LSGNT London School of Geometry & Number Theory

G_2-instantons on Non-Compact Manifolds

First year mini-project

By: Daniel Platt
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Advisors: Dr. Jason Lotay
Prof. Simon Salamon

Abstract. The main areas of operation in the field of special holonomy geometry deal with the following two tasks: (1) constructing manifolds with special holonomy, and (2) constructing invariants on manifolds with special holonomy. One attempt to construct such invariants utilises instantons.

In this project, we first revise the definitions of G_2-manifolds and G_2-instantons. We outline the context from which the interest in studying G_2-instantons arises and summarise the main results about them.

After that we look at G_2-manifolds of the form $I \times N$, where N is a 6-manifold with a family of SU(3)-structures. We review how G_2-instantons on $I \times N$ can arise from data on N (cf. [LM17]). As an example, we choose N to be the Iwasawa 6-manifold. A family of G_2-structures on $I \times N$ was constructed in [AS04]. To allow for some basic observations about the moduli space of G_2-instantons on $I \times N$ we construct several G_2-instantons on the G_2-manifold $I \times W$.

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1. What are G_2-manifolds and why are they interesting?

1.1. What are G_2-manifolds?

Definition 1.1. Let $(x_1, \dots, x_7)$ be coordinates on $\mathbb{R}^7$ and let

$$\varphi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} - \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} + \mathrm{d}x^{347} - \mathrm{d}x^{356}, \tag{1}$$

then we denote

$$G_2 := \{A \in \mathrm{End}(\mathbb{R}^7) : A^*\varphi_0 = \varphi_0\}. \tag{2}$$

Equivalently, one may characterise G_2 to be the compact real form of the complex Lie group whose Lie algebra has Dynking diagram of type G_2. Alternatively, G_2 is also isomorphic to the automorphism group of the octonions.

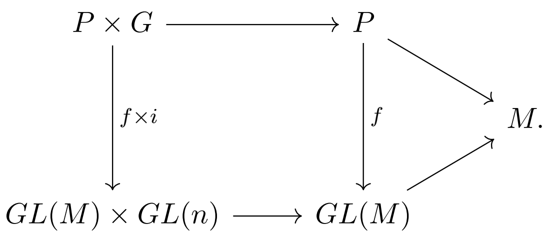
Proposition 1.2 (Section 11.1 of [Joy07])**.** *G_2 is a compact, 14-dimensional Lie group that also stabilises the 4-form*

$$*\varphi_0 = \mathrm{d}x^{4567} - \mathrm{d}x^{2345} + \mathrm{d}x^{2367} - \mathrm{d}x^{3146} + \mathrm{d}x^{3175} - \mathrm{d}x^{1247} + \mathrm{d}x^{1256} \tag{3}$$

and the standard Euclidean metric g_0 on $\mathbb{R}^7$.

Definition 1.3. Let M be a smooth, orientable manifold of dimension n with frame bundle $\mathrm{GL}(M)$ and oriented frame bundle $\mathrm{GL}_+(M)$.

1. Let $G \subset \mathrm{GL}(n)$. A principal G-bundle P is called *G-structure*, if there exists a principal bundle morphism $f : P \to \mathrm{GL}(M)$, i.e. a map making the following diagram commutative:



(4)

2. Let g be a Riemannian metric on M. A G-structure is called *torsion-free*, if $\mathrm{Hol}(M, g) \subset G$. A manifold with torsion-free G_2-structure is called *G_2-manifold.*

3. For $p \in M$, define

$$\mathcal{P}^3_p M := \{\varphi \in \Lambda^3 T^*_p M : \text{ex. } L \in \mathrm{GL}_+(M)_p \text{ s.t. } L^*\varphi = \varphi_0\} \tag{5}$$

and let $\mathcal{P}^3 M$ be the fibre bundle over M with fibre $\mathcal{P}^3_p M$. For φ a section of $\mathcal{P}^3 M$ define

$$(P_\varphi)_p := \{L \in \mathrm{GL}_+(M)_p : L^*\varphi_p = \varphi_0\} \tag{6}$$

and let P_φ be the fibre bundle over M with fibre $(P_\varphi)_p$. Then P_φ is called the *G_2-structure induced by φ.*

4. $\varphi \in \mathcal{P}^3 M$ is called *torsion-free*, if $\mathrm{d}\varphi = 0$ and $\mathrm{d}(*\varphi) = 0$, where $*$ is the Hodge-star with respect to the metric defined by φ.

Proposition 1.4 (Section 11.1 of [Joy07], Lemma 11.5 of [Sal89])**.** *Let (M, g) be a Riemannian manifold and $\varphi \in \mathcal{P}^3 M$.*

1. *P_φ is a G_2-structure.*
2. *P_φ is torsion-free if and only if φ is torsion-free.*

In the light of this proposition, we will often refer to forms $\varphi \in \mathcal{P}^3 M$ as G_2-structures and (if $\mathrm{d}\varphi = 0$ and $\mathrm{d}(*\varphi) = 0$) as torsion-free G_2-structures.

1.2. Why are G_2-manifolds interesting?

1. A celebrated theorem of Berger ([Ber55]) states that not all Lie groups can appear as the holonomy group of an irreducible, non-symmetric, simply-connected Riemannian manifold. In fact, on such a manifold of dimension $n = 2m$, only the groups $\mathrm{SO}(n)$ (*full holonomy*), $\mathrm{U}(m)$ (*Kähler manifold*), $\mathrm{SU}(m)$ (*Calabi-Yau manifold*), $\mathrm{Sp}(n)$ (*Hyperkähler manifold*), $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ (*Quaternion-Kähler manifold*), G_2 (if $n = 7$), and $\mathrm{Spin}(7)$ (if $n = 8$) can occur.

   A common theme in geometry is the search for invariants of manifolds, which motivated homology and cohomology theories as well as homotopy theory. When additional structure is present, finer invariants can be constructed, such as the Hodge numbers on a Kähler manifold, or Casson invariants on Calabi-Yau 3-folds. One may hope that the presence of a G_2 structure allows for such refined invariants (cf. [DT98] and [DS11]).

2. M-theory is a variant of string theory, in which the universe is locally assumed to be of the form $\mathbb{R}^{1,3} \times M$, where $\mathbb{R}^{1,3}$ is the Minkowski 4-space and M is a compact G_2-manifold.

