Geometry advanced class
Construction of non-locally symmetric Einstein manifolds with negative curvature, Part I

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1 Introduction

The goal of these two classes is to describe the construction of Einstein metrics with negative sectional curvature which are not locally symmetric by Hamenstädt–Jäckel [3] (in dimension $n \geq 4$) on a class of compact manifolds constructed by Gromov–Thurston [2], building upon earlier work of Fine–Premoselli [1] (in dimension 4). More precisely, we have the following theorem:

Theorem 1.1. *For any $n \geq 4$ there exists infinitely many pairwise non-homeomorphic smooth closed Riemannian manifolds (X, g) of dimension n satisfying:*

(i) $\mathrm{Ric}_g = -(n-1)g$.

(ii) $\sec_g < 0$.

(iii) g is not locally homogeneous.

Moreover, if $n = 4$ the manifolds X do not admit any *locally homogeneous Einstein metrics, and if $n \geq 5$ they do not admit a locally symmetric Einstein metric.*

In a sense, these are Einstein manifolds which have "nothing special" (no special holonomy or high degree of symmetries, even locally). This rules out the possibility of constructing them explicitly. Instead, the idea (which is classical for constructing solutions of nonlinear geometric PDEs) is use a *gluing-perturbation method.* Schematically this involves three steps:

Step 1: Build a countable family (X_i^n, g_i) of Einstein manifolds which have $\sec_{g_i} \leq -\delta < 0$ and $\| \mathrm{Ric}_{g_i} + (n-1)g_i \| \to 0$ as $i \to \infty$. This is done by gluing different Einstein metrics (constructed using for instance homegeneity or cohomogeneity one methods) together.

Step 2: Show that g_i can be deformed to a nearby Einstein metric $\tilde{g}_i$; if $\tilde{g}_i$ is close enough to g_i (say in the C^2-sense) then the negativity of the sectional curvature is preserved, as will be other properties of the sectional curvature which will imply that the metric cannot be locally homogeneous.

Step 3: Find obstructions to the existence of locally symmetric Einstein metrics on X_i. These considerations usually have a more topological flavour.

Before explaining how each of these steps work, let me give a word of warning about branched covers. If M and X are two n-manifolds and $\Sigma \subset X$ is a submanifold of codimension 2, we will say that a continuous map $X \to M$ is a covering map of degree l ramified along Σ if

1. $\pi : X\backslash\pi^{-1}(\Sigma) \to M\backslash\Sigma$ is a *smooth* covering map of degree l.
2. $\pi : \pi^{-1}(\Sigma) \to \Sigma$ is a diffeomorphism.

But we do *not* require that π be a smooth map globally. The reason is the following issue. Consider the map $\pi' : \mathbb{C} \to \mathbb{C},\ z \mapsto z^l$. This is a globally smooth branched cover of $\mathbb{C}$ with branching locus $\Sigma = \{0\}$. The problem with this covering map is that $d\pi$ vanishes at $z = 0$, and hence the pull-back on any Riemannian metric (or indeed any other tensor) on $\mathbb{C}$ by π degenerates at $z = 0$: for instance $\pi^*|dz|^2 = l^2|z|^{2(l-1)}|dz|^2$. But if one instead considers the map $\pi : \mathbb{C} \to \mathbb{C}, z \mapsto z^l/|z|^l$, which is a branched cover of degree l ramified at 0 but not a smooth map, then the pull-back of a Riemannian metric on $\mathbb{C}$ is a singular metric with cone angle $2\pi l$ at $z = 0$; e.g. $\pi^*|dz|^2 = dr^2 + l^2r^2d\theta^2$ in polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{R}/(2\pi\mathbb{Z})$.

2 Topological construction

Definition 2.1. A *hyperbolic manifold* of dimension $n \geq 3$ is a complete Riemannian manifold (M, g) with constant sectional curvature $\sec_g \equiv -1$.

