Einstein manifolds: introduction & review

(M^n, g) Riem. mfld. $\rightsquigarrow$ Einstein: $Ric(g) = \lambda g$ $(\lambda \in \mathbb{R})$ $(= (n-1)K g,\ K \in \mathbb{R})$

- Scale-invariance $\rightsquigarrow$ can choose $\lambda \in \{-1, 0, 1\}$ (or $K \in \{-1, 0, 1\}$)
- $n \leq 3 \Rightarrow$ constant curvature $\rightsquigarrow$ assume $n \geq 4$

This class: M^n compact, $n \geq 4$

Suitable local coords (harmonic): $Ric(g) = \frac{1}{2} \underbrace{\Delta_L}_{\text{Lichnerowicz Laplacian}}(g) + \text{l.o.t.}$

$\rightsquigarrow$ Einstein eq$^{\underline{n}}$ $\longleftrightarrow$ nonlinear elliptic PDE
(modulo diffeos since $Ric(f^* g) = f^* Ric(g)$)

$\rightsquigarrow$ ∞-ly many local solutions.

Examples
- $\lambda > 0$: S^n/Γ, $(S^{k+2} \times S^{l+2})/\Gamma$, $\mathbb{C}P^n$ (Note: $(M_1, g_1) \times (M_2, g_2)$ Einstein if $Ric(g_j) = \lambda g_j$ same λ)
- $\lambda = 0$: T^n, $K3$, $\{[z_0, \ldots, z_n] \in \mathbb{C}P^n : z_0^{n+1} + \ldots + z_n^{n+1} = 0\} \subset Y$, G_2, $Spin(7)$
- $\lambda < 0$: $\mathcal{H}^n/\Gamma$, $-$ve $K \in \{[z_0, \ldots, z_n] \in \mathbb{C}P^n : z_0^d + \ldots + z_n^d = 0\}$ $d > n+1$

Remark: most known Einstein metrics satisfy stronger conditions, e.g. constant curvature or reduced holonomy (Kähler, for example)

This class: focus on generic holonomy, and not locally symmetric.

- $\mathcal{E}(M) = \{\text{Einstein } g \text{ on } M\} / Diff_{(0)}(M)$
- $\mathcal{E}(M, \lambda) = \{g : Ric(g) = \lambda g \text{ on } M\} / Diff_{(0)}(M)$

Q1: $\mathcal{E}(M) \neq \phi$? < Existence / Obstructions $\longleftarrow$ focus of this class.

Q2: Structure of $\mathcal{E}(M, \lambda)$?

Obstructions
- restrictions on $\pi_1(M)$ (eg. Bonnet-Myers $\Rightarrow$ $\pi_1(M)$ finite if $\lambda > 0$)
- $n = 4$: Hitchin-Thorpe inequality ($\chi(M) \geq \frac{3}{2} |\underbrace{\sigma(M)}_{\text{signature}}|$), Gromov simplicial volume, ... $\Rightarrow_{\text{e.g.}}$ $\nexists$ Einstein on $k\mathbb{C}P^2$, $k \geq 4$ (i.e. $\mathcal{E}(k\mathbb{C}P^2) = \phi$) & $\mathcal{E}(\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}) = \phi$ for $k \geq 9$

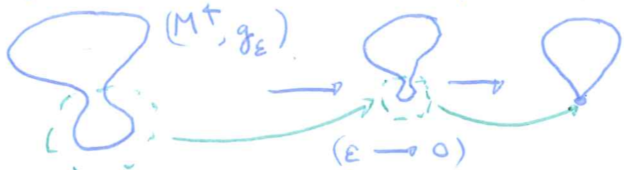
Little else $\rightsquigarrow$ Q: Is $\mathcal{E}(M^n) \neq \phi$ $\forall n \geq 5$ if $\pi_1(M) = 1$?

Structure • (Koiso) $\mathcal{E}(M,\lambda)$ real analytic subset of real analytic finite dim. mfld.

(Note: all known examples have $\mathcal{E}(M,\lambda)$ smooth mfld)

• $n=4$: (Anderson, Bando–Kasue–Nakajima, ...) $\rightsquigarrow$ given $D>0$, $v>0$

$\{ g \in \mathcal{E}(M^4,\lambda) : \mathrm{diam}(M,g) \leq D,\ \mathrm{vol}(M,g) \geq v \}$ compact up to Einstein orbifolds



$(\hat{M}, \hat{g})$ Einstein 4-orbifold.