   To adapt such a model to experimental data, one needs to test different choices of M. However, not many explicit examples of compact G_2-manifolds are known which makes this step difficult. [YN10, p. 150]

3. Physics motivated the study of Ricci-flat metrics, because they arise as the solutions to the Einstein field equations under extra conditions. The construction of Ricci-flat metrics which are non-flat is a difficult task that is pursued in many areas of geometry. G_2-manifolds are Ricci-flat [Joy07, Prop. 11.1.5] and non-flat, if the holonomy is non-trivial, and thus are example solutions to this problem.

2. What are G_2-instantons and why are they interesting?

2.1. What are G_2-instantons?

Let $P \to M$ be a G-structure on M, with $G \subset \mathrm{SO}(n)$. Then

$$\Lambda^2(\mathbb{R}^n)^* \simeq \mathfrak{so}(n) \simeq \mathfrak{g} \oplus \mathfrak{g}^\perp. \tag{7}$$

Thus

$$\Gamma(\Lambda^2 T^*M) = \Omega^2(T^*M) \simeq \mathrm{GL}(M) \times_\varrho \Lambda^2(\mathbb{R}^n)^* \simeq \mathrm{GL}(M) \times_\varrho (\mathfrak{g} \oplus \mathfrak{g}^\perp), \tag{8}$$

where ϱ is the standard representation of $\mathrm{GL}(n)$.

Definition 2.1. Let $Q \to M$ be a K-principal bundle with bundle connection $\omega \in \Omega^1(Q, \mathfrak{k})$ and curvature $F_\omega = \Omega$. Then ω is called *G-instanton*, if locally Ω is of the form

$$\Omega = \sum_i \eta_i \otimes X_i, \text{ for } \eta_i \in (\mathrm{GL}(M) \times_\varrho \mathfrak{g}) \subset \Omega^2(M) \text{ and } X_i \in \mathfrak{k}, \tag{9}$$

i.e. the 2-form part of Ω takes values in the subbundle associated to $\mathfrak{g} \subset \Lambda^2 T^*M$.

If we choose $G = G_2$, we obtain from the representation theory of $\mathfrak{g}_2$ the following two equivalent characterisations of G_2-instantons:

Proposition 2.2. *Let M be a smooth manifold with torsion-free G_2-structure φ and $Q \to M$ a K-principal bundle on M with connection $\omega \in \Omega^1(Q, \mathfrak{k})$. Then the following are equivalent:*

1. *ω is a G_2-instanton.*
2. $$\varphi \wedge \Omega = - * \Omega. \tag{10}$$
3. $$*\varphi \wedge \Omega = 0. \tag{11}$$

2.2. Why are G_2-instantons interesting?

It follows from the pigeonhole principle and the generalized twisted connected sum construction described in [CHNP15] that there exist manifolds with non-isomorphic torsion-free G_2-structures. Using bordism theory examples of homotopic and non-homotopic G_2-structures that are not connected in the moduli space of G_2-structures have been constructed in [CN15, CGN15]. Associating numerical invariants to G_2-structures may be able to refine these results and help to identify G_2-structures belonging to different connected components of the moduli space of G_2-structures in a general setting.

In 4-dimensional geometry the moduli space of anti-self-dual (ASD) instantons can be used to assign numerical invariants to smooth 4-manifolds. The theory of this process is known as *Donaldson theory* and was first described in [Don83] and is the topic of the monograph [DK90]. It is conjectured that one may employ similar methods to construct numerical invariants of G_2-manifolds, a hope that was expressed in [DT98, DS11].

The two steps needed to achieve this are:

1. Find a compactification $\overline{\mathcal{M}}(\varphi)$ of the moduli space $\mathcal{M}(\varphi)$ of G_2-instantons on a G_2-manifold with a suitable fixed gauge group that is a compact, 0-dimensional manifold, i.e. a finite set of points.

2. The moduli space $\mathcal{M}(\varphi)$ is known to behave badly under deformation of φ and it is expected that $\#\overline{\mathcal{M}}(\varphi)$ is not invariant under deformation of φ.

   However, it is conjectured that one may construct a counterterm by counting triples (ω, L, k) of a G_2-instanton ω, an associative submanifold L, and a multiplicity k with some weights, so that the count of G_2-instantons together with the counterterms is invariant under deformation of φ ([DS11, Section 6.2], [Wal13b, Remark 4.9]).

The following theorem describes how a sequence of G_2-instantons may degenerate. The theorem therefore suggests a candidate for $\mathcal{M}(\varphi)$ mentioned in step 1 above.

Theorem 2.3 (Theorem 4.1 in [Wal13b])**.** *Let M be a compact 7-manifold, let (t_i) be a null-sequence, let (φ_{t_i}) be a sequence of torsion-free G_2-structures on M converging to a G_2-structure φ_0, let Q be a K-bundle over M where K is a compact Lie group and let (ω_{t_i}) be a sequence of connections on Q with uniformly bounded L^2-norm of curvature such that ω_{t_i} is an irreducible G_2-instanton over (M, φ_{t_i}). Then there exist subsets $\operatorname{sing}(\omega_0) \subset S \subset M$ and a connection ω_0 on $Q|_{Y \setminus S}$ such that after passing to a subsequence the following holds:*

1. *S is closed, countably $\mathcal{H}^3$-rectifiable and satisfies $\mathcal{H}^3(S) < \infty$. Moreover, for $\mathcal{H}^3$-a.e. $x \in S$, $\varphi_0|_{T_x S}$ is a volume form on $T_x S$.*

2. *ω_0 is a G_2-instanton over $(M \setminus S, \varphi_0)$ and the current $p_1(\omega_0)$ defined by*

$$\langle p_1(\omega_0), \alpha \rangle := -\frac{1}{8\pi^2} \int_M \operatorname{tr}(F_{\omega_0} \wedge F_{\omega_0}) \wedge \alpha \tag{12}$$

   is closed.

3. *The connection ω_0 extends to a G_2-instanton on a K-bundle $\tilde{Q}$ over $(Y \setminus \operatorname{sing}(A_0), \varphi_0)$ which is isomorphic to Q over $Y \setminus S$; moreover, $\mathcal{H}^3(\operatorname{sing}(A_0)) = 0$.*

4. *There exists a sequence of gauge transformations (g_{t_i}) of $Q|_{M \setminus S}$ such that $g_{t_i}^* \omega_{t_i}$ converges to ω_0 in C^∞_{loc} on $M \setminus S$.*

A visualisation of Theorem 2.3 is given in Figure 2.2.

