New Einstein metric on S^{12} — Buttsworth — Hodgkinson
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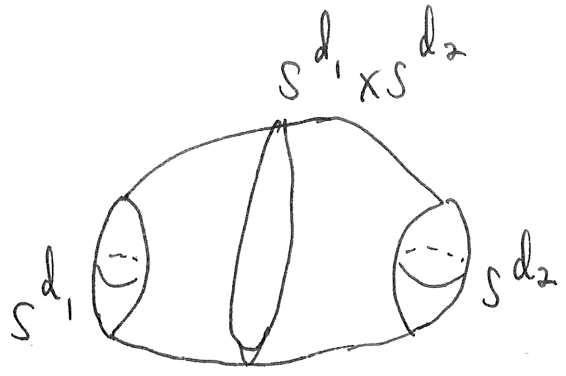
$S^{d_1+d_2+1}$ under the usual action of $O(d_1+1)\times O(d_2+1)$.

If $d_1=1$ or $d_2=1$, any invariant Einstein metric is round.

Let $n=d_1+d_2+1$. If $n=5,6,7,8,9$, $\exists$ ∞-ly many non-round metrics (Böhm).

3 new non-round metrics on S^{10}, for $(d_1,d_2)\in\{(2,7),(3,6),(4,5)\}$, (Nienhaus – Wink).



$$g = dt^2 + f_1^2\, Q_{d_1} + f_2^2\, Q_{d_2}.$$

$$f_1(0)=\frac{\sqrt{d_1-1}}{\alpha},\quad f_2(0)=0,\quad f_2'(0)=1$$

$$f_1(T)=0,\quad f_1'(T)=-1,\quad f_2(T)=\frac{\sqrt{d_2-1}}{\omega}.$$

Stopping condition $\dfrac{d_1 f_1'}{f_1}+\dfrac{d_2 f_2'}{f_2}=0$ reached at time t_α, t_ω

Let $A(\alpha)=\left(\dfrac{\sqrt{d_2-1}}{f_2(t_\alpha)},\ \dfrac{-d_2 f_2'(t_\alpha)}{f_2(t_\alpha)}\right)$, $\Omega(\omega)=\left(\dfrac{\sqrt{d_2-1}}{f_2(T-t_\omega)},\ \dfrac{-d_2 f_2'(T-t_\omega)}{f_2(T-t_\omega)}\right)$

If $A(\alpha)=\Omega(\omega)$, then $\exists$ smooth Einstein metric.

(A priori 4 functions to match.
2 in $A(\alpha)=\Omega(\omega)$
1 in the conservation law
1 in the stopping condition).

$$W = \frac{\sqrt{d_2 - 1}}{f_2}, \quad X = \frac{\sqrt{d_1 - 1}}{f_1}, \quad Y = \frac{d_1 f_1'}{f_1}, \quad Z = d_1 \frac{f_1'}{f_1} + d_2 \frac{f_2'}{f_2}$$

$$X(t) = \alpha + \eta_1(t), \quad Y(t) = \eta_2(t), \quad Z(t) = \frac{d_2}{t} + \eta_3(t)$$

Initial value problem

$$\eta'(t) = \frac{1}{t} L_{d_1 d_2} \eta(t) + B_{d_1 d_2}(\eta(t), \eta(t)), \quad \eta(0) = (0, 0, 0, \alpha, \sqrt{\lambda})$$

$$L_{d_1 d_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

B is some symmetric bilinear fun depending only on d_1, d_2, and is zero on the last 2 components.

Assumption 1. $\exists\ \hat{\alpha}, \hat{\omega}, \hat{t}_{\hat{\alpha}}, \hat{t}_{\hat{\omega}} \in \mathbb{R}$, smooth functions

$$\hat{\eta} : [0, \hat{t}_{\hat{\alpha}} + 10^{-3}] \longrightarrow \mathbb{R}^5, \qquad \hat{\xi} : [0, \hat{t}_{\hat{\omega}} + 10^{-3}] \longrightarrow \mathbb{R}^5$$

such that the following are $< \varepsilon$:

- The H^3 norms of

$$\hat{E}_{1,\text{start}}(t) = \frac{L_{29}\, \hat{\eta}(t)}{t} + B_{29}(\hat{\eta}(t), \hat{\eta}(t)) - \hat{\eta}'(t)$$

$$\hat{E}_{1,\text{end}}(t) = \frac{L_{92}\, \hat{\xi}(t)}{t} + B_{92}(\hat{\xi}(t), \hat{\xi}(t)) - \hat{\xi}'(t)$$

on $[0, \hat{t}_{\hat{\alpha}} + 10^{-3}]$ and $[0, \hat{t}_{\hat{\omega}} + 10^{-3}]$ respectively.