It follows from the classification of space-forms that the universal cover $(\widetilde{M}, \widetilde{g})$ of a hyperbolic n-manifold (M, g) is isometric to the hyperbolic space $(\mathbb{H}^n, g_n)$. It can be described as $\mathbb{H}^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | x_0 > 0, Q(x, x) = -1\}$ where the quadratic form Q is defined by

$$Q(x, x) = -\sqrt{2}x_0^2 + x_1^2 + \cdot + x_n^2 \tag{2.1}$$

and the metric g_n is induced by restriction of Q. The hyperbolic space is symmetric: this is the symmetric space of $SO_0(Q)$. If S is a complete totally geodesic submanifold of $\mathbb{H}^n$, then $(S, g_S) \simeq (\mathbb{H}^{n-2}, g_{n-2})$. Moreover, $\mathbb{H}^n$ is diffeomorphic to $S \times \mathbb{R}^2$, and if one uses polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{R}/(2\pi\mathbb{Z})$ on $\mathbb{R}^2\backslash\{0\}$ then one can write the hyperbolic metric (on the complement of S) as

$$g_n = dr^2 + \sinh^2(r)d\theta^2 + \cosh^2(r)g_S. \tag{2.2}$$

Using these coordinates, we can define a branching cover $p_l : S \times \mathbb{R}^2 \to S \times \mathbb{R}^2$ ramified along S by

$$p_l(x, r, \theta) = (x, r, l\theta). \tag{2.3}$$

This construction has a "global version": namely, if (M^n, g) is a compact, oriented, hyperbolic manifold and $\Sigma \subset M$ a closed totally geodesic submanifold of codimension 2, we want to construct a branched l-fold cover $X \to M$ ramified along Σ based on the above local model. This is not possible in general: there are global *topological obstructions*. But it can be done in the following situation: assume that there are two totally geodesic hypersurfaces H, H' such that $\Sigma = H \cap H'$ and H, H' separate M: $M\backslash H = U \coprod V$, $\overline{U} \cap \overline{V} = H$. Then $\Sigma = \partial(U \cap H')$ – in particular $[\Sigma] = 0 \in H_{n-2}(M)$ – and one can build a cyclic l-fold cover $X_l \to M$ ramified along Σ (basically by gluing cyclically l copies of $M\backslash(H' \cap U)$ – just think of the $n = 2$ case). These are particular examples of Gromov–Thurston manifolds [2].

The following existence result is a combination of [1, Prop. 1.1] and [3, Prop. 3.1]:

Proposition 2.2. *For each $n \geq 4$, there exists a sequence of compact hyperbolic manifolds (M_k^n, g_k) containing such codimension 2 totally geodesic submanifolds Σ_k such that*

(i) The injectivity radius $R_k \to \infty$.

(ii) The normal injectivity radius of $\Sigma_k \subset M_k$, $R_k^\nu \geq R_k/2$.

(iii) Σ_k has a most two connected components.

(iv) $\frac{\operatorname{diam}(\Sigma_k)}{R_k^\nu} \to 0$.

Remark 2.3. In the next lecture, it will be useful to note that since $\Sigma_k \simeq \mathbb{H}^{n-2}/\Gamma_k$ for some discrete group of isometries Γ_k, the volume of Σ_k is bounded by

$$\operatorname{vol}(\Sigma_k) \leq C_n \exp((n-3)\operatorname{diam}(\Sigma_k))$$

for some universal constant C_n. This is a special easy case of the Bishop–Gromov inequality, which holds more generally for Riemannian manifolds with $\operatorname{Ric} \geq -(n-3)$.

Remark 2.4. I will not attempt to reproduce the proof; but basically the idea is just to find a discrete subgroup of $SO(Q)$ satisfying appropriate properties.

With this theorem we get of family of manifolds $X_{k,l}$ with hyperbolic metrics $g_{k,l} = \pi_l^* g_k$ which have conical singularities along Σ_k, with cone angle $2\pi l$. The next step is to find model Einstein metrics to *smooth out* these singularities.

3 Local model along the branching locus

To smooth out the previously constructed conically singular metrics near the branching locus, we seek Einstein metrics h_l on $S \times \mathbb{R}^2$ ($S \simeq \mathbb{H}^{n-2}$) such that

$$h_l \sim dr^2 + l^2 \sinh^2(r) d\theta^2 + \cosh^2(r) g_S \tag{3.1}$$

as $r \to \infty$. Using the cyclic l-fold branched cover p_l, we can rephrase this problem and instead look for an asymptotically hyperbolic metric on $S \times \mathbb{R}^2$ which has conical singularities with angle $2\pi/l$ along S. In order to do this it is convenient to introduce the coordinate $u = \cosh(r)$ and rewrite the hyperbolic metric