Figure 1: A sequence of functions $f_i \in C^\infty(\mathbb{R}^2)$ converging to 0 pointwise on $\mathbb{R}^2 \setminus \{0\}$ and converging to 0 smoothly on $\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$. In the example, $f_i|_{(\{0\}\times\mathbb{R})} \approx \frac{1}{i}\sin(ix)$ which converges pointwise but not smoothly to 0. In the language of Theorem 2.3, S is marked blue, and $\mathrm{sing}(\omega_0) = \{0\}$ is marked green.

For the second step mentioned above, no progress has been made beyond the initial conjecture in [DS11, Section 6.2].

Current and recent work tries to better understand the moduli space of G_2-instantons $\mathcal{M}(\varphi)$. A very practical first problem is to construct a family of G_2-instantons that actually has the degeneration property predicted by Theorem 2.3. In particular it is conjectured that the singular set $\mathrm{sing}(\omega_0)$ from the previous theorem has Hausdorff dimension at most one, cf. [Wal13b, Remark 4.5], [Tia00, Section 5.3].

In [Wal13b, Theorem 4.7] general conditions have been formulated under which a family of instantons converges only outside an associative submanifold. In the non-compact case, explicit examples have been constructed in [LO18]. In the compact case, no explicit example has been constructed as of now.

2.3. Examples of instantons

Figure 2.3 shows a list of construction methods for G_2-manifolds and the described instantons on them. The list is not exhaustive.

Example 2.4. Flat connections are G_2-instantons.

Example 2.5 (Example 1.95 in [Wal13b]). The Levi-Civita connection on a G_2-manifold is a G_2-instanton.

	Reference for:	
Name of construction method	Construction	Instantons
Bryant-Salamon metrics	[BS89]	[Cla14], [LO18]
Twisted connected sum	[Kov03, CHNP15]	[SEW15, Wal16, MNE15]
Generalized Kummer construction	[Joy96, JK17]	[Wal13a]
Evolved half-flat SU(3)-structures	[Hit01]	[LM17]

Figure 2: A summary of construction methods for G_2-manifolds and G_2-instantons constructed on them.

3. G_2-instantons on manifolds arising from evolution of 6-manifolds

3.1. How does evolution of 6-manifolds give G_2-manifolds?

In this section we explain how one may evolve particular SU(3)-structures on a 6-manifold to generate G_2-manifolds. To this end we will use Corollary 3.5 that gives a characterisation of SU(n)-structures in terms of forms on a manifold, analog to part 1 of Proposition 1.2.

Lemma 3.1 (Section 7.1 in [Joy07]). *Consider the complex coordinates* $(z_1, \ldots, z_n)$ *on* $\mathbb{C}^n$ *and the according real coordinates* $(z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n)$ *on* $\mathbb{R}^{2n}$. *Define a metric* g_0, $\sigma_0 \in \Lambda^2(\mathbb{R}^{2n})^*$ *and* $\Gamma \in \Lambda^n(\mathbb{R}^{2n})^*$ *by*

$$g_0 = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_n|^2, \tag{13}$$

$$\sigma_0 = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 + \cdots + \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n), \tag{14}$$

$$\Gamma = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n. \tag{15}$$

The subgroup of $\mathrm{GL}(2n, \mathbb{R})$ *preserving* g, σ *and* Γ *is* $\mathrm{SU}(n)$.

In the case $n = 3$ the group SU(n) is defined by less data, as follows from Corollary 3.5:

Proposition 3.2 (Proposition 12 in [Bry06]). *Consider the complex coordinates*

$$(z_1 = x_1 + ix_2, z_2 = x_3 + ix_4, z_3 = x_5 + ix_6)$$

on $\mathbb{R}^6$. *Then the 3-form*

$$\gamma = \mathrm{d}x^{135} - \mathrm{d}x^{146} - \mathrm{d}x^{236} - \mathrm{d}x^{245} = \mathrm{Re}(\mathrm{d}z^{123}) \tag{16}$$

has oriented stabiliser $\mathrm{Stab}_{\mathrm{GL}^+(6,\mathbb{R})}(\gamma)$ *isomorphic to* $\mathrm{SL}(3, \mathbb{C})$.

Corollary 3.3. *Let V be an oriented 6-dimensional real vector space and $\eta \in \Lambda^3 V^*$. Then the following are equivalent:*

1. *There exists a basis $(x_1, \dots, x_n)$ of V with dual basis $(x^1, \dots, x^n)$ such that η has the form given in line 16.*
2. $\mathrm{Stab}_{\mathrm{GL}^+(V)}(\eta) \simeq \mathrm{SU}(3)$.
3. *There exists a unique complex structure $J \in \mathrm{Aut}(V)$ for which η is the real part of a $(3,0)$-form.*

Corollary 3.4. *Let V be an oriented 6-dimensional real vector space. Let $\sigma \in \Lambda^2 V^*$ be a non-degenerate form and let $\gamma \in \Lambda^3 V^*$ be a form satisfying one of the equivalent conditions of the previous Corollary. Then*

$$\mathrm{Stab}_{\mathrm{GL}(V)}(\sigma) \cap \mathrm{Stab}_{\mathrm{GL}^+(V)}(\gamma) = \mathrm{SU}(3). \tag{17}$$

Corollary 3.5. *Let N be an oriented 6-manifold and let $\sigma \in \Omega^2(N)$ and $\gamma \in \Omega^3(N)$ such that for all $p \in N$ there exists $L \in \mathrm{GL}_+(N)_p$ s.t. $L^*\sigma_p = \sigma_0$, $L^*\gamma_p = \mathrm{Re}(\Gamma)$.*

Then the collection of all such L is an $\mathrm{SU}(3)$-structure on N.

Because of this lemma, we refer to such a pair (σ, γ) as an SU(3)-structure.

Definition 3.6. An SU(3)-structure (σ, γ) is called *half-flat*, if

$$\mathrm{d}\sigma^2 = 0 = \mathrm{d}\gamma. \tag{18}$$

Theorem 3.7. *Let N be a 6-manifold and*

$$I \ni t \mapsto (\sigma(t), \gamma(t)) \tag{19}$$

a family of half-flat $\mathrm{SU}(3)$-structures on N satisfying "Hitchin's flow equations" *(cf. [Hit01])*

$$\begin{aligned} \frac{\partial \gamma}{\partial t} &= \mathrm{d}\sigma, \\ \frac{\partial \sigma^2}{\partial t} &= -2\,\mathrm{d}\hat{\gamma}, \end{aligned} \tag{20}$$

where $\hat{\gamma} \in \Omega^3(N)$ is given by requiring that $\gamma + i\hat{\gamma}$ is a holomorphic volume form with respect to the complex structure defined by (σ, γ). Then

$$\begin{aligned} \varphi &= \sigma \wedge \mathrm{d}t + \gamma, \\ *\varphi &= \hat{\gamma} \wedge \mathrm{d}t + \frac{1}{2}\sigma^2 \end{aligned} \tag{21}$$

is a torsion-free G_2-structure on $M = I \times N$.