- $|\hat{A}(\hat{\alpha}) - \hat{\Omega}(\hat{\omega})|$ (the shooting problem error)
- The stopping time error $\left|\frac{9}{\hat{t}_{\hat{\alpha}}} + \hat{\eta}_3(\hat{t}_{\hat{\alpha}})\right|$ and $\left|\frac{2}{\hat{t}_{\hat{\omega}}} + \hat{\xi}_3(\hat{t}_{\hat{\omega}})\right|$.

Furthermore, we assume some knowledge (~7 digits) of $\hat{\eta}(t_\alpha)$, $\hat{\xi}(t_\omega)$, α, ω, t_α, t_ω, and upper bounds on the L^2, H^1, H^2, H^3 norms of η, ξ.

Assumption 2. ~~$\exists$ $\hat{\alpha}$, $\hat{\omega}$, t~~ Let $\hat{\eta}$, $\hat{\xi}$ be as in Ass. 1.

$\exists$ functions $\hat{\mu}_{Ic}$, $\hat{\nu}_{Ic}$, initially $(0,0,0,1,0)$, with good (7 digits) control on $\mu_{Ic}(t_\alpha)$, $\mu_{Ic}'(t_\alpha)$, $\nu_{Ic}(t_\omega)$, $\nu_{Ic}'(t_\omega)$ $\|\mu_{Ic}\|_{H^3}$, $\|\nu_{Ic}\|_{H^3}$, and the diff. eq. errors, below have H^3 norm $< \varepsilon$:

$$\hat{F}_{2,start}(t) = (\hat{\mu}_{Ic})'(t) - \frac{1}{t} L_{2\xi} \hat{\mu}_{Ic}(t) - 2 B_{2\xi}(\hat{\eta}(t), \hat{\mu}_{Ic}(t))$$

etc.

C^k estimates.

$$k_0 = f(t_0) \quad t_0 \in I$$

Sobolev embeddings.

$$\|f - f_0\|_{C^0} \le \frac{5}{4} \|f'\|_{L^2}$$

$$\|f\|_{C^0} \le 2 \|f\|_{H^1}$$

Lemma 2. C^0, C^1, C^2 control on $\hat{\eta}$, $\hat{\xi}$.

Lemma 3. C^2 control on $\hat{\mu}_{Ic}$, $\hat{\nu}_{Ic}$.

Lemma 4. $\hat{E}_1$, $\hat{F}_2$ (start/end) are ~~C^2-small~~ C^2-ε-small.

Linearisation. Let $\mu(\alpha,t) = \eta(\alpha,t) - \hat\eta(t)$, $\nu(w,t) = \xi(w,t) - \hat\xi(t)$, so that
$$\mu'(t) = \frac{L_{24}\mu(t)}{t} + 2B_{24}(\hat\eta(t), \mu(t)) + \hat{E}_{1,\text{start}}(t) + B_{24}(\mu(t),\mu(t)),$$
$\mu(0) = (0, 0, 0, \alpha - \hat\alpha, 0)$, etc.

We linearise to $\mu'(t) = \dfrac{L_{24}\,\mu(t)}{t} + 2B_{24}(\hat\eta(t), \mu(t)) + F_{\text{start}}(t)$,

and write $\mu = S_{24} F_{\text{start}}$.

Prop. 4. $S_{24} : L^2([0, \hat t_{\hat\alpha} + 10^{-3}]) \longrightarrow C^0([0, \hat t_{\hat\alpha} + 10^{-3}])$ is bounded, with norm $\leq M = e^{250}$.

