Thm: $\Gamma \subset SU(2)$ finite group acting freely on S^3 and (M_0^4, g_0) metric suspension over S^3/Γ endowed with a metric of constant curvature with ~~Λ~~ constant curvature $\Lambda > 0$. Let (N, g) be a hyperkähler ALE asymptotic to $\mathbb{C}^2/\Gamma$ and $k_\Gamma = \dim_{\mathbb{R}} H^2_c(N)$. Denote by M the manifold obtained from glueing (N, g) ~~to~~ (M_0, g_0) at its orbifold points. Then M is either oriented homeo to $\#k_\Gamma(S^2 \times S^2)$ if $k_\Gamma > 1$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, $\overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$ or $S^2 \times S^2$ and $\exists$ $C(k_\Gamma)$-parameter family of non-isometric solutions $(g(t))_{t \in (-\infty, 0]}$ on M to the renormalised Ricci flow

$$\partial_t g = -2 \operatorname{Ric}(g(t)) + 2\Lambda g(t)$$

on $M \times (-\infty, 0]$.

The metric converges locally smoothly on M_0 to g_0 outside the orbifold points of M_0 as $t \to -\infty$ and Gromov–Hausdorff converge to (M_0, g_0).

$(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$r = \sqrt{x_1^2 + \cdots + x_4^2}$

$\omega_1^\pm = dx^1 \wedge dx^2 \pm dx^3 \wedge dx^4$

$\omega_2^\pm$, $\omega_3^\pm$ obtained by cyclic perms of $\{2, 3, 4\}$.

(form orthogonal basis of space of selfdual & antiselfdual 2-forms)

$\alpha_i^\pm := \omega_i^\pm \left(\frac{\partial r}{r}\right)$ basis of $SU(2)$ invariant 1-forms on S^3.

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4 manifold (M, g)

Riemannian curvature decomposes using selfdual and anti-selfdual 2-forms):

$$R = \begin{pmatrix} R^+ & \mathring{Ric} \\ \mathring{Ric} & R^- \end{pmatrix}$$

constant curvature $Ric(g) = \lambda g \Rightarrow R^\pm = \frac{\lambda}{3} I_3$.

$T^* S^2$:

$$eh = \sqrt{\frac{r^4}{1+r^4}}\left(dr^2 + r^2 \alpha_1^2\right) + \sqrt{1+r^4}\left(\alpha_2^2 + \alpha_3^2\right)$$

asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$.

$\zeta = (x, y, z) \in \mathbb{R}^3 \setminus \{0\}$

$\zeta_1^+ = x\omega_1^+ + y\omega_2^+ + z\omega_3^+$ and ζ_2^+, ζ_3^+

so they form orthogonal basis of constant norm.

$$\epsilon = \sqrt[4]{x^2+y^2+z^2}\ , \quad \phi_i^+ = \epsilon^{-2}\zeta_i^+$$

$$eh_\zeta = \sqrt{\frac{r^4}{\epsilon^4+r^4}}\left(dr^2 + \phi_1^+(\partial_r)^2\right) + \frac{\sqrt{\epsilon^4+r^4}}{r^2}\left(\phi_2^+(\partial_r)^2 + \phi_3^+(\partial_r)^2\right)$$

1) $eh_0 = e$, $eh_{(1,0,0)} = eh$

2) eh_ζ is isometric to $|\zeta| \cdot eh$

$\shortparallel$

$|\zeta|\,(s_\zeta)_* eh$

s_ζ composition of $r \mapsto \epsilon r$

& rotation $\omega_i^+ \mapsto \phi_i^+$

Infinitesmal Ricci flat ALE deformations of Eguchi Hanson

$O(eh)$ set of smooth symmetric 2-tensor

$h \in C^2(eh)$ s.t.

$d_{eh}\,Ric(h) = 0, \quad div_{eh} h = 0, \quad tr_{eh} h = 0$

$O(eh)$ is 3-dim.

$O(eh_3)$ is spanned by $O_i(3) = \partial_t\big|_{t=0} eh_{3+t3_i}$.

