References:

1) Orbifold singularity formation along ancient and immortal Ricci flow (Deruelle, Ozuch)

2) Noncollapsed degeneration of Einstein 4-manifolds I & II (Ozuch)

3) Integrability of Einstein deformations and desingularisation

M^4 smooth manifold

$$\mathcal{E}(M) = \left\{ (M,g) \;\middle|\; \exists \Lambda \in \mathbb{R} \text{ s.t. } \begin{array}{l} \mathrm{Ric}(g) = \Lambda g, \ \mathrm{vol}(M^4, g) = \text{[illegible]} \\ \mathrm{diam}(M,g) \leq \alpha \end{array} \right\} \Big/ \mathrm{Diff}(M)$$

compact up to Einstein orbifolds (see Jason's talk).

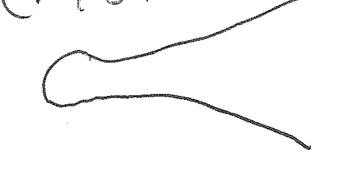
Natural question: Are all Einstein orbifolds limits of smooth Einstein (G-H sense)?

This is related to the problem of desingularizing an Einstein orbifold.

Let (M_o^4, g_o) be a Einstein orbifold with isolated singularities. Then the singularities look like $\mathbb{R}^4/\Gamma$, for Γ a finite group acting on $\mathbb{R}^4$.

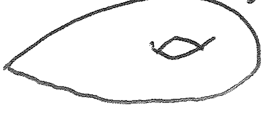
Naive desingularisation: Glue in an ALE Ricci flat metric (N, g) into (M_o, g_o):

(N, g)



$\#$

(M_o, g_o)



$=$

$(N \# M_o, g \# g_o) := (N \# M_o, g^D_t)$

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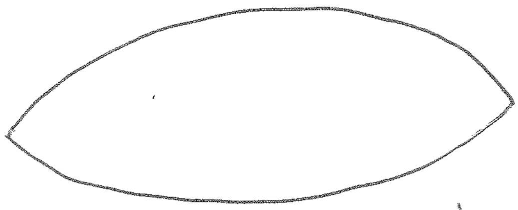
Example 1:

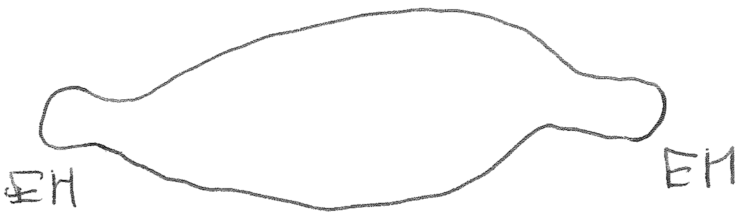
$S^4 \subset \mathbb{R}^5$ quotient by $\mathbb{Z}_2$

$(x_1, x_2, x_3, x_4, x_5) \sim (x_1, -x_2, -x_3, -x_4, -x_5)$



two $\mathbb{R}^4/\mathbb{Z}_2$ singularities

Glue into two Eguchi–Hansons:



$S^2 \times S^2$.

Example 2:

Glue in EH into $\mathbb{H}^4/\mathbb{Z}_2$

Q: Can we perturb $g \# g_0 =: g^D$ into an Einstein metric on $N \# M_0$?

Einstein metrics are zeros of

$$E(g) = Ric(g) - \frac{R(g)}{2} g + \frac{1}{4}\bar{R}(g)\, g .$$

~~Not~~ $0 = E(g)$ is not elliptic

gauge fix: $0 = E(g) + \underbrace{\mathcal{F}_{g^D}(g)}_{\substack{\text{term to} \\ \text{gauge} \\ \text{fix}}} =: E(g) + \mathcal{E}_{g^D}(g),$

Study the linearization on volume preserving deformations at g^D (h s.t. $\int_M \mathrm{tr}_{g^D} h \, dV_{g^D} = 0$)

$$\bar{P}_{g^D}(h) = d_{g^D}(\mathcal{E}_{g^D}(h)) .$$

If g^D Einstein and h is divergence free then the operator reduces to

$$P_{g^D} := \frac{1}{2}\nabla^*_{g^D}\nabla_{g^D} - \mathring{R}_{g^D} .$$

$O(g)$ kernel of P_g Hölder spaces

$O(g_0)$ kernel of P_{g_0}

$\tilde{O}(g^D_t)$ space of truncated obstructions

(see ~~page~~ definition 4.4 in second reference).

Thm (Ozuch 1, 2) Let (M_0^4, g_0) be a compact Einstein orbifold and (N, g) an ALE Ricci flat metric. ~~with~~ Assume t is small enough. Then there is $\epsilon > 0$ and a unique metric g on $N \# M_0^4$ solving $g^D(g) \in \tilde{O}(g^D)$ and:

i) $\|g - g^D\| \leq 2\epsilon$

ii) $g - g^D$ is $L^2(g^D)$ orthogonal to $\mathcal{O}(g^D_t)$.

Thm (Ozuch 1, 2): Let (M_0, g_0) be an Einstein 4-orbifold. Then there exists $\delta > 0$ s.t. if (M, g) Einstein s.t.

$$d_{GH}((M_0, g_0), (M, g)) \leq \delta$$

then (M, g) is isometric to a result of the gluing procedure described earlier but also allowing for trees of singularities.

Thm [Ozuch, 3]:

A spherical or hyperbolic 4-orbifold with at least one isolated singularity $\mathbb{R}^4/\mathbb{Z}_2$ cannot be a limit of smooth Einstein metrics.

Ricci flow:

$$M, \quad \partial_t g = -2 \mathrm{Ric}_g$$

- curvature blows up if there is a finite time singularity
- singularity models are noncollapsed ancient solutions
- immortal solutions expected to describe the ~~[illegible]~~ ~~[illegible]~~ core model geometric of the initial topology

Renormalized Ricci flow

$$\partial_t g = -2(\mathrm{Ric} - \Lambda), \quad \Lambda \in \mathbb{R}$$

- In dimension 2, 3 all ancient noncollapsed solutions are know
- Dimension 4 is ~~[illegible]~~ more complicated

FIK : shrinking soliton on $\mathcal{O}(-1)$ where the curvature along the Ricci flow only blows up on an embedded $S^2 \subset \mathcal{O}(-1)$ and converges everywhere else to its asymptotic cone

(also is cohomogeneity one $U(2)$-symmetric)