$S_{24} : C^k \longrightarrow C^{k+1}$, $k = 0, 1$ are also bdd.

proof. L_{24} is negative definite, and $|B_{d_1 d_2}(X, Y)| \leq 3(d_1+1)|X||Y|$, so the result follows by Grönwall's inequality on $|\mu(t)|^2$.

From the Appendix:

Consider $\mu'(t) = \frac{1}{t} L_{d_1 d_2}\mu(t) + F(t)$, $\mu(0) = 0$.

The $\mathcal{L}_{d_1 d_2} : C^k \to C^{k+1}$ on $[0, t^*]$, $t^* \leq 1.5$, $d_2 \in \{2, 4\}$, defined by $\mu = \mathcal{L}_{d_1 d_2} F$, is bdd for $k = 0, 1$.

$\mu = \mathcal{L}_{24}(F_{\text{start}} + 2B_{24}(\hat\eta, \mu))$.

Then $\|\mu\|_{C^1} \leq C_1 \|F_{\text{start}}\|_{C^0} + C_2 \|\mu\|_{C^0}$

$\leq C_3 \|F_{\text{start}}\|_{C^0}$.

Similarly for the C^2 estimate on μ.

Consider the expansion

$$\mu(\alpha,t)=\mu^{(0)}(t)+(\alpha-\hat{\alpha})\,\mu^{(1)}(t)+(\alpha-\hat{\alpha})^2\mu^{(2)}(t,\alpha).$$

If we put this into the ODEs, morally speaking, the 0^{th} order term in $\alpha-\hat{\alpha}$ satisfies the model

$$(\mu^{(0)})'(t)=\frac{L_{2q}\mu^{(0)}(t)}{t}+2B_{2q}(\hat{\eta}(t),\mu^{(0)}(t))+\hat{\tilde{E}}_{1,\text{start}}(t)+B_{2q}(\mu^{(0)}(t),\mu^{(0)}(t))$$

$$\mu^{(0)}(0)=0.$$

<u>Lemma 5</u>. If $M^2\varepsilon\le\frac{1}{2001}$ then we have

$$\|\mu^{(0)}\|_{C^k}\le 10^{c_k}M\varepsilon,\quad c_k=1,\,6,\,20.$$

<u>proof</u>.

$$\mu^{(0)}=S_{2q}\left(\tilde{E}_{1,\text{start}}+B_{2q}(\mu^{(0)},\mu^{(0)})\right).$$

As $S_{2q}:L^2\to C^0$ is bdd, a solution $\mu^{(0)}$ can be constructed from the Schauder fixed point theorem, with $\|\mu^{(0)}\|_{C^0}$ ε-controlled. Bootstrap up to get C^1, C^2 estimates on $\mu^{(0)}$.

Morally speaking, the 1^{st} order terms in $\alpha-\hat{\alpha}$ satisfy

$$\mu_{IC}'(t)=\frac{1}{t}L_{2q}\mu_{IC}(t)+2B_{2q}(\hat{\eta}(t),\mu_{IC}(t)).$$

<u>Lemma 6</u>. $\|\mu^{(1)}-\mu_{IC}\|_{C^k}\le 10^{c_k}M^2\varepsilon,\quad k=0,1,2.$

<u>proof</u>. We get the fixed-point equations

$$\mu^{(1)}-\mu_{IC}=S_{2q}\left(2B_{2q}(\mu^{(0)},\mu^{(1)}-\mu_{IC})+2B_{2q}(\mu^{(0)},\mu_{IC})\right)$$

to which we can apply the Schauder fixed point theorem.
to get a solution $\mu^{(1)} - \mu_{IC}$ which is small in C^0.

<u>Lemma 7</u>. For small enough $|\alpha - \hat{\alpha}|$ and ε, $\|\mu^{(2)}\|_{C^K}$ is bdd.

Write in fixed-point form again & get $\mu^{(2)}$ using Schauder.

