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Ref.

WZ — M. Wang, W. Ziller
Invent Math. 1986
Existence + nonexistence of homogeneous Einstein metrics

BWZ — C. Böhm, M. Wang, W. Ziller
GAFA 2004
A variational approach for compact homogeneous Einstein manifolds.

C. Böhm — JDG 2004
Homogeneous Einstein metrics and simplicial complexes

M Wang — Einstein metrics from symmetry and bundle constructions: a sequel
IntlPress 2012

Homogeneous Einstein metrics

0. look for G-invariant metric on homogeneous space G/K

Algebraic eqns – but nonlinear

need 'positive real' solns.

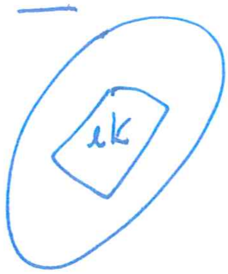
Take G cpct, so $\lambda > 0$

Historical example

Jensen S^{4n+3}

$Sp(n+1)Sp(1)$ invt

1. Basics



$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$
$\curvearrowleft Ad_K$

G-invt metric $\leftrightarrow$ Ad_K-invt $\langle , \rangle$ on $\mathfrak{p}$

so decompose $\mathfrak{p} = \mathfrak{p}_1 \oplus .. \oplus \mathfrak{p}_r$ into K-modules

note Schur $\Rightarrow$ $\mathfrak{p}_i \perp \mathfrak{p}_j$ if $\mathfrak{p}_i \not\cong_K \mathfrak{p}_j$

in fact even if there are repeated summands we can find some decomposition wrt which $\langle , \rangle$ is diagonal.

"Complications of space of decompositions"

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consider diagonal metric

$$\langle , \rangle = x_1 Q|_{p_1} \underset{\perp}{\oplus} \dots \underset{\perp}{\oplus} x_r Q|_{p_r}$$

Q is a background Ad-inv metric eg Killing form if G simple

$$\left([{}^{k}_{ij}] = \sum_{\substack{e_\alpha \in m_i \\ e_\beta \in m_j \\ e_\gamma \in m_k}} Q([e_\alpha, e_\beta], e_\gamma)^2 \right.$$

$\left. e_s \text{ } Q\text{-onb for } p \text{ adapted to } p = \oplus p_i \right)$

Ex $r = 1$: isotropy irreducible now $Ric = \lambda g$ by Schur! Examples (WZ).

Idea Direct approach to Einstein eqns is tricky for the reasons given above.

Use a variational approach, exploiting the fact that

$$\text{Einstein (vol 1)} \iff \text{critical for } T : g \mapsto \int S(g)\, dvol_g \text{ on metrics of vol 1}$$

('total scalar' ; scalar curv.)

$$\boxed{\text{so in homogeneous case, } \iff \text{critical for scalar curvature on vol 1 } G\text{-inv metrics}}$$

[**note** $dT_g(h) = \int_M \langle \frac{S}{n} g - \mathrm{Ric}(g), h \rangle \, d\mathrm{vol}_g$

$h \in \mathrm{Sym}^2$, $\int \mathrm{tr}_g h = 0$

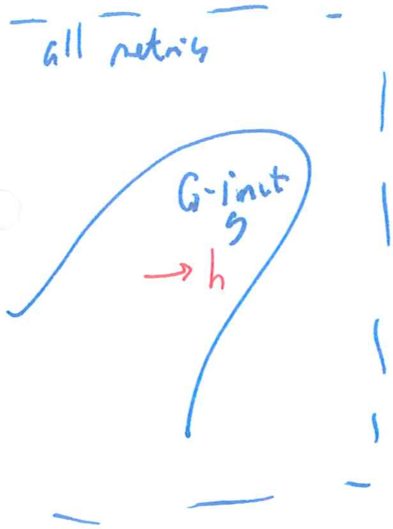
and can choose h to be G-invt, so if g is critical for $T|_{\text{G-invt vol 1 metrics}}$

$h = \frac{S}{n} g - \mathrm{Ric}(g)$

"symmetric criticality"

then $\frac{S}{n} - \mathrm{Ric}(g) = 0$

so g is Einstein]



In general, variational methods have not been very successful in proving existence of Einstein metrics

In homogeneous case, things are easier

as M_G^1 = vol 1 G-invt metrics is a finite-dimensional space.