Rescale around orbifold singularity $\rightsquigarrow$ Ricci-flat asymptotically locally Euclidean (ALE) "bubble"

e.g. T^*S^2 Eguchi–Hanson Ricci flat ALE asymptotic to $\mathbb{R}^4/\pm 1$

(Q : Is Eguchi–Hanson unique Ricci-flat ALE asymptotic to $\mathbb{R}^4/\pm 1$)

$\rightsquigarrow \left(K3 = T^4/\pm 1 \ \# 16\, T^*S^2,\ g_\varepsilon \right) \xrightarrow[\varepsilon \to 0]{} \left(T^4/\pm 1,\ \hat{g} \right)$ flat orbifold

[Kummer construction] (T^*S^2's "shrink")

• Cheeger–Naber: Codimension 4 conjecture $g_i \in \mathcal{E}(M^n,\lambda)$ $\forall i$, $\mathrm{vol}(M,g_i) \geq v > 0$

$\Rightarrow$ (up to subsequence) $g_i \to g \in \mathcal{E}(\hat{M},\lambda)$ smooth away from codim 4 set (Hausdorff)

This class : 3 themes

(A) Symmetries – Weeks 2–4

(B) Ricci flow – Weeks 5–6

(C) Perturbation/gluing – Weeks 7–8

Recall : M^n compact, $n \geq 4$

(A) Symmetries

$g \in \mathcal{E}(M,\lambda)$: Bochner formula $\Rightarrow$

• $\lambda < 0 \Rightarrow \nexists$ non-zero Killing fields $\Rightarrow \mathrm{Isom}(M,g)$ finite

• $\lambda = 0 \Rightarrow$ Killing fields parallel (so harmonic)
$\pi_1(M)$ finite $\Rightarrow \mathrm{Isom}(M,g)$ finite

$\rightsquigarrow$ want $\lambda > 0$ for continuous symmetries

Homogeneous : Einstein eq$^n \longleftrightarrow$ algebraic

Finiteness conjecture for $\{\text{homogeneous Einstein metrics on } M\}/\sim$ ③

Recent: (Böhm–Lafuente) <u>Alekseevskii conj</u>: Homogeneous (M^n, g) with $\lambda < 0 \Rightarrow M^n$ <u>diffeo</u> to $\mathbb{R}^n$.

Cohomogeneity one: Einstein eq$^{\underline{n}}$ $\longleftrightarrow$ ODE system

Böhm: ∞-ly many Einstein g with $\lambda > 0$ on S^n, $n = 5, \dots, 9$

Nienhaus–Wink: $\exists$ new Einstein g with $\lambda > 0$ on S^{10}

<u>This class</u>: Buttsworth–Hodgkinson: new Einstein g with $\lambda > 0$ on S^{12}
(uses numerics & perturbation)

(B) <u>Ricci flow</u>

$t \in I \subseteq \mathbb{R} \longmapsto g(t)$ on $M \rightsquigarrow \dfrac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) \left(= -\Delta_L(g) + \text{l.o.t.} \right)$ "heat eq$^{\underline{n}}$"

Challenging for $n \geq 4$ & non-Kähler

<u>Stability</u>

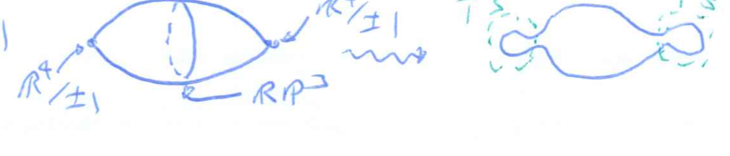
- round S^n <u>stable</u> but $\mathbb{CP}^n$ $(n \geq 2)$ <u>unstable</u>
- Einstein crit pt of $\lambda(g) = \inf\limits_{\int e^{-f} dV_g = 1} \int_M (\mathrm{scal}_g + |\nabla f|_g^2) e^{-f} dV_g$ (Perelman)
- Buzano–Haslhofer: $\hat{g} \in \mathcal{E}(M, 0)$.
  - local max of $\lambda \Rightarrow$ stable
  - not local max $\Rightarrow \exists (g_t)_{t \in (-\infty, 0)}$ s.t. $g_t \to \hat{g}$ as $t \to -\infty$ $\rightsquigarrow$ unstable.

