Cohomogeneity One Einstein metrics

G acts with generic ("principal") orbits G/K of (real) codimension 1.

Einstein eqn. $Ric_g = \lambda g$ for (M, g) cohom. 1

Special orbits $G/H \rightsquigarrow H/K$ sphere

Soliton eqn.

$Ric_g + \underbrace{L_X g}_{\text{Hess}(u)} = \lambda g$

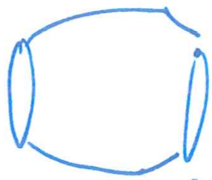
if $X = \nabla u$

$k > 0$

$\underline{\lambda > 0}$ Now M compact

2 special orbits

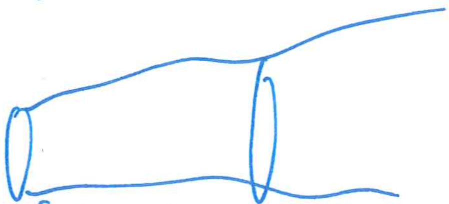
[$M \leftarrow G/K$
$\downarrow$
S^1 would $\Rightarrow$ infinite π_1 $\cancel{\because}$ Myers]



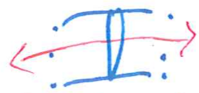
G/H_1 G/H_2

$H_i/K = S^{d_i}$

$\underline{\lambda = 0}$ if irreducible, must be one special orbit as if none, Cheeger-Gromoll $\Rightarrow \exists$ line



G/H G/K



which splits off as $Ric \geq 0$

$g = dt^2 + g_t$: $\dot{g} = 2gL$

↑ shape op.

Eqns.

tgt to G/K

$$\boxed{-\dot{L} - (\mathrm{tr}\, L) L + r = \lambda I}$$

nonlinear ↑ nonlinear

r is Ricci endomorphism

$Ric_{g_t}(x,y) = g_t(r(x),y)$

Also conservation law $((\mathrm{tr}\, L)^2 - \mathrm{tr}(L^2) - S + (n-1)\lambda = 0)$

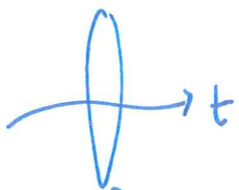
normal to G/K

$-\mathrm{tr}(\dot{L}) - \mathrm{tr}(L^2) = \lambda$

follows from above eqn if $\exists$ special orbit

Hamiltonian system

Diagonalisability?



G/K

$t = t_0$

Can diagonalise at some $t = t_0$

in some cases

$Ric(X, \partial/\partial t) = 0 \Rightarrow$ diagonal $\forall$ time

eg Bianchi IX

but this is not always the case.

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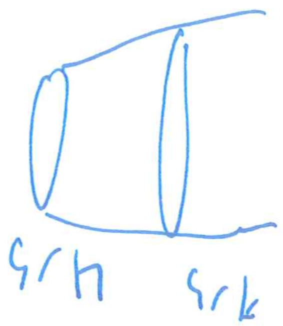
initial conditions at special orbit



$$\mathfrak{g} = \underbrace{\underbrace{\mathfrak{k} \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s}_{\mathfrak{h}}}_{\text{collapsing part}} \oplus \underbrace{\mathfrak{p}_{s+1} \oplus \cdots \oplus \mathfrak{p}_r}_{T(G/H)}$$

initial value problem. Eschenburg-Wang.

$1/t$ term

$\exists$ local soln. in neighbourhood of special orbit

(subj. to mild conditions)

Important Example

Bianchi IX

dim $M = 4$

$SU(2)$ acts, 3-dim principal orbit

special orbits — nut pt

bolt S^2 or $\mathbb{RP}^2$

write $g = (abc)^2 d\eta^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2$

[recall can diagonalize $\forall$ time]

$Ric = 0 \iff$

$$2\left(\frac{a'}{a}\right)' = a^4 - (b^2-c^2)^2 \qquad \circlearrowleft$$

[conservation law]

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Difficult in general!

Subsystems

1) Enhance $SU(2)$ to $U(2)$ action.

ie. S^3 orbits are Berger spheres –

S^1 submersion over $S^2 = \mathbb{CP}^1$

$$dt^2 + f^2 d\theta^2 + h^2 g_{S^2}$$ 2 fns.