$$g_n = \frac{du^2}{u^2-1} + (u^2-1)d\theta^2 + u^2 g_S. \tag{3.2}$$

The idea is to perturb slightly the potential $V(u) = u^2 - 1$. In fact, we consider the family of metrics given by

$$h_a = \frac{du^2}{V_a(u)} + V_a(u) d\theta^2 + u^2 g_S. \tag{3.3}$$

where

$$V_a(u) = u^2 - 1 + \frac{a}{u^{n-3}}. \tag{3.4}$$

Here a is a real parameter taking values in the interval $(-\infty, a_{\max}(n)]$ and this defines a positive metric on $(u_a, \infty) \times \mathbb{R}/(2\pi\mathbb{Z}) \times S$ where $u_a > 0$ is the maximal root of V_a. Fine–Premoselli proved that these metrics satisfy the Einstein equation $\operatorname{Ric}_{h_a} = -(n-1)h_a$ [1, Prop. 3.2]. Evidently, h_a is asymptotically hyperbolic. Moreover, by doing a Taylor expansion near $u = u_a$, one can easily prove that the metrics are conically singular along S, with cone angle c_a which can be given as an explicit function of u_a. In fact, one can prove that the map $a \mapsto c_a$ is one-to-one, and there exists a sequence $a_l \to a_{\max}$ such that $c_{a_l} = 2\pi/l$ [1, Prop. 3.3]. Finally, by an explicit computation Fine–Premoselli obtained [1, Lem. 3.4]

$$\sec_{h_a} \leq -1 + \frac{n-2}{2} a u_a^{1-n} \tag{3.5}$$

where the RHS is negative when $a \to a_{\max}$.

Remark 3.1. We can take a quotient $\Sigma = S/\Gamma$ where Γ is a discrete group of isometries of S and consider the metrics h_a as defined on $\Sigma \times \mathbb{R}^2$ as the potential only depends on the normal radial coordinate.

Remark 3.2. We obtain a smooth metric $p_l^* h_{a_l}$ on $S \times \mathbb{R}^2$, asymptotic to $p_l^* g$ and symmetric under the action of the cyclic groups $\mathbb{Z}_l$ (in fact under the action of a full circle). The fixed-point set of this action is the branching locus S, end hence S is totally geodesic for this metric. In particular, the sectional curvature of $p_l^* h_{a_l}$ along S is $-u_{a_l}^2$, which can be shown to be strictly smaller than -1.

4 A family of approximately Einstein metrics

Let (M_k, g_k), Σ_k and $\pi_{k,l} : X_{k,l} \to M_k$ be as before. On M_k, we define the functions $r(x) = d(\Sigma_k, x)$ and $u = \cosh(r)$. Let us moreover denote $U_{k,\max} = \cosh(R_k^\nu/2)$ and pick a sequence $U_k \to \infty$ with $U_k \leq U_{k,\max}$. We aim to prove the following proposition ([3, Prop. 2.4], after [1, Prop. 3.1]):

Proposition 4.1. *There exists a family of metrics $g_{k,l}$ on $X_{k,l}$ and constants $C(m,n,l)$, $c(m,n,l)$ depending on the dimension n and an integers m, l such that (for k large enough):*

(i) $\| \operatorname{Ric}(g_{k,l}) + (n-1)g_{k,l} \|_{C^m} \leq C U_k^{-(n-1)}$ *for any $m \in \mathbb{N}$.*

(ii) $\sec(g_{k,l}) \leq -c < 0$.

(iii) $\operatorname{vol}(\{\frac{1}{2}U_k < u < U_k\}) \leq C \operatorname{vol}(\Sigma_k) U_k^{n-1}$.

The metrics $g_{k,l}$ can be constructed as follows. We may identify the tubular neighbourhood $\{u < U_k\}$ of Σ_k with a tubular neighbourhood of $\Sigma_k \times \mathbb{R}^2$ for the hyperbolic metric. Now fix a cutoff function $\chi : \mathbb{R} \to [0,1]$ such that $\chi(t) = 1$ for $t \leq 1/2$ and $\chi(t) = 1$ of $t \geq 1$. On $\Sigma_k \times \mathbb{R}^2$ define the metrics

$$\overline{g}_{k,l} = \frac{du^2}{V_{k,l}(u)} + V_{k,l}(u) d\theta^2 + u^2 g_{\Sigma_k} \tag{4.1}$$

where

$$V_{k,l}(u) = u^2 - 1 + \chi(u/U_k) \frac{a}{u^{n-3}}. \tag{4.2}$$

This metric can be extended to all of M_k by the hyperbolic metric, and it has conical singularities along Σ_k with angle $2\pi/l$. Then we might define $g_{k,l} = \pi_{k,l}^* \overline{g}_{k,l}$ on $X_{k,l}$. The various estimates follow from straightforward (though tedious to write down explicitly) computations; we refer the interested reader to the original articles for more details.