~~Approximate kernel~~ $\cancel{\tilde{O}(t)}$

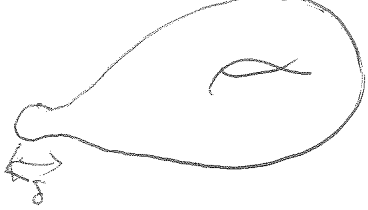
$\cancel{O_i(t) \approx}$

(M_0, g_0) compact spherical Einstein orbifold

$Ric(g) = \Lambda g$, $\Lambda > 0$ with $\mathbb{R}^4/\mathbb{Z}_2$ singularity

$$(N, g) = (T^*S^2, eh)$$

On $N \# M_0$:



$$\tilde{g} = \chi_\delta(eh_s + h_{2,3}) + (1-\chi_\delta)(g_0 + h_7^4)$$

for $h_{2,3}$, h_7^4 two-tensors to match up eh_7 & g_0 to an appropriate order (see section 3 of "orbifold singularity formation along ancient & immortal Ricci flow").

Under the Ricci flow $\tilde{g}$ approximately evolves by simple equation

$$\frac{d\mathfrak{z}(t)}{dt} = R_p^+(\mathfrak{z}),$$

where R_p^+ is $R^+(p)$ for p the singularity point of (M_0, g_0).

$R_p^+ > 0 \Rightarrow |\zeta|$ will increase under the Ricci flow

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Approximate Ricci flow:

Fix $\zeta_0 \in \mathbb{R}^3 \setminus \{0\}$.

$$g(t) = \chi_{\delta(t)}\left(e h_{\eta(t)} + h_{2,\zeta}\right) + (1-\chi_\delta)(g_0 + h_\zeta^4)$$

for $\delta(t) \to 0$ as $t \to -\infty$, $t \in (-\infty, T)$

for some $T < 0$ and

$$\dot{\zeta}(t) = 2R_p^+(\zeta) + \eta, \quad \zeta(T) = e^{2R_p^+(\zeta_0)}.$$

where ~~$\eta(t)$ is~~ $\eta(t) \in \mathbb{R}^3$

will appear in our map Φ later.

Approximate kernel $\tilde{\sigma}(t)$:

1) $\tilde{\sigma}_i(t) = \chi_{s(t)}\, \sigma_i(z(t))$

2) $\hat{g}(t)$

3) $\exists$ function v_0 on M_0 s.t. $\Delta_{g_0} v_0 = -(1+1) v_0$

so $\mathcal{L}_{\nabla^{g_0} v_0}\, g_0 = \frac{1}{4}\Delta_{g_0}(v_0 g_0) = -v_0 g_0$.

Set $\hat{v}(t) = (1-\chi_s) v_0 + \chi_{s(t)} v_{\hat{g}(t)}(t)$

where $v_{\hat{g}(t)}(t) = 1 - \epsilon^2(t)(S_{z(t)})_3\, U$

for $\Delta_{g_0} U = 4$, $U = \frac{1}{2} r^2 + O((1+r)^{-2})$.

set $\tilde{c}(t) = \frac{1}{2}\mathcal{L}_{\nabla^{\hat{g}(t)} \hat{v}(t)}\, \hat{g}(t)$.

$$\tilde{\sigma}(t) = \langle \tilde{\sigma}_i(t), \hat{g}(t), \tilde{c}(t) \rangle.$$

~~Rem~~ Prop :

Let $\psi \in C^{0,\alpha}_{\gamma,\sigma+2,T}$. Then $\exists!\, h \in h^{2,\alpha}_{\gamma,\sigma,T}$ orthogonal to $\tilde{o}(t)$ $\forall t \leq T$ and s.t

$$(\partial_t - \Delta_{L,\tilde{g}(t)} - 2\Lambda)h(t) - \psi(t) \in \tilde{\sigma}(t)$$

i.e. $\exists\, \lambda_i, \nu, \mu\, (t)$ s.t

$$(\partial_t - \Delta_{L,\tilde{g}(t)} - 2\Lambda)h(t) = \psi(t) + \lambda_i(t)\,\tilde{o}_i(t) + \nu(t)\,\tilde{g}(t) + \mu(t)\,\tilde{c}(t).$$

