 - \gamma_t\overline{\gamma'_t}\right) - \frac{1}{|\gamma_t|^2}\langle\gamma_t, \frac{\gamma'_t}{|\gamma'_t|}\rangle\frac{\gamma'_t}{|\gamma'_t|} \\
&= -\frac{\gamma_t}{|\gamma_t|^2} + 2\frac{1}{|\gamma_t|^2}\langle\gamma_t, \frac{\gamma'_t}{|\gamma'_t|}\rangle\frac{\gamma'_t}{|\gamma'_t|} - \frac{1}{|\gamma_t|^2}\langle\gamma_t, \frac{\gamma'_t}{|\gamma'_t|}\rangle\frac{\gamma'_t}{|\gamma'_t|} \\
&= -\frac{\gamma_t}{|\gamma_t|^2} + \frac{1}{|\gamma_t|^2}\langle\gamma_t, \frac{\gamma'_t}{|\gamma'_t|}\rangle\frac{\gamma'_t}{|\gamma'_t|} = -\frac{\gamma_t^\perp}{|\gamma_t|^2}.
\end{aligned}$$

This finish the proof because as pointed out in the beginning of the proof $\frac{d\gamma_t}{dt} = \frac{i\theta'_t}{|\gamma'_t|^2}\gamma'_t$. □

Although the term $\frac{z^\perp}{|z|^2}$ is not well defined at the origin the quantity has its meaning even when a curve goes through the origin as we can see below.

Lemma 4.7. *Let $\gamma : [-a, a] \to \mathbb{R}^2$ a smooth regular curve such that $\gamma(0) = 0$. Then*

$$\lim_{s\to 0} \frac{\gamma^\perp}{|\gamma|^2}(s) = \frac{1}{2}\overrightarrow{k}(0).$$

Proof. Let's write the right hand side as

$$\frac{\gamma^\perp}{|\gamma|^2}(s) = \frac{1}{|\gamma|^2}\langle\gamma, i\frac{\gamma'}{|\gamma'|}\rangle i\frac{\gamma'}{|\gamma'|} = \frac{1}{|\gamma|^2}\langle\gamma, i\frac{\gamma'}{|\gamma'|} - i\frac{\gamma'(0)}{|\gamma'(0)|}\rangle i\frac{\gamma'}{|\gamma'|} + \frac{1}{|\gamma|^2}\langle\gamma, i\frac{\gamma'(0)}{|\gamma'|(0)}\rangle i\frac{\gamma'}{|\gamma'|}.$$

Now we apply the fundamental theorem of calculus on the second term in the right hand side

$$\frac{\gamma^\perp}{|\gamma|^2}(s) = \frac{1}{|\gamma|^2}\langle\gamma, i\frac{\gamma'}{|\gamma'|} - i\frac{\gamma'(0)}{|\gamma'(0)|}\rangle i\frac{\gamma'}{|\gamma'|} + \frac{1}{|\gamma|^2}\langle\gamma, i\frac{\gamma'}{|\gamma'|}\rangle i\frac{\gamma'}{|\gamma'|} - \frac{1}{|\gamma|^2}\langle\gamma, \int_0^s i\frac{\gamma'(t)}{|\gamma'|(t)}\rangle i\frac{\gamma'}{|\gamma'|}$$

This implies

$$2\frac{\gamma^\perp}{|\gamma|^2}(s) = \frac{1}{|\gamma|^2}\langle\gamma, i\frac{\gamma'}{|\gamma'|} - i\frac{\gamma'(0)}{|\gamma'(0)|}\rangle i\frac{\gamma'}{|\gamma'|} - \frac{1}{|\gamma|^2}\langle\gamma, \int_0^s i\frac{\gamma'(t)}{|\gamma'|(t)}\rangle i\frac{\gamma'}{|\gamma'|}.$$

The proof follows by taking the limit and using

$$\lim_{s\to 0}\frac{1}{s}\Big(\frac{\gamma'}{|\gamma'|}-\frac{\gamma'(0)}{|\gamma'(0)|}\Big)=\overrightarrow{k}(0)|\gamma'(0)| \quad\text{and}\quad \lim_{s\to 0}\frac{\gamma(s)}{s}=\gamma'(0).$$