Brendle–Kapouleas: $\left(M^4 = T^4/\pm1 \# 8\, \overline{T^*S^2} \# 8\, \overline{T^*S^2}, g_t\right)_{t \in (-\infty, 0)} \xrightarrow[t \to -\infty]{} (T^4/\pm1, g_0)$ flat
((extended by Ozuch) $\rightsquigarrow$ suggests no Ricci-flat g on M^4 "near" $(T^4/\pm1, g_0)$.
(parabolic gluing))

<u>This class</u>: Deruelle–Ozuch (M^4, g_t) for $t \in (-\infty, 0)$ or $t \in (0, \infty)$
"ancient" "immortal"

s.t. $(M^4, g_t) \xrightarrow[t \to \pm\infty]{} (\hat{M}, \hat{g})$ Einstein 4-orbifold & Ricci-flat ALE "bubble off"
(cf. structure of $\mathcal{E}(M, \lambda)$)

Stability / instability notion for $(\hat{M}, \hat{g}) \longleftrightarrow$ ancient / immortal

e.g. "American football" $S^4/\pm1$ 

$S^4/\pm1 \# 2T^*S^2 = S^2 \times S^2$

<u>Q</u>: Is $\mathcal{E}(S^2 \times S^2, 0) \neq \emptyset$?

(C) Perturbation/gluing

Strategy: $(M,\hat{g})$ "approximate" Einstein ⇝ perturb to $g=\hat{g}+h$ with h "small" s.t. g Einstein

$$Ric(g)-\lambda g = \underbrace{Ric(\hat{g})-\lambda\hat{g}}_{\text{small}} + \frac{1}{2}\Delta_L(h)-\lambda h + \underbrace{Q(h)}_{\text{nonlinear}} \quad \left(\text{modulo diffeos}\right)$$

$\boxed{\text{If}}$ $\frac{1}{2}\Delta_L-\lambda$ has right inverse R_L, $g\in\mathcal{E}(M,\lambda)$ <=>

$$h = R_L\left(\lambda\hat{g}-Ric(\hat{g})-Q(h)\right)$$ ⇝ fixed point/IFT.

This class: Hamenstädt–Jäckel (cf. Fine–Premoselli dim 4)

$\forall n\geq 4$ $\exists$ ∞-ly many non-homeo M^n with $Ric(g)=\lambda g$ with $\lambda<0$ s.t. $K(g)<0$ but not locally symmetric ($\Rightarrow$ not hyperbolic)

M^n: Gromov–Thurston manifolds – certain covers of arithmetic hyperbolic manifolds N^n branched over totally geodesic codimension 2 $\Sigma^{n-2}\subseteq N^n$ with $[\Sigma]=0\in H_{n-2}(N)$.

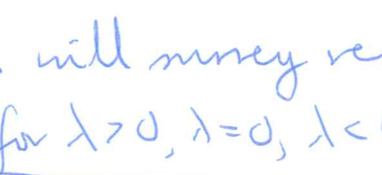
Summary: This class will survey recent progress in Einstein manifolds for $\lambda>0$, $\lambda=0$, $\lambda<0$ with generic holonomy.

Addendum: Variational approach

Einstein ⟷ crit pts of $\mathcal{S}:\{g: Vol(M,g)=1\}\mapsto\int_M scal(g)\,dV_g$ on compact M^n $(n\geq 3)$ (Einstein–Hilbert)

- $\mathcal{S}$ neither bdd above nor below
- $\mathcal{S}(g)$ bounded below for $g\in[g_0]$ fixed conformal class ⇝ crit pt exists in $[g_0]$ has $scal_g$ constant (Yamabe problem)

⇝ suggests min-max



$$\mathcal{Y}(M)=\sup_{[g_0]}\inf_{g\in[g_0]}\mathcal{S}(g) \quad \text{(Yamabe invariant)}$$

Petean: $\pi_1(M^n)=1$, $n\geq 5$ $\Rightarrow$ $\mathcal{Y}(M)\geq 0$ ⇝ min-max fails if $\mathcal{E}(M,\lambda)=\emptyset$ for $\lambda\geq 0$ ⇝ min-max approach not viable in many cases (but $n=4$ is special)