For diagonal metric

$$\langle\,,\rangle = x_1 Q|_{p_1} \underset{\perp}{\oplus} \cdots \underset{\perp}{\oplus} x_r Q|_{p_r}$$

$(x_i > 0)$

$$\left[{k \atop ij}\right] = \sum_{\substack{e_\alpha \in m_i \\ e_\beta \in m_j \\ e_\gamma \in m_k}} Q([e_\alpha, e_\beta], e_\gamma)^2$$

e_i Q-onb for p
adapted to $p = \oplus p_i$

Also b_i $Q|_{m_i} = B|_{m_i}$: $d_i = \dim m_i$

scalar curvature is

$$\boxed{S = \frac{1}{2}\sum_{i=1}^{r} \frac{d_i b_i}{x_i} - \frac{1}{4}\sum_{i,j,k} \left[{k \atop ij}\right]\frac{x_k}{x_i x_j}}$$

<u>Note</u> 1) This is homogeneous of degree -1

2) $\left[{k \atop ij}\right] \geq 0$, $=0$ iff $[m_i, m_j] \perp_Q m_k$

$b_i \geq 0$

3) coeffs of $\frac{x_k}{x_i x_j}$ i,j,k distinct are ≤ 0

" $\frac{x_k}{x_j^2}$ ≤ 0

" $\frac{1}{x_i}$ ≥ 0
$=0$ iff $m_i \subset m_0$
$[m_i, m_j] \subseteq m_j$
$\forall j$

4) submersions

$$\begin{array}{ll} G/K \leftarrow H/K & t \cdot g_V \\ \downarrow & \\ G/H & g_H \end{array}$$

$$S = t^{\frac{\dim H/K}{\dim G/K}}\left(\frac{1}{t} S(H/K) + S(G/H) - t\,\|A\|^2\right)$$

$\boxed{\text{so} \to \infty \text{ as } t \to 0 \text{ if } S(H/K) > 0 \text{ as normal}}$ $\boxed{\to 0 \text{ if tom fibre}}$

we see the structure of the lattice of subgroups intermediate between K and G will be important!

<u>Variational approach</u> look for critical points for S on M^1_G = G-invt vol 1 metrics

One problem is – M_G is a real cone – no topology! even true for M^1_G

<u>Boundedness</u>?

S is hardly ever bounded below (products of isotropy ireds).

But we have

<u>Thm</u> (WZ)

if K is <u>maximal</u> (connected) in G

then S is bounded <u>above</u> and proper, so has a global max which is Einstein.

[actually this is iff]

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idea

maximal

$\Rightarrow$

lots of nonzero

$\left[{k \atop ij}\right]$

$\exists a > 0$:

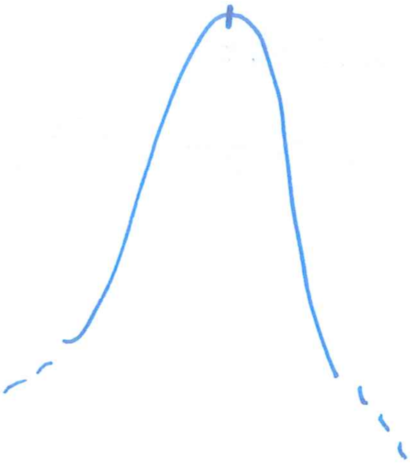
$\forall I \subsetneq \{1 \dots r\}$

$\neq \emptyset$

$\exists i, j \in I$

$k \notin I$

$\left[{k \atop ij}\right] \geq a$



Lots of examples!

Dynkin eg. $\frac{SO(pq)}{SO(p)SO(q)}$: $p, q \geq 3$, $(p,q) \neq (4,4)$

$\frac{SO(n)}{\pi(H)}$

π irrep of H

Nonexistence

Can find 2-summand examples $\mathfrak{g} = \underbrace{\mathfrak{k} \oplus \mathfrak{p}_1}_{\mathfrak{h}} \oplus \mathfrak{p}_2$ where $\nexists$ inv't Einstein metric

lowest-dim example $\frac{SU(4)}{SU(2)}$ where $SU(2) \subset Sp(2) \subset SU(4)$

More general situation – saddle points rather than maxima

Graph Thm (Böhm–Wang–Ziller)

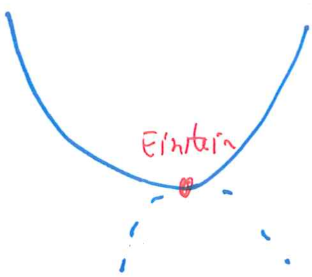
Introduce graph of Ad_k-inv strictly intermediate subalgebras $k \subset h \subset g$

[if $\exists$ its family take 1 rep. in each component]

A vertex is toral if h/k is Abelian

Thm (BWZ)

If the graph has ≥ 2 non-toral components then $\exists$ invariant Einstein metric.