□

Proposition 4.8. *Let $\gamma_{i,t}:[-a,a]\to\mathbb{R}^2$ $i=1,2$ and $0\le t\le T$ smooth regular curves satisfying*

1. *$\gamma_{i,t}(-s)=-\gamma_{i,t}(s)$ for all $0\le t\le T$ and for every $s\in[-a,a]$.*
2. *The curves γ_t solve the equation*

$$\frac{d\gamma}{dt}=\overrightarrow{k}-\frac{z^{\perp}}{|z|^2}.$$

3. *$\gamma_{1,0}\cap\gamma_{2,0}=\{0\}$ (non-tangential intersection) and $\partial\gamma_{1,t}\cap\gamma_{2,t}=\partial\gamma_{2,t}\cap\gamma_{1,t}=\emptyset$ for all t.*

Then for all $0\le t\le T$ we have $\gamma_{1,t}\cap\gamma_{2,t}=\{0\}$.

Proof. If outside a small ball around the origin the hypothesis on the lemma hold true then the proof follows by the maximum principle by just choosing the first time of tangential intersection. The problem is when this tangential intersection can not be attained, and so that is why we can restrict ourselves to what happen at the origin.

There is a $\delta>0$ such that $\gamma_{i,t}$ is a graph on $[-\delta,\delta]$. So we can write $\gamma_{i,t}(s)=(s,f_{i,t}(s)$, now we define $\alpha_{i,t}(s)=\frac{f_{i,t}(s)}{s}$. Notice that this function is smooth by the hypothesis in the lemma, finally we consider the function $u_t(s)=\alpha_{1,t}-\alpha_{2,t}$. It also satisfies $u_0>0$ by item 3 in the lemma and $u_t(s)=u_t(-s)$, let's assume that u_t is not always positive and so there exist the first moment T_1 where this function has a zero. Recall that for $\gamma(x)=(x,f(x))$ we have

$$\gamma'=(1,f'),\quad \nu=\frac{(f',-1)}{\sqrt{1+(f')^2}}\quad\text{and}\quad \overrightarrow{k}=-\frac{f''}{(1+(f')^2)^{\frac{3}{2}}}\nu.$$

Besides,

$$\frac{z^{\perp}}{|z|^2}=\frac{sf'-f}{s^2+f^2}\frac{1}{\sqrt{1+(f')^2}}\nu.$$

Therefore, the equation $\frac{d\gamma}{dt}^{\perp} = \overrightarrow{k} - \frac{z^{\perp}}{|z|^2}$ implies

$$\frac{df}{dt} = \frac{f''}{1+(f')^2} + \big(\arctan\alpha\big)'.$$

A bit more of computations provides

$$\frac{d\alpha_{i,t}}{dt} = \frac{\alpha''_{i,t}}{1+(s\alpha'_{i,t}+\alpha_{i,t})^2} + \frac{\alpha'_{i,t}}{s}\frac{2}{1+(s\alpha'_{i,t}+\alpha_{i,t})^2} + \frac{\alpha'_{i,t}}{s}\frac{1}{1+\alpha_{i,t}^2}.$$

A good remark is that the $\frac{\alpha_{i,t}}{s}$ is also smooth. Now we proceed to find the equation for $\frac{du_t}{dt}$. From a standard argument (see the proof of (2.6)) we have

$$\frac{du_t}{dt} = C_1^2 u''_t + C_2 u'_t + C_3 u_t + C_4^2 \frac{u'_t}{s},$$

where each C_k is a smooth and bounded functions.

Now we are ready to prove the lemma, the proof is just almost the same argument we have to prove the scalar maximum principle. Suppose T_1 is the first time where u_t has a zero at s_0 , consider the function $v_t = u_t e^{-Ct} + \varepsilon(t - T_1)$ where C is very large and ε is very small. So at (s_0, T_1)

$$0 \geq \frac{dv_t}{dt}(s_0,T_1) = \frac{du_t}{dt}(s_0,T_1)e^{-CT_1} + \varepsilon$$

$$0 \geq \varepsilon + C_4^2 \frac{u'_t(s_0)}{s_0} e^{-CT_1}.$$

We have used that at s_0 $u''_t \leq 0$ and $u'_t = 0$ because it is a minimum point. If $s_0 \neq 0$ then the second term in the right hand side is zero and we get a contradiction. If $s_0 = 0$ then that term is just $u''_t(0)e^{-CT_1}$ which is non-negative and we get a contradiction again.