Remark 4.2. From part (iv) of Proposition 2.2 and Proposition 4.1 we obtain the L^2-estimate

$$\| \operatorname{Ric}(g_{k,l} + (n-1)g_{k,l} \|_{L^2} \to 0$$

as $k \to \infty$ (for fixed l). The importance of this estimate will be clear in the next lecture.

5 Non-existence of locally symmetric Einstein metrics

Before making a lot of effort to deform the approximately Einstein metrics to genuine one, let us justify that these efforts will not be in vain and that we will indeed end up with metrics which are not locally symmetric. In fact the author of these note has a PhD thesis to finish writing so I will not actually explain anything in detail; we will just sketch the argument of Fine–Premoselli in the case $n = 4$ and refer the reader to the article of Hammendstädt–Jäckel (which is more subtle) for the higher-dimensional case.

In the remainder of this section we shall assume that the dimension $n = 4$ unless otherwise noted. As mentioned in the introduction, the idea is to rule out the existence of locally homogeneous Einstein metrics on $X_{k,l}$ by topological consideration. First of all, notice that the fundamental group of $X_{k,l}$ is infinite (they admit negatively curved metric so the universal cover must be diffeomorphic to $\mathbb{R}^4$, and by compactness of $X_{k,l}$ this implies the result). The specificity of dimension 4 is that there is a classification of homogeneous Einstein metrics due to Jensen [4]; compact manifolds with infinite fundamental group the only possibilities are flat, hyperbolic and complex-hyperbolic metrics (metrics which are locally isometric to the symmetric space of $PU(1,m) - m = 2$ if $n = 4$), which are in particular automatically locally symmetric.

The first possibility can be ruled out by considering the fundamental group. The fundamental group of any compact flat 4-manifold manifold is isomorphic to $F \ltimes \mathbb{Z}^4$; in particular it has an abelian subgroup isometric to $\mathbb{Z}^4$. On the other hand, Preisman's theorem implies that any nontrivial abelian subgroup of a negatively curved manifold is isometric to $\mathbb{Z}$, and so $X_{k,l}$ cannot admit a flat metric.

The remaining possibilities are hyperbolic and complex-hyperbolic metrics. Such metrics are self-dual: that is, the *Weyl tensor* W is self-dual. Therefore, Chern-Weil theory and Hirzebruch's signature formula yield

$$\int_{X_{k,l}} |W|^2 d\,\mathrm{vol} = C\sigma(X_{k,l}) \tag{5.1}$$

for some combinatorial constant $C \neq 0$, where $\sigma(X_{k,l})$ is the signature of the intersection form on $H^2(X_{k,l})$. One can prove that $\sigma(X_{k,l}) = 0$ which implies that any self-dual metric has vanishing Weyl tensor, which implies that it must be conformally flat. This rules out complex-hyperbolic metrics, which are not. Finally, the fact that $X_{k,l}$ admit no hyperbolic metrics was noted by Gromov and Thurston in their original article [2, Remark 3.6].

The main difference when $n \geq 5$ is that there is no classification of locally homogeneous Einstein metrics, and Hamenstädt–Jäckel could only prove a weaker result: for any k, only a finite number of cyclic covers can admit a locally symmetric metric. The idea of proof is similar: one can prove that the only possibility would be rank one symmetric spaces (using Preissmann's theorem again), and hence the only cases to consider are hyperbolic, complex-hyperbolic, quaternionic-hyperbolic and Cayley-hyperbolic metrics. The last three possibilities are relatively easy to rule out on topological grounds, but the situation is a lot more subtle in the hyperbolic case.

References

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[2] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Inv. Math. **89** (1987), 1–12.

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[4] G. R. Jensen, *Homogeneous Einstein spaces of dimension four*, J. Differ. Geom. (1969), no. 3, 309–349.