<u>Cor. 2</u>.

$$\mu(\alpha, t) = \mu^{(0,0)} + \mu^{(1,0)}(\alpha - \hat{\alpha}) + \mu^{(0,1)}(t - \hat{t}_\alpha) + \mu^{(1,1)}(\alpha - \hat{\alpha})(t - \hat{t}_{\hat{\alpha}})$$
$$+ (t - \hat{t}_{\hat{\alpha}})^2 \left(\mu^{(0,2)}(t) + (\alpha - \hat{\alpha})\mu^{(1,2)}(t)\right)$$
$$+ (\alpha - \hat{\alpha})^2 \mu^{(2,0)}(\alpha, t), \quad \text{with}$$

$|\mu^{(0,0)}|$, $|\mu^{(0,1)}|$, $|\mu^{(1,0)} - \hat{\mu}_{IC}(\hat{t}_{\hat{\alpha}})|$, $|\mu^{(1,1)} - \hat{\mu}_{IC}'(\hat{t}_{\hat{\alpha}})|$, $\|\mu^{(0,2)}\|_{C^0}$ ε-small, and $\|\mu^{(1,2)}\|_{C^0}$, $\|\mu^{(2,0)}\|_{C^2}$ bdd.

Next, note that the stopping times satisfy

$$0 = \eta_3(\alpha, t_\alpha) + \frac{q}{t_\alpha}.$$

$\varepsilon, |\rho^{(0)}|$

We then get estimates on $|t_{\hat{\alpha}} - \hat{t}_{\hat{\alpha}}| \leq 10^3 M\varepsilon$,

as well as expansions $t_\alpha - t_{\hat{\alpha}} = (\alpha - \hat{\alpha})\rho^{(1)} + (\alpha - \hat{\alpha})^2\rho^{(2)}(\alpha)$

Combined to be $t_\alpha - \hat{t}_{\hat{\alpha}} = \rho^{(0)} + (\alpha - \hat{\alpha})\rho^{(1)} + (\alpha - \hat{\alpha})^2\rho^{(2)}(\alpha)$.

Existence. Expand $\eta(t_\alpha) = \eta^{(0)} + (\alpha - \hat{\alpha})\eta^{(1)} + (\alpha-\hat{\alpha})^2 \eta^{(2)}(\alpha)$

and get ε-control on $|\eta^{(0)} - \hat{\eta}(\hat{t}_{\hat{\alpha}})|$, and bds on $\eta^{(1)}, \eta^{(2)}$.

We then get

$$A(\alpha) = A^{(0)} + (\alpha - \hat{\alpha}) A^{(1)} + (\alpha - \hat{\alpha})^2 A^{(2)}(\alpha),$$

$$\Omega(\omega) = \Omega^{(0)} + (\omega - \hat{\omega}) \Omega^{(1)} + (\omega - \hat{\omega})^2 \Omega^{(2)}(\omega).$$

$|A^{(0)} - \Omega^{(0)}|$ ε-small, $A^{(1)}, \Omega^{(1)}$ known,

$\| A^{(2)}(\alpha) | + | \Omega^{(2)}(\omega) \|$ bounded.

$\longrightarrow$ $\exists$ solution by Schauder fixed-point on

$$\begin{pmatrix} \alpha - \hat{\alpha} \\ \omega - \hat{\omega} \end{pmatrix}.$$

Numerical scheme.

In practice, they need $\varepsilon \sim 10^{-350}$ precision.

Order k finite difference methods (Runge-Kutta)

give $\mathcal{O}(\delta^k)$ ~~prec~~ precision, when δ is the timestep.

$\hookrightarrow$ not good enough!

Solution: Use power series methods.

$\hookrightarrow$ Frobenius' method near 0

$\hookrightarrow$ use half the radius of convergence, estimated w/ root test.

Polynomial interpolation for rigorous numerics.

$\rightsquigarrow$ Chebyshev polynomials $T_n(\cos\theta) = \cos(n\theta)$.

They are an $\perp$ basis on $L^2([-1,1])$.

By Riemann-Lebesgue the coefficients decrease.

$\rightsquigarrow$ products, integrals, derivatives of T_n's are T_n's.

Take finite diff. approx $\hat{\mu}_{IC}(t,\alpha) = \dfrac{\hat{\eta}(t,\hat{\alpha}+\delta) - \hat{\eta}(t,\alpha-\delta)}{2\delta}$

works well for Assumption 2.