□

One consequence of this lemma is that the Whitney sphere stay with only one self-intersection at the origin while the flow exist. Any straight line through the origin is a stationary solution for the equivariant flow and the curve γ which generates the Whitney sphere intersect, apart from the origin, each line exacly once at the inicial time. We can apply the lemma to conclude that it stays like this under the flow until one possibly tangential intersection at the origin. A different way to see this is by saying that for each time t there exist $\varepsilon_t > 0$ where we can parametrize $\gamma_t : (-\varepsilon_t, +\varepsilon_t) \to \mathbb{C}$ by $\gamma_t(s) = r_t(s)e^{is}$.

The curves γ_t are symmetric with respect to the $o\overrightarrow{x}$ axis and so

$$r_t(\frac{\pi}{2} - u) = r_t(\frac{\pi}{s} + u),$$

this implies $r_t'(\frac{\pi}{2}) = 0$ for all $t \leq T$.

Lemma 4.9. *For all $0 \leq t \leq T$, $r_t(u)$ is non-decreasing for all $u < \frac{\pi}{2}$ and non-increasing for all $u > \frac{\pi}{2}$.*

Proof. On the interval $[\varepsilon, \frac{\pi}{2}]$ we consider $v_t = r_t'$ then we have the following equation

$$\partial_t v_t = \frac{1}{|\gamma'|^2}\partial^2_{uu} v_t + a_1(r_t, v_t, v_t')v_t' + a_2((r_t, v_t, v_t')v_t.$$

The functions a_1 and a_2 are smooth and bounded. Notice that by lemma (4.8)looking the curve as a graph we see that $f_t'(0) > 0$, so $v_t'(\varepsilon) > 0$, besides as pointed out before $v_t(\frac{\pi}{2}) = 0$. The result follows by the maximum principle because at $t = 0$ the function is non-decreasing ($v_0' \geq 0$). The same argument works on the interval $[\frac{\pi}{2}, \pi - \varepsilon]$. □

There is another way to see the above lemma by comparing two solutions of the equivariant flow. By the work of Angenent the number of intersection of two solutions does not increase in time, so comparing our solution with the self shrinker circle centered at the origin we see that this number is two. Therefore, the distance from the origin is monotone (as in above lemma) up to symmetry.

From properties discussed above we expect that the enclosed area will go to zero at the singular time. If that is the case, we have two possibilities, the curve collapse into a point or the limit curve is a line segment with singularity at $x_0 = (a, 0)$ for some $a \in \mathbb{R}$. The symmetries of the solutions rule out the second case to happen. Indeed, writing the curve, which lies above the x axis and far away from the origin and the singular point x_0, as a graph (f_t) we have

$$\partial_t f_t = \frac{f_t''}{1 + (f')^2} + \big(\arctan\frac{f_t}{x}\big)' = \frac{f_t''}{1 + (f')^2} + \big(\frac{x^2}{x^2 + f^2}\big)\big(\frac{f'}{x} - \frac{f}{x^2}\big).$$

The strong maximum principle for parabolic equations tell us that for a subsolution of a uniformly parabolic equation $\partial_t u \leq a(x,t)u'' + b(x,t)u' + c(x,t)u$ with $c \leq 0$ there exist no interior non-negative maximum unless the solution is constant, this gives a contradiction since at $t = T$ the solution is identically zero. Let's now discuss what might happen to the curves regarding its asymptotic behaviour around the singularity which we are assuming to be the origin.

The symplectic form $\omega = \sum dx^i \wedge dy^i$ in $\mathbb{C}^n$ is exact and its integral is $\lambda = \sum x_i dy^i - y_i dx^i$. This $1-$form on a equivariant lagrangian, $L = (\gamma(s)\cos, \gamma(s)\sin)$, can be represented as $\lambda = \langle \gamma, i\gamma' \rangle ds$. Integrating it along γ and using Stoke's theorem we get $\int_\gamma \langle \gamma, i\frac{\gamma'}{|\gamma'|} \rangle d\gamma = 2\,\mathrm{Area}(\gamma)$, meaning the area enclosed by γ. If β is the angle between γ (Whitney curve) and the normal vector then

$$\mathrm{Area}(\gamma) = \frac{1}{2}\int_\gamma \cos\beta |\gamma||\gamma'| du \quad \text{and} \quad \mathrm{Area}(\Sigma) = \int_\gamma 2\pi|\gamma||\gamma'|du.$$