Example 3.8 (The Iwasawa manifold). Denote by

$$H_{\mathbb{C}} := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\} \tag{22}$$

the *complex Heisenberg group* and denote by H_Γ the discrete subgroup of $H_{\mathbb{C}}$, whose entries z_1, z_2, $z_3 \in \mathbb{Z}[i]$ are Gaussian integers. H_Γ acts on $H_{\mathbb{C}}$ by left-multiplication and the quotient

$$W := H_{\mathbb{C}}/H_\Gamma \tag{23}$$

is called *Iwasawa manifold.* Then

$$\pi : W \to \mathbb{T}^4 \tag{24}$$
$$H_{\mathbb{Z}[i]} \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (z_1, z_2)$$

defines a non-trivial $\mathbb{T}^2$-principal bundle over $\mathbb{T}^4$. Denote the coordinates $(\lambda, \mu, l, m, x, y)$ on $H_{\mathbb{C}}$ given by

$$z_1 = \lambda + i\mu = x_1 + ix_2, \quad z_2 = l + im = x_3 + ix_4, \quad z_3 = x + iy = x_5 + ix_6. \tag{25}$$

Then the frame

$$\begin{aligned} \mathrm{d}\lambda &=: e^1, & \mathrm{d}\mu &=: e^2, \\ \mathrm{d}l &=: e^3, & \mathrm{d}m &=: e^4, \\ -\mathrm{d}x + \lambda\,\mathrm{d}l - \mu\,\mathrm{d}m &=: e^5, & -\mathrm{d}y + \mu\,\mathrm{d}l + \lambda\,\mathrm{d}m &=: e^6 \end{aligned} \tag{26}$$

is left-invariant on $H_{\mathbb{C}}$ and therefor descends to a frame on W. We then have

$$\begin{aligned} de^1 = de^2 = de^3 = de^4 &= 0, \\ de^5 &= e^{13} + e^{42}, \\ de^6 &= e^{14} + e^{23}. \end{aligned} \tag{27}$$

Let t be a parameter in $(0, \infty)$ and denote $z = t^4$, $\tau = \frac{1}{3}t^3$, then

$$\begin{aligned} \varrho &= z^{\frac{1}{2}}(e^{12} + e^{34}) + z^{-\frac{1}{2}} e^{56}, \\ \phi^+ &= t\,\mathrm{d}\,(e^{56}), \\ \phi^- &= t(e^5 \wedge \mathrm{d}\,(e^5) + e^6 \wedge \mathrm{d}\,(e^6)) \end{aligned} \tag{28}$$

defines a family of half-flat SU(3)-structures parametrised by τ that satisfies Hitchin's flow equations 20. Note that this is a special case of the family of solutions given in [AS04, Section 4]. Thus, $\varphi = \varrho \wedge \mathrm{d}\tau + \phi^+$ defines a torsion-free G_2-structure on $(0, \infty) \times W$.

The frame

$$(te^1, te^2, te^3, te^4, t^{-1}e^5, t^{-1}e^6, t^2 dt) \tag{29}$$

is an orthonormal frame of $(0, \infty) \times W$.

3.2. How does one compute G_2-instantons on manifolds arising from evolution of 6-manifolds?

The following lemma describes the curvature form of connections on $I \times N$ defined by a family of connections on N:

Lemma 3.9 (Lemma 1.5 in [LM17]). *Let ω be a connection on a principal K-bundle over $I \times N$. Then ω can be identified with a one-parameter family $I \ni t \mapsto A(t)$ of connections on a principal K-bundle over N. In particular, if $F_A = F_A(t)$ is the curvature of $A(t)$ then the curvature 2-form Ω of ω can be expressed as*

$$\Omega = \mathrm{d}t \wedge A' + F_A. \tag{30}$$

Figure 3.2 illustrates the lemma by displaying a family of connections on N and the corresponding connection on $I \times N$.

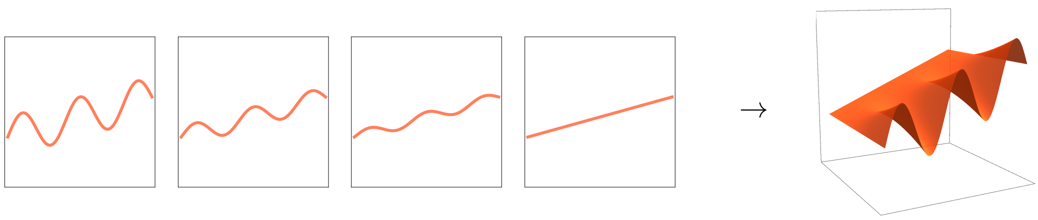


Figure 3: A family of connections on the $\mathbb{R}$-bundle over $N = (0, 1)$, indicated by integral manifolds for the respective horizontal distributions, and the corresponding connection on $I \times N$.

The following proposition translates the instanton equations 10 and 11 from Proposition 2.2 into conditions on a family of principal bundle connections on the evolving manifold:

Proposition 3.10 (Lemma 3.1, Proposition 3.2 in [LM17]). *Let N be a 6-manifold and let*

$$I \ni t \mapsto (\sigma, \gamma) \tag{31}$$

be a family of half-flat $\mathrm{SU}(3)$-structures satisfying Hitchin's flow equations 20. Denote by $M = I \times N$ the corresponding G_2-manifold and let ω be a connection on a K-principal bundle over M. Let $I \ni t \mapsto A(t)$ be the associated family of principal bundle connections on N according to Lemma 3.9. Then the following are equivalent:

1. *ω is a G_2-instanton.*
2. $$\begin{cases} A' = *_t(F_A \wedge \gamma) \\ \sigma \wedge F_A + *_t F_A = \gamma \wedge *_t(F_A \wedge \gamma) \end{cases} \tag{32}$$
3. $$\begin{cases} A' \wedge \sigma^2 = 2F_A \wedge \hat{\gamma} \\ F_A \wedge \sigma^2 = 0 \end{cases} \tag{33}$$

4.
$$\begin{cases} A' = *_t(F_A \wedge \gamma) \\ F_A(t_0 \wedge \sigma^2(t_0) = 0 \textit{ for some } t_0 \in I \end{cases} \tag{34}$$

In the previous theorem $*_t$ denotes the Hodge-star on N defined by the metric induced by the SU(3)-structure $(\sigma(t), \gamma(t))$.