$(h^2(\sigma_2^2 + \sigma_3^2))$

Solvable — Taub-Bolt.

includes Taub-NUT, Eguchi-Hanson

nut. $\mathbb{R}^4$ — S^2, $T^*S^2 = \mathcal{O}(-2)$ over $\mathbb{P}^1$

Also $\lambda > 0$ example

Page $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

Hermitian, not Kähler

2) Hyperkähler (loc. $\Leftrightarrow$ $Ric = 0$ and $W_+ = 0$) $\Leftrightarrow$ Ricci flat Kähler in $\dim_{\mathbb{R}} = 4$)

Two cases - have $\mathbb{R}^3$ of cov.c $\parallel$ 2-forms, so $SU(2)$ can act trivially or by rotation

i) $SU(2)$ acts trivially on $\mathbb{R}^3$ of $\parallel$ 2-forms — contrast to generic Kähler

Triholomorphic case

$$2\frac{a'}{a} = b^2 + c^2 - a^2 \quad \circlearrowleft$$

$\Leftrightarrow$ $\dot{w}_1 = w_2 w_3$ $\circlearrowleft$ Jacobi fns BGPP metrics

generically incomplete

Complete solns given by hyperbolic fns $\therefore$

Eguchi-Hanson

ALE

S^2

$T^*\mathbb{CP}^1$

I

J, K affine

$xy = z^2 + c$

(flat space $\mathbb{C}^2/\mathbb{Z}_2$ limit)

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ii) $SU(2)$ rotates $\mathbb{R}^3$ of ASD 2-forms

$$2a'/a = (b-c)^2 - a^2$$

Halphen

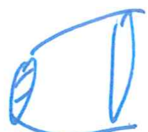
Harder, but can solve via elliptic $\int$

Complete solns are

1) Taub-NUT on $\mathbb{R}^4$ ($U(2)$-invt) (and flat)

ALF

2) Atiyah-Hitchin on $SO(3)/\mathbb{Z}_2\times\mathbb{Z}_2$ (triaxial)



$\mathbb{RP}^2$

$= S^4 - \mathbb{RP}^2$

$\Lambda < 0$ selfdual Einstein

Twistor – isomonodromy – Painlevé VI

Generalisations

Bérard-Bergery : generalises $U(2)$-inv't B $\underline{X}$

$$\begin{array}{ccc} S^3 & \overset{+}{\leftarrow} & S^1 \\ \downarrow & & \\ h \; S^2 & & \end{array} \quad \text{replaced by} \quad \begin{array}{ccc} P & \overset{+}{\leftarrow} & S^1 \\ \downarrow & & \\ h \; N & & \end{array}$$

Fano K-E

[Bundle construction not nec. cohom1]

$\underline{\text{solvable}}$ system. Get compact Hermitian $\lambda > 0$ examples

generalising Page metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

noncompact complete $Ric = 0, < 0$ some Hermitian, Kähler

Wang-Wang generalise to $P \leftarrow S^1$, $\downarrow$ $N_1 \times \dots N_r$

D-Wang Ricci $\underline{\text{solitons}}$ $Ric_g + L_X g = \lambda g$

($r=1$ case FIK) $\mathbb{CP}^n$

Existence results

Böhm multiple warped products $dt^2 + f_1^2(t) g_{M_1} \oplus \cdots \oplus f_r^2(t) g_{M_r}$

complete Ricci flat examples

D – Wang: Ricci solitons from multiple warped product

steady, expanders

$Ric_g + Hess(u) = \lambda g$

$\lambda = 0$ steady
$\lambda < 0$ expand.
$\lambda > 0$ shrinker

($r = 1$ is Bryant I)

$Ric = 0$ special structure if

$r = 2$ $(d_1, d_2) = (3,6)$, $(5,5)$, $(2,8)$

Sble : superpotential

$r = 3$ $(d_1, d_2, d_3) = (2,4,4)$, $(3,3,3)$, $(2,3,6)$

superpotential

solvable subsystem

exceptional holonomy : Gibbons, Page, Lu, Cvetic etc

Bryant Salamon

$\mathbb{C}P^2$ $\frac{SU(3)}{T^2}$ biaxial

eg $\mathbb{C}P^2$ $\frac{SU(3)}{U(1)}$ $\boxed{Spin(7)}$

AW

ALC (generic) $<$

AC $<$

$S^{4k+3} = \frac{Sp(k+1)U(1)}{Sp(k)U(1)}$ $k=1$ $S^7 = \frac{Sp(2)U(1)}{Sp(1)U(1)}$

$\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{H}^k$

Foscolo-Haskins G_2 nearly Kähler S^6, $S^3 \times S^3$.

special orbits : pt or S^4

$\mathbb{A}_8$ $\mathbb{B}_8$

Compact examples $\lambda > 0$

Böhm examples on S^n : $5 \leq n \leq 9$

$S^3 \times S^2 \ldots$

Failure of Palais-Smale

Cohomol , generic holonomy

$SO(k+1) \times SO(l+1)$

$\curvearrowright S^{k+l+1}$ principal orbit $S^k \times S^l$

Wink-Nienhaus S^{10} three exotic Einstein metrics

H. Chi examples $\mathbb{H}P^{m+1} \# \mathbb{H}P^{m+1}$: S^{4m+3} principal orbit