Moreover, $\|h\| \leq C\|\psi\|$

...

$$\|s\|_{C^{k,\alpha}_{w,\rho}(g)} = \sup_M w\left(\sum_{i=0}^{k} \rho^i |\nabla^{g,i}s|_g + \rho^{k+\alpha}[\nabla^{g,k}s]_{C^\alpha(g)}\right).$$

Will define a map

$(k, n, \beta, \tau) \longmapsto (h, \gamma, v, \mu)$

<u>Hölder norms:</u>

1) Given $n \in C^{0,\alpha}_{n_0,T}$ define

$$\dot{z}(t) = 2R_p^+(z(t)) + n(t), \quad z(T) = e^{2TR_p^+} z_0$$

2) $$\psi(t) := -Q_{g(t)}(k(t)) + \beta(t)k(t) + \tfrac{1}{2}\tau(t) \mathcal{L}_{\nabla^{\tilde{g}} \tilde{\tau}} k$$
$$- (\partial_t \tilde{g} + 2\mathrm{Ric}\,\tilde{g} - 2\Delta \tilde{g})$$

$$Q_{\tilde{g}}(k(t)) = 2\mathrm{Ric}(\tilde{g}+k) - 2\mathrm{Ric}(\tilde{g})$$
$$+ \Delta_{L,\tilde{g}} k - \mathcal{L}_{\mathrm{div}_{\tilde{g}} k - \frac{1}{2} d\,\mathrm{tr}_{\tilde{g}} k} (\tilde{g}+k)$$

3) ~~[illegible]~~ Choose the unique h s.t.

$$\partial_t h - \Delta_{L,\tilde{g}} h - 2\Delta h = \psi(t) + \lambda_i \hat{o}_i + v\tilde{g} + \mu \hat{c}.$$

4) $$d_{z(t)} g(\gamma(t)) = d_{z(t)} g(n(t)) - \sum_i \lambda_i(t) \|\hat{o}_i(t)\|_{L^2_{\tilde{g}(t)}} \frac{o_i(z(t))}{\|o_i(z(t))\|^2_{L^2(g_{z(t)})}}$$

Set

$$X = C^{2,\alpha}_{Y,\sigma,T} \times C^{0,\alpha}_{h_0,T} \times C^{0,\alpha}_{\beta_0,T} \times C^{0,\alpha}_{\tau_0,T}$$

$$X' = C^{2,\alpha'}_{Y,\sigma,T} \times C^{0,\alpha'}_{h_0',T} \times C^{0,\alpha'}_{\beta_0',T} \times C^{0,\alpha'}_{\tau_0,T}$$

for $\alpha' < \alpha$, $h_0' < h_0$, $\beta_0' < \beta_0$.

<u>Thm</u>: Fix $\xi_0 \in \mathbb{R}^3 \setminus \{0\}$, $h_0 > \Lambda$ s.t. $(h_0 - \Lambda)|\xi_0| > c_0$ for some c_0. Then, for the ball $B \subset X$ of radius 1,

1) $Id : X \to X'$ compact embedding
2) $\exists T < 0$ negative so that $\Phi(B) \subset B$
3) Φ is cts on B w.r.t to the topology induced by that of X'.

Therefore, by the Schauder fixed point theorem,

$\exists\, g(t) = \hat{g}(t) + k(t)$ s.t.

$$\partial_t g = -2Ric(g(t)) + (2\Lambda + \beta(t))g(t) + \mathcal{L}_{V(t)}(g(t))$$

where $g(t)(V(t), \cdot) = \frac{\tau(t)}{2} g(t)(\nabla^g \hat{V}, \cdot) + B_{\hat{g}}(g - \hat{g})$

where $B_{\hat{g}}$ is the Bianchi operator.

$g(t)$ ~~it~~ satisfies the Ricci flow up to rescaling and reparametrization, with limit at $t = -\infty$ being (M_0, g_0).