Example 3.11 (G_2-instanton on flat $\mathbb{R}^7$, section 3.3 in [LM17], p.3 in [GN95], section 8 in [IP93]). To find an instanton on the trivial G_2 bundle over flat $\mathbb{R}^7$, we will construct an instanton on $S^6 \times (0, \infty) \subset \mathbb{R}^7$ and show that it extends to the origin.

To this end we first describe the half-flat SU(3)-structure on S^6. S^6 is equipped with the canonical Riemannian metric g and an almost complex structure J induced by octonionic multiplication. For $x \in S^6$, J is given by

$$\begin{aligned} J : T_xM &\to T_xM \\ Z &\mapsto x \cdot Z, \end{aligned} \tag{35}$$

where $x \cdot Z$ denotes octonionic multiplication. The symplectic form σ and the real part of the holomorphic volume form γ are then given as

$$\sigma(X, Y) = g(X, JY), \tag{36}$$

$$\gamma(X, Y, Z) = g(Y, (\nabla_X J)Z) \text{ for } X, Y, Z \in \mathfrak{X}(S^6). \tag{37}$$

Fix $x_0 = (0, 0, 0, 0, 0, 0, 1) \in S^6$ and denote by $(X_1, \ldots, X_6)$ the canonical basis of $T_{x_0}S^6$, i.e. $X_1 = (1, 0, 0, 0, 0, 0, 0)$, $X_2 = (0, 1, 0, 0, 0, 0, 0)$, etc. Direct calculation then shows that

$$\gamma = dX^{123} - dX^{145} + dX^{246} - dX^{356}, \tag{38}$$

$$\sigma = dX^{16} + dX^{25} + dX^{34}. \tag{39}$$

Write a general element of $\mathfrak{g}_2$ as

$$\xi = \begin{pmatrix} 0 & a_3 & -a_2 & a_5 & -a_4 & -a_7 & -a_6 + b_6 \\ -a_3 & 0 & a_1 & a_6 & -a_7 + b_7 & a_4 - b_4 & a_5 + b_5 \\ a_2 & -a_1 & 0 & -b_7 & b_6 & b_5 & b_4 \\ -a_5 & -a_6 & b_7 & 0 & -a_1 + b_1 & -a_2 + b_2 & -a_3 + b_3 \\ a_4 & a_7 - b_7 & -b_6 & a_1 - b_1 & 0 & b_3 & -b_2 \\ a_7 & -a_4 + b_4 & -b_5 & a_2 - b_2 & -b_3 & 0 & b_1 \\ a_6 - b_6 & -a_5 - b_5 & -b_4 & a_3 - b_3 & b_2 & -b_1 & 0 \end{pmatrix} \tag{40}$$

as suggested in [Are05]. Let $(A_1, \ldots, A_7, B_1, \ldots, B_7) = (e_1, \ldots, e_{14})$ be the corresponding basis of $\mathfrak{g}_2$, i.e. A_1 is the above matrix with $a_1 = 1$ and all other variables set to zero, etc. We have $\mathfrak{g}_2 \subset \mathfrak{so}(7) \simeq \Lambda^2(\mathbb{R}^7)^*$, so denote by $(\beta_1, \ldots, \beta_{14})$ the elements in $\Lambda^2(\mathbb{R}^7)^*$ corresponding to $e_1, \ldots, e_{14}$.

For $1 \leq j \leq 14$ let

$$\alpha_j := \partial r \lrcorner \beta_j|_{S^6} \in \Omega^1(S^6),$$

where $\partial r \in \mathfrak{X}(\mathbb{R}^7 \setminus \{0\})$ is the canonical radial vector field, and use the following ansatz for $A(t)$:

$$A(t) = ta(t) \sum_{j=1}^{14} \alpha_j \otimes e_j \in \Omega^1(S^6, \mathfrak{g}_2) \simeq \Omega^1(S^6 \times G_2, \mathfrak{g}_2). \tag{41}$$

We used here that the bundle $S^6 \times G_2$ is trivial. Thus, defining a right-invariant 1-form on that bundle is equivalent to defining a 1-form on the base space S^6.

We find that

$$\begin{aligned} \mathrm{d}\alpha_j &= \mathrm{d}\iota_{\partial r} \beta_j \\ &= -\iota_{\partial r} \underbrace{\mathrm{d}\beta_j}_{=0} + \underbrace{\mathcal{L}_{\partial_r} \beta_j}_{=\beta_j \text{ on } S^6} \\ &= \beta_j \in \Omega^2(S^6). \end{aligned} \tag{42}$$

We therefore find the curvature $F_A \in \Omega^2(S^6, \mathfrak{g}_2)$ to be

$$\begin{aligned} F_A &= \mathrm{d}A + \frac{1}{2}[A, A] \\ &= ta(t)\beta_i \otimes e_i + \frac{1}{2} t^2 a(t)^2 \sum_{1 \leq j,k \leq 14} (\alpha_j \wedge \alpha_k)[e_j, e_k] \end{aligned} \tag{43}$$

and have for the derivative

$$A'(t) = (a(t) + ta'(t)) \cdot \sum_{j=1}^{14} \alpha_j \otimes e_j. \tag{44}$$

After writing A' and $*_r(F_A \wedge \gamma)$ in terms of the canonical basis $(X_1, \ldots, X_7)$ a lengthy calculation shows that

$$*_r(F_A \wedge \gamma) = (ta(t)(1 - ta(t)) \cdot \eta, \tag{45}$$

$$A' = (a(t) + ta'(t)) \cdot \eta \tag{46}$$

for some $\eta \in \Omega^1(S^6, \mathfrak{g}_2)$. Thus, solutions $a(t)$ to the equation

$$ta(t)(1 - ta(t)) = a(t) + ta'(t) \tag{47}$$

define a G_2-instanton where they exist. The general solution is of the form

$$a(t) = \frac{e^t}{t(e^t + C)} \tag{48}$$

with $C \in \mathbb{R}$. For $C \geq 0$ we obtain a solution which is defined on $(0, \infty)$. Note that

$$\lim_{t \to 0} ta(t) = \frac{1}{1 + C} \tag{49}$$

is well-defined. Thus, the solution closes up and for $C \geq 0$ we obtain an instanton on all of $S^6 \times [0, \infty) = \mathbb{R}^7$.