Let's choose a sequence of times $t_i \to T_{\max}$ and a sequence os scale factors $\lambda_i \to +\infty$ such that $\mathrm{Area}(\lambda_i \gamma_{t_i}) = 1$. In fact, $\lambda_i = \frac{1}{\sqrt{\mathrm{Area}(\gamma_{t_i})}}$ and they converge to infinity because the curves are collapsing into a point.

The rescaled surfaces $\Sigma_s^i = \lambda_i \Sigma_{T+\frac{s}{\lambda_i^2}}$ also satisfie the mean curvature equation (2) and induce the rescaled curves $\gamma_s^i = \lambda_i \gamma_{T+\frac{s}{\lambda_i^2}}$ defined in $[-T\lambda_i^2, 0)$. The time s_1^i where $\mathrm{Area}(\gamma^i_{s_1^i}) = 1$ is $s_1^i = -\lambda_i^2(T - t_i)$. Since the exterior angle α_t at the point where γ_t is not smooth lives in $[-\pi, \pi]$, the Gauss- Bonnet theorem gives

$$\int_{\gamma_t} \langle \overrightarrow{k}, \nu \rangle d_{\gamma_t} + \alpha_t = 2\pi \Longrightarrow \pi \leq \int_{\gamma_t} \langle \overrightarrow{k}, \nu \rangle d_{\gamma_t} \leq 3\pi,$$

ν stands for the inward unit normal of the curve. Now by Stoke's theorem we have $\mathrm{Area}(\gamma_t) = -\frac{1}{2}\int_{\gamma_t}\langle \gamma_t, \nu\rangle d_{\gamma_t}$ and this implies

$$
\begin{aligned}
\mathrm{Area}'(t) &= -\frac{1}{2}\int_{\gamma_t}\langle \partial_t\gamma, \nu\rangle d_{\gamma_t} + \langle \gamma, \partial_t\nu\rangle d_{\gamma_t} + \langle \gamma, \nu\rangle \partial_t d_{\gamma_t} \\
&= -\frac{1}{2}\int_{\gamma_t}\langle \overrightarrow{k} - \frac{z^\perp}{|z|^2}, \nu\rangle d_{\gamma_t} + \langle \gamma, i\partial_t\gamma_t'\rangle ds \\
&= -\frac{1}{2}\int_{\gamma_t}\langle \overrightarrow{k} - \frac{z^\perp}{|z|^2}, \nu\rangle d_{\gamma_t} + \partial_s\langle \gamma, i\partial_t\gamma_t\rangle ds - \langle i\gamma_t', \partial_t\gamma\rangle ds \\
&= -\int_{\gamma_t}\langle \overrightarrow{k} - \frac{z^\perp}{|z|^2}, \nu\rangle d_{\gamma_t} = -\int_{\gamma_t}\langle \overrightarrow{k}, \nu\rangle d_{\gamma_t}.
\end{aligned}
$$

Therefore, $-3\pi \leq \mathrm{Area}'(\gamma_t) \leq -\pi$ and integrating this inequality from t to T we get $\pi(T-t) \leq \mathrm{Area}(\gamma_t) \leq 3\pi(T-t)$. From this we have $s_1^i \in [-\frac{1}{\pi}, -\frac{1}{3\pi}]$ taking $s^* = -\frac{1}{3\pi}$ we see that $\lim_{i\to\infty}\mathrm{Area}(\gamma_{s^*}^i) \leq 1$.

By theorem (3.8), the sequence of lagrangian is converging, in the sense given in (3.8), to a union of special lagrangian cones and by symmetries the sequence of curves is converging to a the respective curves, which have to be half-lines through the origin. Because the enclosed area is bounded the lemma (4.9) implies the limit is the half-line in which the curves are symmetric with, in particular the enclosed area converges to zero.

Therefore, this discussions support the expectation that the tangent flow of the Whitney sphere is a lagrangian plane with multiplicity two and lagrangian angle $\theta = 0$.

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