Idea each component, if non-toral gives a region in which $S \to \infty$

[cf. p. 4 uses eg normal metrics to get $S(F) > 0$]

now use Mountain-Pass and Palais-Smale, valid on $S \geq a$ for $a > 0$

(ie $S(g_i)$ bounded and $\nabla S(g_i) \to 0$ in L^2 $\Rightarrow \exists$ subseq. converging in M_1^G in C^∞)

$\uparrow$

Cheeger–Gromov

This produces metrics of coindex 0 or 1

Böhm generalised these ideas,

producing not just a graph

but a <u>simplicial complex</u>

Roughly, i-simplices are <u>flags</u> of length i

$\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \dots \subset \mathfrak{h}_i$

of intermediate algebras

[need to modify this if $N(K)/K$ has +ve dim]

<u>Thm</u> (Böhm) if this simplicial complex is non-contractible,

$\exists$ Einstein metric

Looking at H_0 gives graph thm

In general $H_q \neq 0 \rightsquigarrow$ coindex $\geq q+1$

Ex (BWZ)

$G = U(p+q+r)$

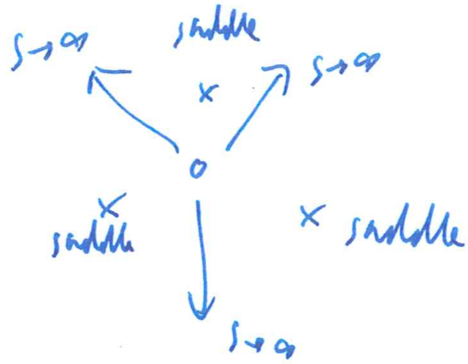
$K = U(p)U(q)U(r)$

graph is : three isolated vertices

$U(p+q)U(r)\,\bullet$ $\quad\bullet\, U(p)U(q+r)$

$\bullet\, U(p+r)U(q).$

$\exists$ inv't Einstein metrics



$\exists$ saddles are Kähler Einstein

o is a non-Kähler min.

Ex $G_2/U(2) = Gr_2^+(\mathbb{R}^7)$

graph is

$\bullet\, SU(3)$

$\bullet\, SO(4)$

$G_2/SU(3) = S^6$

$G_2/SO(4) =$ 8-dim Wolf space

$\exists$ crit. pt of coindex 1

but also 2 other critical pts (one of them symmetric)

8b)

Ex.

$$M_r = \frac{S^3 \times S^3}{S^1_r} \overset{\text{diffeo}}{\cong} S^3 \times S^2$$

S^1_r is embedded by integers p, q with $r = p/q$

$0 \le r \le 1$

$r = 0 \rightsquigarrow S^3 \times S^2$

$r = 1 \rightsquigarrow \frac{SO(4)}{SO(2)}$ Stiefel.

$$S = \frac{8}{b} + \frac{8}{c} - \frac{2a}{(1+r^2)b^2} - \frac{2ar^2}{(1+r^2)c^2}$$

$$\begin{array}{l} M_r \overset{a}{\leftarrow} S^1 \\ \downarrow \\ \frac{S^3 \times S^3}{T^2} = \underset{b}{S^2} \times \underset{c}{S^2} \end{array}$$

0-Palais-Smale sequence from $a \to 0$ directions

$\exists$ Einstein metric even though graph is connected.

Another Ex with graph connected

$$\frac{SO(9)}{SO(3)U(1)SO(3)} \qquad \frac{SO(10)}{SO(3)U(1)SO(4)}$$

graph connected

$U(3)SO(k)$ ——— $SO(6)SO(k)$

$k = 3, 4$

S takes same form with different constants

$\nexists$ inv't Einstein

$\exists$ 2 inv't Einstein

so existence is 'unstable'

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Negative λ case

so G/K noncompact.

Work of Heber, Lauret, Böhm, Lafuente
Alekseevski,

Alekseevski Conjecture

proved recently
by Böhm-Lafuente

$(G/K, g)$ homogeneous Einstein $\lambda < 0$ is diffeomorphic to $\mathbb{R}^n$

and isometric to a
solvmanifold
(homogeneous WRT
solvable group of
isometries)

Use of real GIT