3.3. G_2-instantons on the evolved Iwasawa manifold

The aim of this project was to find a family of G_2-instantons that exhibits the degeneration behaviour described in Theorem 2.3. Note that the theorem is a statement on compact manifolds, but one may expect a similar degeneration for families of instantons with locally uniformly bounded L^2-norm of their curvature on non-compact manifolds.

Several instantons were constructed, but none of them display the desired degeneration behaviour. The constructed instantons will be listed throughout the remainder of the section.

3.3.1. Solutions with gauge group $\mathrm{U}(1)$

Consider the trivial $\mathrm{U}(1)$-bundle over N and make the ansatz

$$A = \sum_{i=1}^{6} f_i(\lambda, \mu, l, m, t) e^i, \tag{50}$$

where f_i are functions defined on $I \times W$ that are independent of the z_3-coordinate, i.e. they can be considered as functions on $(0, \infty) \times (\mathbb{C}/\mathbb{Z}[i])^2$. Then

$$A' = \sum_{i=1}^{6} f_i' e^i, \tag{51}$$

$$F_A = f_5 \,\mathrm{d}(e^5) + f_6 \,\mathrm{d}(e^6) + \sum_{i=1}^{6} \sum_{k=1}^{4} \frac{\partial f_i}{\partial x_k} e^{ki}, \tag{52}$$

$$\begin{aligned} F_A \wedge \phi^+ = t \Big(& 2 f_5 e^{12346} - 2 f_6 e^{12345} \\ & + \left(\frac{\partial f_6}{\partial m} + \frac{\partial f_5}{\partial l} \right) e^{23456} + \left(\frac{\partial f_5}{\partial m} - \frac{\partial f_6}{\partial l} \right) e^{13456} \\ & + \left(-\frac{\partial f_6}{\partial \mu} - \frac{\partial f_5}{\partial \lambda} \right) e^{12456} + \left(-\frac{\partial f_5}{\partial \mu} + \frac{\partial f_6}{\partial \lambda} \right) e^{12356} \\ & + \left(\frac{\partial f_2}{\partial m} - \frac{\partial f_1}{\partial l} - \frac{\partial f_4}{\partial \mu} + \frac{\partial f_3}{\partial \lambda} \right) e^{12346} \\ & + \left(\frac{\partial f_1}{\partial m} + \frac{\partial f_2}{\partial l} - \frac{\partial f_3}{\partial \mu} - \frac{\partial f_4}{\partial \lambda} \right) e^{12345} \Big), \end{aligned} \tag{53}$$

$$
\begin{aligned}
*_t(F_A \wedge \phi^+) = t \Bigg(& 2f_5 t^{-3} e^5 - 2f_6 t^{-3} e^6 + \left(\frac{\partial f_6}{\partial m} + \frac{\partial f_5}{\partial l}\right) e^1 + \left(\frac{\partial f_5}{\partial m} - \frac{\partial f_6}{\partial l}\right) e^2 \\
& + \left(-\frac{\partial f_6}{\partial \mu} - \frac{\partial f_5}{\partial \lambda}\right) e^3 + \left(-\frac{\partial f_5}{\partial \mu} + \frac{\partial f_6}{\partial \lambda}\right) e^4 \\
& + \left(\frac{\partial f_2}{\partial m} - \frac{\partial f_1}{\partial l} - \frac{\partial f_4}{\partial \mu} + \frac{\partial f_3}{\partial \lambda}\right) t^{-3} e^5 \\
& + \left(\frac{\partial f_1}{\partial m} + \frac{\partial f_2}{\partial l} - \frac{\partial f_3}{\partial \mu} - \frac{\partial f_4}{\partial \lambda}\right) t^{-3} e^6 \Bigg).
\end{aligned} \tag{54}
$$

Assuming that $f_i(\lambda, \mu, l, m, t) = f_i(\lambda, \mu, l, m) \cdot g(t)$ we can separate variables and obtain the system

$$
\begin{aligned}
\frac{\partial f_6}{\partial m} + \frac{\partial f_5}{\partial l} &= cf_1, \\
\frac{\partial f_5}{\partial m} - \frac{\partial f_6}{\partial l} &= cf_2, \\
-\frac{\partial f_6}{\partial \mu} - \frac{\partial f_5}{\partial \lambda} &= cf_3, \\
-\frac{\partial f_5}{\partial \mu} + \frac{\partial f_6}{\partial \lambda} &= cf_4, \\
2f_5 + \frac{\partial f_2}{\partial m} - \frac{\partial f_1}{\partial l} - \frac{\partial f_4}{\partial \mu} + \frac{\partial f_3}{\partial \lambda} = cf_5 \text{ or } \frac{\partial f_2}{\partial m} - \frac{\partial f_1}{\partial l} - \frac{\partial f_4}{\partial \mu} + \frac{\partial f_3}{\partial \lambda} &= c' f_5, \\
-2f_6 + \frac{\partial f_1}{\partial m} + \frac{\partial f_2}{\partial l} - \frac{\partial f_3}{\partial \mu} - \frac{\partial f_4}{\partial \lambda} = cf_6 \text{ or } \frac{\partial f_1}{\partial m} + \frac{\partial f_2}{\partial l} - \frac{\partial f_3}{\partial \mu} - \frac{\partial f_4}{\partial \lambda} &= c'' f_6,
\end{aligned} \tag{55}
$$

for some constant $c \in \mathbb{R}$, $c' = c - 2$, $c'' = c + 2$. The general PDE system is too difficult to solve. We impose some extra conditions to get an easier PDE system that can be solved.

To this end, set $c = 0$, $c' = -2$, $c'' = 2$. The first four lines of equations 55 then become the Cauchy-Riemann equations for $i \cdot f_5 + f_6$.

If $i \cdot f_5 + f_6$ satisfies the Cauchy-Riemann equations *and* satisfies the periodicity $(i \cdot f_5 + f_6)(\lambda + k_1, \mu + k_2, l + k_3, m + k_4)$ for $k_i \in \mathbb{Z}$, then $i \cdot f_5 + f_6$ must be a rational function in the Weierstraß $\mathcal{P}$-function and its first derivative by the characterisation of elliptic functions.

1. If $i \cdot f_5 + f_6 \equiv 0$ then an example family of solutions is given by $f_1(\lambda, \mu)$, $f_2(\lambda, \mu)$, $f_3(l, m)$, and $f_4(l, m)$, i.e. solutions depending on four freely chosen functions in two variables. Another example family is obtained after choosing $f_3(\lambda, \mu, l)$,

$f_4(\lambda,\mu,l)$ freely, which determines

$$f_1(\lambda,\mu,l) = \int_0^l -\frac{\partial}{\partial\mu} f_4(\lambda,\mu,s) + \frac{\partial}{\partial\lambda} f_3(\lambda,\mu,s)\,\mathrm{d}s, \tag{56}$$

$$f_2(\lambda,\mu,l) = \int_0^l -\frac{\partial}{\partial\lambda} f_4(\lambda,\mu,s) + \frac{\partial}{\partial\mu} f_3(\lambda,\mu,s)\,\mathrm{d}s. \tag{57}$$

In the case of functions depending on four variables one computes the Cartan characters (cf. sections A.1 and A.2) in this case to be $(s_1,s_2,s_3,s_4)=(4,4,4,3)$, i.e. the general real-analytic solution will locally depend on 3 functions of 4 variables. The `pdsolve` method of the software package *Maple* can explicitly compute a general solution depending on 2 functions of 4 variables.

2. The assumption $i\cdot f_5+f_6\equiv c\neq 0$ contradicts the periodicity of (f_1,f_2,f_3,f_4), thus no such solution can exist on W.

3. If $i\cdot f_5+f_6$ is non-constant, then it must have a pole by the characterisation of elliptic functions. Thus no smooth solution can exist in this case.

3.3.2. Solutions with gauge group SU(2)

Consider the trivial SU(2)-bundle over W. Writing

$$\mathfrak{su}(2) = \operatorname{span}\left(\underbrace{-\frac{1}{2}i\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{=v_1}, \underbrace{-\frac{1}{2}i\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{=v_3}, \underbrace{-\frac{1}{2}i\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{=v_5}\right) \tag{58}$$

we have $[v_1,v_3]=v_5$ and the identities arising from cyclic permutation.

The ansatz

$$A = a_1 e^1\otimes v_1 + a_3 e^3\otimes v_3 + a_5 e^5\otimes v_5 \tag{59}$$

with a_1, a_2, a_3 functions defined on $(0,\infty)$ has

$$\begin{aligned}
A' &= a_1' e^1\otimes v_1 + a_3' e^3\otimes v_3 + a_5' e^5\otimes v_5,\\
F_A &= a_1a_3 e^{13}\otimes v_5 - a_1a_5 e^{15}\otimes v_3 + a_3a_5 e^{35}\otimes v_1 + a_5\cdot(e^{13}+e^{42})\otimes v_5,\\
*_t(F_A\wedge\phi^+) &= t\left((a_1a_3+2a_5)t^{-3}e^5\otimes v_5 + a_1a_5 e^3\otimes v_3 + a_3a_5 e^1\otimes v_1\right)
\end{aligned}$$

resulting in the ODE system

$$\begin{aligned}
a_1' &= a_3a_5t,\\
a_3' &= a_1a_5t,\\
a_5' &= a_1a_3t^{-2} + 2t^{-2}a_5.
\end{aligned} \tag{60}$$

It admits an obvious solution $a_1 = a_3 = 0$, and $a_5 = e^{-2t^{-1}}$.

Non-obvious solutions are given by formal power series. According to Theorem 7.1 of [Mal72] *every* such solution exists on $(0, \infty)$.

4. Outlook

As a next step of investigation, one would want to answer the following questions:

Question 4.1. **Does there exist an instanton with gauge group** U(1) **that displays the degeneration behaviour predicted in Theorem 2.3?**

To this end one may try to choose other values for the constant $c \in \mathbb{R}$ from equations 55 first and later try a more general ansatz than in line 50, with functions f_i depending also on the fibre coordinates.

Question 4.2. **Does there exist an instanton with non-commutative gauge group that displays the degeneration behaviour predicted in Theorem 2.3?**

If A is an instanton with commutative gauge group, then $\lambda \cdot A$ is again an instanton for all $\lambda \in \mathbb{R}$. This is not the case for instantons with non-commutative gauge groups, unless the connection takes values in a Cartan subalgebra, thus this class of instantons is a good candidate to allow the definition of a numerical invariant. It is therefore desirable to understand their limit behaviour better.

A. Appendix

A.1. Exterior Differential Systems

In the whole section let M be a smooth manifold of dimension m.

Definition A.1 (Section 1.2 in [BCG+91]).

1. An *exterior differential system* is given by an ideal $\mathcal{I} \subset \Omega^*(M)$ that is closed under exterior differentiation;

2. an *integral manifold* of the system is given by an immersion $f : N \to M$ such that $f^*\alpha = 0$ for all $\alpha \in \mathcal{I}$.

Definition A.2 (Section 1.3 in [BCG+91]). Let $f : N \to M$ be a smooth map and $p \in N$. Fix $r \in \mathbb{N}$, $r \geq 0$. Denote by $j^r_p(f)$ the *r-jet of f in p*. We write that $j^r_p(f) = j^r_p(g)$, if the partial derivatives of f and g coincide up to order r in some (and therefore in any) chart.

Denote by

$$J^r(M,N) = \left\{ j^r_p(f) \mid p \in N, f : N \to M \text{ smooth} \right\} \tag{61}$$

the *r-jet bundle of maps from N to M*. Let U be a coordinate patch on N with coordinates $(x^1, \dots, x^m)$ and V be a coordinate patch on M with coordinates $(z^1, \dots, z^n)$, then we have coordinates h on $J(M,N)$ given by

$$h(j^r_p(f)) = \left(x^i(p), z^j(f(p)), \frac{\partial (z \circ f)}{\partial x^\alpha} \right), 1 \le i \le m, 1 \le j \le n, 1 \le |\alpha| \le r. \tag{62}$$

These are called *natural coordinates.*

Remark A.3 (Section 2.1 in [BCG+91]). Using the jet bundle formalism, PDEs can be formulated in the language of exterior differential systems. As an example, consider the first order partial differential equation

$$F\left(x^i, z, \frac{\partial z}{\partial x^i} \right) = 0, \quad 1 \le i \le n \tag{63}$$

for a function z defined on $\mathbb{R}^n$. Denote by (x^i, z, p_i) the natural coordinates on $J(\mathbb{R}^n, \mathbb{R})$ and consider the exterior differential system

$$\begin{aligned} F(x^i, z, p_i) &= 0 \\ \mathrm{d}z - \sum p_i \, \mathrm{d}x^i &= 0 \end{aligned} \tag{64}$$

together with their exterior derivates

$$\begin{aligned} \sum (F_{x^i} + F_z p_i) \, \mathrm{d}x^i + \sum F_{p_i} \, \mathrm{d}p_i &= 0 \\ \sum \mathrm{d}x^i \wedge \mathrm{d}p_i &= 0. \end{aligned} \tag{65}$$

A parametrisation $\varphi(x) = (x, z(x), p_i(x))$ of the integral manifold for this system over an open set $U \subset \mathbb{R}^n$ then gives the solution $z(x)$ to the original PDE.

So far, nothing has been said about how to construct integral manifolds for a given exterior differential system. We will make good for this now.

Definition A.4 (Section 3.1, Definition 1.1 in [BCG+91]). A subspace $E \subset T_xM$ is called *integral element* if $\varphi_E = 0$ for all $\varphi \in \mathcal{I}$.

Definition A.5 (Section 3.1, Definition 1.5 in [BCG+91]). Let M be a smooth manifold, $\mathcal{I}$ an exterior differential system and $(e_1, \dots, e_p)$ a basis of $E \subset T_xM$. Define the *polar space* of E to be the vector space

$$H(E) = \{ v \in T_xM : \varphi(v, e_1, \dots, e_p) = 0 \text{ for all } \varphi \text{ in } \mathcal{I} \text{ of degree } p+1 \}. \tag{66}$$

Definition A.6 (Section 3.1, Definition 1.7 in [BCG+91]). For $\Omega, \varphi \in \Omega^n(M)$ denote

$$G_n(TM, \Omega) := \{E \subset T_xM : x \in M, \Omega_E \neq 0\} \tag{67}$$

and define the function

$$\begin{aligned} \varphi_\Omega : G_n(TM, \Omega) &\to \mathbb{R} \\ E &\mapsto \varphi_\Omega(E) \end{aligned} \tag{68}$$

characterised by $\varphi_E = \varphi_\Omega(E)\Omega_E$.

The subspace $E \subset T_xM$ is called *Kähler-ordinary* if there exists an n-form Ω on M such that $\Omega_E \neq 0$ and E is an ordinary zero of

$$\mathcal{F}_\Omega(\mathcal{I}) := \{\varphi_\Omega : \varphi \text{ lies in } \mathcal{I} \text{ and has degree } n\}. \tag{69}$$

Denote by

$$V_n(\mathcal{I}) := \{E \in TM : E \text{ integral and } \dim E = n\}, \tag{70}$$

$$V_n^0(\mathcal{I}) := \{E \in V_n(\mathcal{I}) : E \text{ Kähler-ordinary}\}. \tag{71}$$

E is called *Kähler-regular* if it is Kähler-ordinary and the map

$$\begin{aligned} r : V_n(\mathcal{I}) &\to \mathbb{Z} \\ E &\mapsto \dim H(E) - (n+1) \end{aligned} \tag{72}$$

is locally constant on $V_n^0(\mathcal{I})$.

Remark A.7. The notion of Kähler-regularity is well-defined, i.e. independent of the choice of Ω.

Definition A.8 (Section 3.1, Definition 1.9 in [BCG+91]). A nested sequence of subspaces

$$(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_zM \tag{73}$$

where each E_k is of dimension k and E_n is an integral element of $\mathcal{I}$ is called an *integral flag* of $\mathcal{I}$ of length n based at z.

The integral flag is called *ordinary flag* if each E_k is Kähler-regular.

Definition A.9 (Section 3.2 in [BCG+91]). Let $\mathcal{I}$ be a real analytic differential system on M that contains no 0-forms. Let $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_zM$ be an ordinary integral flag of $\mathcal{I}$. For $0 \leq k \leq n-1$ denote by c_k the codimension of $H(E_k)$ in T_zM and set $c_{-1} = 0$, $c_n = \dim M - n$. The (non-negative) number

$$s_k = c_k - c_{k-1} \tag{74}$$

is called the *k-th Cartan character* of the given flag.

Proposition A.10 (Proposition 2.4 and Section 3.2 in [BCG$^+$91])**.**

1. *The sequence of Cartan characters* $(s_0, s_1, \dots, s_n)$ *does not depend on the choice of integral flag.*

2. *Locally the real analytic* n*-dimensional integral manifolds of* $\mathcal{I}$ *depend on* s_0 *constants,* s_1 *functions of* 1 *variable,* s_2 *functions of* 2 *variables,* $\dots$*, and* s_n *functions of* n *variables.*

Example A.11*.* As an example take the Cauchy-Riemann equations for two functions u, $v : \mathbb{R}^2 \to \mathbb{R}$:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned} \tag{75}$$

The induced exterior differential system on the manifold $\mathbb{R}^6$ with coordinates (x, y, u, v, u_x, u_y) is (after eliminating 0-forms)

$$\begin{aligned} \mathcal{I} = (\, &\mathrm{d}u - (u_x \,\mathrm{d}x + u_y \,\mathrm{d}y), \\ &\mathrm{d}v - (-u_y \,\mathrm{d}x + u_x \,\mathrm{d}y)) \end{aligned} \tag{76}$$

and has Cartan characters $(s_1, s_2) = (2, 0)$, i.e. an integral manifold depends locally on 2 functions of 1 variable.

This is another formulation of the fact that a power series $f : \mathbb{R} \to \mathbb{C}$ in one real variable with complex coefficients has a unique analytic continuation in a neighbourhood of $\mathbb{R} \subset \mathbb{C}$.

A.2. Software Packages

For this project the computer software *Maple* was used to solve partial differential equations and execute EDS calculations in three ways:

1. Using the built-in function `pdsolve` to compute explicit solutions for PDE systems such as 55.

2. Using the function `IntegralManifold` of the *ExteriorDifferentialSystems* package to compute integral manifolds for a given exterior differential system. Note that this function "uses `pdsolve` to integrate" (cf. [Map13]) which means that this function does not give a more powerful method to solve PDEs than `pdsolve`.

3. Using the function `CartanKahler` of the *Cartan* package (cf. [Cle10]) to compute Cartan characteristics of an EDS. While not computing an integral manifold, the function does compute Cartan characters in cases where the previously mentioned `pdsolve` and `IntegralManifold` fail to provide any solution.

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